EISENSTEIN CONGRUENCES FOR UNITARY GROUPS

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Abstract. We present a general conjecture on congruences between Hecke eigenvalues of induced and cuspidal automorphic representations, of a connected reductive group over $\mathbb{Q}$ that splits over an imaginary quadratic extension, modulo divisors of critical values of certain L-functions. We examine the consequences for various unitary groups, including some fragments of numerical evidence.

1. Introduction

Ramanujan discovered the congruence $\tau(p) \equiv 1 + p^{11} \pmod{691}$ (for all primes p), where $\Delta = \sum_{n=1}^{\infty} \tau(n)q^n = q\prod_{n=1}^{\infty}(1-q^n)^{24}$. We may view this as being a congruence between Hecke eigenvalues, for $T(p)$ acting on the cusp form Δ of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$, and on the Eisenstein series E_{12} of weight 12. The modulus 691 comes from a certain L-function evaluated at a critical point depending on the weight; specifically it divides the numerator of the rational number $\frac{\zeta(12)}{\pi^{12}}$. Conjecture 4.2 in this paper is a very wide generalisation of Ramanujan's congruence, to congruences between Hecke eigenvalues of automorphic representations of $G(\mathbb{A})$, where $\mathbb{A}$ is the adele ring and $G/\mathbb{Q}$ is any connected, reductive group. On one side of the congruence is a cuspidal automorphic representation $\tilde{\Pi}$. On the other is one induced from a cuspidal automorphic representation Π of the Levi subgroup M of a maximal parabolic subgroup P. The modulus of the congruence comes from a critical value of a certain L-function, associated to Π and to the adjoint representation of the L-group ${}^LM$ on the Lie algebra $\hat{\mathfrak{n}}$ of the unipotent radical of the maximal parabolic subgroup $\hat{P}$ of the Langlands dual $\hat{G}$. Starting from Π, we conjecture the existence of $\tilde{\Pi}$, satisfying the congruence. Ramanujan's congruence is an instance of the case $G = \mathrm{GL}_2$, $M = \mathrm{GL}_1 \times \mathrm{GL}_1$.

This paper is a sequel to [BD], which dealt exclusively with split reductive groups. The reader is referred to that paper for further introduction and motivation. Here we consider the case that G splits over an imaginary quadratic extension $F/\mathbb{Q}$; in all the examples, G is a quasi-split unitary group.

In §2 we introduce some notation and basic facts on reductive groups, characters and cocharacters, automorphic representations, Satake parameters and infinitesimal characters. In §3 we look at the L-functions mentioned above, connected with the adjoint action of $\hat{M}$ on $\hat{\mathfrak{n}}$. In §4, after introducing what we need on the Bloch-Kato conjecture, we state the main conjecture on congruences. The connection with the Bloch-Kato conjecture is examined further in the last section of [BD], at least in the split case. Here it suffices to say that a prime divisor of a suitably normalised L-value should be a divisor of the order of a Selmer group, and the congruence seems to lead to the construction (in the manner of Ribet [Ri]) of the required

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element. This construction has been worked out by Berger for Asai L-functions, and his Corollary 6.2 should apply to the situation in §7 (discussed below) [Be].

§5 deals with the relation between Hermitian modular forms (in general vector-valued) and automorphic representations of $U(n,n)$, while in §6 we see that the Langlands functoriality conjecture implies the existence of a lift from $\mathrm{SO}(n+1,n)$ to $U(n,n)$, which is especially relevant when $n=1$ or $n=2$, since $\mathrm{SO}(2,1)\simeq \mathrm{PGL}_2$ and $\mathrm{SO}(3,2)\simeq \mathrm{PGSp}_2$.

§7 introduces the first example, with $G=U(2,2)$ and $M=\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$, with P the Siegel parabolic subgroup. In some ways this is the case most closely analogous to the case $G=\mathrm{GSp}_2$, $M=\mathrm{GL}_2\times\mathrm{GL}_1$ featuring in Harder's conjecture on congruences between Hecke eigenvalues of genus-1 and (vector-valued) genus-2 Siegel modular forms [H1, vdG],[BD, §7]. Such an analogue (at least for $G=\mathrm{SU}(2,2)$, $M=\mathrm{Res}_{F/\mathbb{Q}}\mathrm{SL}_2$) was also anticipated in a remark in the introduction to a paper by Hayata and Schwermer [HS]. In the case where Π is the base change of an automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ coming from a modular form f of odd weight and imaginary quadratic character, critical values of the symmetric square L-function are involved, and we have a fragment of numerical evidence, calculated using algebraic modular forms for $U(4)$, as explained in [Du1]. Klosin's congruences between Maass lifts and non-lifts [Kl1, Kl2] may be viewed as a degenerate case of this conjecture, in the same way as the congruences of Brown and Katsurada [Br, Ka1], between Saito-Kurokawa lifts and non-lifts, may be viewed as a degenerate case of Harder's conjecture. In the case of the base change of f of trivial character, if we believe the functorial lift from $\mathrm{SO}(3,2)\simeq\mathrm{PGSp}_2$ to $U(2,2)$ then we can obtain the congruences here by lifting congruences between Klingen-Eisenstein series and cuspidal genus-2 Siegel modular forms, instances of which have been proved, the first by Kurokawa [Ku, Mi, Du2].

In §8, again $G=U(2,2)$, but this time $M\simeq \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_1\times U(1,1)$ and P is the Klingen parabolic subgroup. Here the congruences are between Hermitian Klingen-Eisenstein series and cusp forms, and instances might be provable in a manner similar to that employed by Katsurada and Mizumoto in the case of Siegel modular forms. Moreover, these congruences should be functorial lifts of those in Harder's conjecture. In their work on the main conjecture of Iwasawa theory for modular forms, Skinner and Urban [SU] use congruences between cusp forms and Klingen-Eisenstein series (with scalar weight but coefficients in an Iwasawa algebra) for the group $G=GU(2,2)$.

The group $U(2,1)$, automorphic forms on which are connected with Picard modular forms, is the subject of §9. We recover a recent conjecture of Harder, involving values of L-functions of Hecke characters, and for which Bergström and van der Geer have found some numerical evidence.

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2. Induced representations

For basic notions on reductive groups and automorphic representations, see [Sp, BJ, Bo]. Let $G/\mathbb{Q}$ be a quasi-split, connected, reductive algebraic group. From the outset, we assume that G splits over an imaginary quadratic extension $F/\mathbb{Q}$. Let $T/\mathbb{Q}\subset B/\mathbb{Q}$ be a maximal torus and Borel subgroup. Let $X^*(T)=\mathrm{Hom}_{\overline{\mathbb{Q}}}(T,\mathrm{GL}_1)$ and $X_*(T)=\mathrm{Hom}_{\overline{\mathbb{Q}}}(\mathrm{GL}_1,T)$ be the character and cocharacter groups, respectively, of T. There is a natural pairing $\langle,\rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$. Let $W_G = W = N_G(T)/T$ be the (absolute) Weyl group. Let $\Phi = \Phi_G \subset X^*(T)$ be the set of (absolute) roots, $\Phi^+ = \Phi_G^+$ the set of positive roots (with respect to B), and Δ_G the set of simple positive roots. There are natural actions of $\mathrm{Gal}(F/\mathbb{Q})$ on $X^*(T)$, $X_*(T)$, Φ, Φ^+, W.

Let $T_d/\mathbb{Q}$ be the maximal $\mathbb{Q}$-split subtorus of T. It is a maximal $\mathbb{Q}$-split torus in G, and its centraliser in G is T. Let ${}_{\mathbb{Q}}W = N_G(T_d)/T$, the relative Weyl group. This is the subgroup of W stabilising T_d, and also the subgroup fixed by the action of $\mathrm{Gal}(F/\mathbb{Q})$.

Let ρ_G be half the sum of all the positive roots in Φ^+. Given any root α, there is an associated coroot $\check{\alpha} \in X_*(T)$, with $\langle \alpha, \check{\alpha}\rangle = 2$. If $\langle,\rangle'$ is any W-invariant inner product on $X^*(T/S) \otimes \mathbb{R}$, where $S = Z(G)^0$ is the connected component of the identity in the centre of G, then for any root α, and any $\chi \in X^*(T/S)$, we have $\langle \chi, \check{\alpha}\rangle = \langle \chi, \frac{2\alpha}{\langle \alpha,\alpha\rangle'}\rangle'$. Identifying $\check{\alpha}$ with $\frac{2\alpha}{\langle\alpha,\alpha\rangle'}$ we get an isomorphism $X_*(T/S) \otimes \mathbb{R} \simeq X^*(T/S) \otimes \mathbb{R}$, so from now on we write $\langle,\rangle'$ as $\langle,\rangle$.

If we choose any $\alpha \in \Delta_G$ then there is a maximal $\mathbb{Q}$-parabolic subgroup $P = MN$ of G (containing B), with unipotent radical N and (reductive) Levi subgroup M, characterised by $\Delta_M = \Delta_G - \Delta_\alpha$, where Δ_α is the $\mathrm{Gal}(F/\mathbb{Q})$-orbit of α. Note that P might not be maximal as a parabolic subgroup of G over $\overline{\mathbb{Q}}$. Let $\Phi_N := \Phi_G^+ - \Phi_M^+$ i.e. those elements of Φ_G^+ whose decomposition as a sum of simple roots includes something in Δ_α, and let ρ_P be half the sum of the elements of Φ_N. Let $\tilde{\alpha} = \frac{\rho_P}{\langle \rho_P, \check{\alpha}\rangle}$. Then $\langle \tilde{\alpha}, \check{\alpha}\rangle = 1$, while $\langle \tilde{\alpha}, \check{\beta}\rangle = 0$ for all $\beta \in \Delta_M$. If we replace α by something else in its orbit Δ_α, we get the same P and the same $\tilde{\alpha}$.

Let $\hat{G}$ be the Langlands dual group of G [Ki, Chapter 3],[Bo, I.2]. Then $\hat{G}$ has a maximal torus $\hat{T}$ with $X^*(\hat{T}) \simeq X_*(T)$ and $X_*(\hat{T}) \simeq X^*(T)$. Under these isomorphisms, roots of $\hat{G}$ become coroots of G, and coroots of $\hat{G}$ become roots of G, with $\check{\Delta} := \{\check{\beta} : \beta \in \Delta_G\}$ mapping to a set of simple positive roots for $\hat{G}$. We can define a maximal parabolic subgroup $\hat{P}$ of $\hat{G}$, with Levi subgroup characterised by having set of simple positive roots $\check{\Delta} - \{\check{\beta} : \beta \in \Delta_\alpha\}$, hence identifiable with $\hat{M}$. Let $\hat{N}$ be the unipotent radical of $\hat{P}$, with Lie algebra $\hat{\mathfrak{n}}$. The natural action of $\mathrm{Gal}(F/\mathbb{Q})$ on $(X^*(T), \Delta, X_*(T), \check{\Delta})$ comes from an action by automorphisms on $\hat{G}$, allowing one to form a semi-direct product ${}^LG := \hat{G}(\mathbb{C}) \rtimes \mathrm{Gal}(F/\mathbb{Q})$, and similarly to define ${}^LM$.

Letting A be the maximal $\mathbb{Q}$-split torus in $Z(M)^0$, the restriction map from ${}_{\mathbb{Q}}X^*(M)$ (i.e. $\mathrm{Hom}_{\mathbb{Q}}(M, \mathrm{GL}_1)$) to $X^*(A)$ identifies ${}_{\mathbb{Q}}X^*(M)$ with a finite-index subgroup of $X^*(A)$, thus ${}_{\mathbb{Q}}X^*(M) \otimes \mathbb{R} = X^*(A) \otimes \mathbb{R}$. If $\chi \in {}_{\mathbb{Q}}X^*(M)$ then we can define, for any archimedean or non-archimedean place v of $\mathbb{Q}$, a homomorphism $|\chi|_v : M(\mathbb{Q}_v) \to \mathbb{R}^\times$ by $|\chi|_v(m) = |\chi(m)|_v$. We can extend this to ${}_{\mathbb{Q}}X^*(M) \otimes \mathbb{R}$, or even ${}_{\mathbb{Q}}X^*(M) \otimes \mathbb{C}$, by $|s\chi|_v(m) = |\chi(m)|_v^s$ (in $\mathbb{C}^\times$). (For a finite prime p, $|\cdot|_p$ is normalised so that $|p|_p = p^{-1}$.) Let $\mathbb{A}$ be the adele ring of $\mathbb{Q}$, and let $G(\mathbb{A})$ be the group of adelic points, which is easy to define, since $\mathbb{A}$ is a $\mathbb{Q}$-algebra. Likewise for $M(\mathbb{A})$. Taking a product of the $|\chi|_v$ over all places, we may define, for any $\chi \in {}_{\mathbb{Q}}X^*(M) \otimes \mathbb{R}$, a homomorphism $|\chi| : M(\mathbb{A}) \to \mathbb{R}^\times$. In particular, restricting $2\rho_P$ to A then viewing it in ${}_{\mathbb{Q}}X^*(M) \otimes \mathbb{R}$, we have $|s\tilde{\alpha}| : M(\mathbb{A}) \to \mathbb{C}^\times$, for any $s \in \mathbb{C}$. Note that this character is trivial when restricted to $S(\mathbb{A})$. For any finite prime p, let ${}_{\mathbb{Q}_p}X^*(T) = \mathrm{Hom}_{\mathbb{Q}_p}(T, \mathrm{GL}_1)$, which is $\mathrm{Hom}_{\mathbb{Q}}(T, \mathrm{GL}_1)$ or $X^*(T)$, according as p is inert or splits, respectively. Then for $\chi \in {}_{\mathbb{Q}_p}X^*(T)$ we may define $|\chi|_p : T(\mathbb{Q}_p) \to \mathbb{R}^\times$ similarly.

Let Π be an irreducible, cuspidal, automorphic representation of $M(\mathbb{A})$. We shall assume in addition that Π is unitary. We have $\Pi = \otimes_v \Pi_v$, where each Π_v is an irreducible, admissible representation of $M(\mathbb{Q}_v)$, unramified for all but finitely many v. Then $\Pi \otimes |s\tilde{\alpha}|$ is a representation of $M(\mathbb{A})$. We may parabolically induce it to a representation $\mathrm{Ind}_P^G(\Pi \otimes |s\tilde{\alpha}|)$ of $G(\mathbb{A})$. This induction is as described in [Ki, Chapter 4]. It involves the addition of ρ_P to $s\tilde{\alpha}$, with the consequence that $\mathrm{Ind}_P^G(\Pi \otimes |s\tilde{\alpha}|)$ would be unitary if $s \in i\mathbb{R}$, though we shall always take $s \in \mathbb{R}_{>0}$.

The admissibility of Π_∞ follows from it being unitary and irreducible, by a theorem of Harish-Chandra [Da, Theorem 2.3]. Then, by [Kn, Proposition 5.19] or [Da, §3], the centre $Z(\mathfrak{m}_\mathbb{C})$ of the universal enveloping algebra $U(\mathfrak{m}_\mathbb{C})$ (where $\mathfrak{m}_\mathbb{C}$ is the complexification of the Lie algebra of $M(\mathbb{R})$) acts by a character (the "infinitesimal character"), on the dense subspace of K_∞-finite vectors, where K_∞ is a maximal compact subgroup of $M(\mathbb{R})$. Given any Cartan subalgebra $\mathfrak{h}_\mathbb{C}$ of $\mathfrak{m}_\mathbb{C}$, the Harish-Chandra isomorphism from $Z(\mathfrak{m}_\mathbb{C})$ to $U(\mathfrak{h}_\mathbb{C})^{W(\mathfrak{h}_\mathbb{C})}$ (invariants under the Weyl group) allows us to write the infinitesimal character in the form χ_λ for some $\lambda \in \mathfrak{h}_\mathbb{C}^*$, determined only up to the Weyl group action. See [Kn, Theorem 5.62] or [Da, §3]. In discussions of discrete series representations, "compact" Cartan subalgebras are most directly relevant, but all Cartan subalgebras of $\mathfrak{m}_\mathbb{C}$ are conjugate by $M(\mathbb{C})$ [Kn, Theorem 2.15], and it is convenient to take $\mathfrak{h}$ to be the Lie algebra of $T(\mathbb{R})$ (and $\mathfrak{h}_\mathbb{C} = \mathfrak{h} \otimes_\mathbb{R} \mathbb{C}$), so we may identify λ (and by abuse of notation χ_λ) with an element of $X^*(T) \otimes \mathbb{C}$ (which for us will always be in $X^*(T) \otimes \mathbb{R}$). If Π_∞ has infinitesimal character λ (up to the action of W_M), then $\mathrm{Ind}_P^G(\Pi_\infty \otimes |s\tilde{\alpha}|_\infty)$ (i.e. the $\mathbb{R}$-component of $\mathrm{Ind}_P^G(\Pi \otimes |s\tilde{\alpha}|)$), though not in general unitary, has an infinitesimal character $\lambda + s\tilde{\alpha}$, now only determined up to the action of W_G. Applying an element of W_M if necessary, we may arrange for λ to be dominant with respect to Δ_M, i.e. $\langle \lambda, \check{\beta} \rangle \geq 0$ for all $\beta \in \Delta_M$. This follows from [Kn, Theorem 2.63, Proposition 2.67]. However, λ might not be dominant for Δ_G: it might not be the case that $\langle \lambda, \check{\alpha} \rangle \geq 0$, but similarly there exists some $w \in W_G$ such that $w(\lambda)$ is dominant for Δ_G. Note that the finite-dimensional representation of G with highest weight λ has infinitesimal character $\lambda + \rho_G$.

Lemma 2.1. *Let $\lambda \in X^*(T) \otimes \mathbb{R}$ be the infinitesimal character of a unitary irreducible representation of $M(\mathbb{R})$. Then $c(\lambda)$ is W_M-equivalent to $-\lambda$, where $c \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is complex conjugation.*

Proof. Since the representation is unitary, its conjugate (i.e. $V \otimes_\sigma \mathbb{C}$, where σ is complex conjugation, so all matrix coefficients are conjugated) and its dual are isomorphic. The infinitesimal character of the conjugate is $c(\lambda)$, while that of the dual is $-\lambda$, both up to W_M-equivalence. □

Let p be a finite prime such that G is unramified at p (i.e. G splits over an unramified extension of $\mathbb{Q}_p$, which for us means that $F/\mathbb{Q}$ is unramified at p) and Π_p is unramified (or "spherical"), i.e. has a non-zero K_p^M-fixed ("spherical") vector, where K_p^M is a hyperspecial maximal compact subgroup of $M(\mathbb{Q}_p)$ (which exists by [Ti, 10.2.1]; likewise a hyperspecial maximal compact subgroup K_p of $G(\mathbb{Q}_p)$ exists). Then for some $\chi_p \in {}_{\mathbb{Q}_p}X^*(T) \otimes i\mathbb{R}$, Π_p is isomorphic to a unique irreducible quotient of the (unitarily) parabolically induced representation $\mathrm{Ind}_{B_M(\mathbb{Q}_p)}^{M(\mathbb{Q}_p)}(|\chi_p|_p)$

[Ca, 4.4(d)], where $B_M := B \cap M$. Note that χ_p can be replaced by anything in the same ${}_{\mathbb{Q}_p}W_M$-orbit, and that the character $|\chi_p|_p$ of $T(\mathbb{Q}_p)$ is unramified, i.e. trivial on the maximal compact subgroup ${}^oT(\mathbb{Q}_p)$. Also, $\mathrm{Ind}_{B_M(\mathbb{Q}_p)}^{M(\mathbb{Q}_p)}(|\chi_p|_p)$ is irreducible if χ_p is regular (i.e. if $\langle \chi_p, \check{\beta}\rangle \neq 0$, for every $\beta \in \Phi_M$). The local component at p of $\mathrm{Ind}_P^G(\Pi \otimes |s\tilde{\alpha}|)$ is a subquotient of $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(|\chi_p + s\tilde{\alpha}|_p)$, by transitivity of induction [Ca, I(36)]. Hence it has the spherical subquotient of $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(|\chi_p + s\tilde{\alpha}|_p)$ as an irreducible constituent. Note that $|\chi_p + s\tilde{\alpha}|_p$ is still an unramified character of $T(\mathbb{Q}_p)$, though it is not unitary for $s \notin i\mathbb{R}$. In our application, χ_p will always be regular for M, and s chosen so that $\chi_p + s\tilde{\alpha}$ is regular for G, hence $\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(\Pi_p \otimes |s\tilde{\alpha}|_p)$ will be irreducible.

We refer to χ_p and $\chi_p + s\tilde{\alpha}$ as the Satake parameters at p of Π and $\mathrm{Ind}_P^G(\Pi \otimes |s\tilde{\alpha}|)$, respectively. Let $\mathcal{H} = \mathcal{H}(G(\mathbb{Q}_p), K_p)$ be the Hecke algebra of $\mathbb{C}$-valued, compactly supported, K_p-bi-invariant functions on $G(\mathbb{Q}_p)$. If $f \in \mathcal{H}$ then f acts on $\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(\Pi_p \otimes |s\tilde{\alpha}|_p)$ by $v \mapsto \int_{G(\mathbb{Q}_p)} g(v) f(g)\, dg$, where dg is a left- and right-invariant Haar measure, normalised so that K_p has volume 1. Then $\mathcal{H}$ is a commutative ring under convolution of functions (which corresponds to composition of operators), and is generated by the characteristic functions T'_μ of double cosets $K_p\mu(p)K_p$, where $\mu \in X_*(T_\#)$ is any cocharacter. Here $T_\# = \begin{cases} T & \text{if } p \text{ splits;} \\ T_d & \text{if } p \text{ is inert} \end{cases}$ is a maximal split torus of $T/\mathbb{Q}_p$. If v_0 is a spherical vector then necessarily so is $T'_\mu(v_0)$, but since v_0 is unique up to scalar multiples, $\mathcal{H}$ acts on v_0 by a character. The value of this character on any particular element of $\mathcal{H}$ is a "Hecke eigenvalue".

Given $\chi \in {}_{\mathbb{Q}_p}X^*(T) \otimes \mathbb{C}$, there is $t(\chi) \in \hat{T}(\mathbb{C})$ such that, for any $\mu \in X_*(T_\#) \subseteq X^*(\hat{T})$, $\mu(t(\chi)) = |\chi(\mu(p))|_p$ [Ca, III(3)]. In the case $\chi = s\lambda$, with $\lambda \in {}_{\mathbb{Q}_p}X^*(T) \subseteq X_*(\hat{T})$ and $s \in \mathbb{C}$, we have $t(\chi) = \lambda(p^{-s})$, and $\mu(t(\chi)) = |\chi(\mu(p))|_p = p^{-s\langle \lambda, \mu\rangle}$. If G splits over $\mathbb{Q}_p$ (i.e. if p splits in F), the Hecke eigenvalue for T'_μ, on the spherical representation of $G(\mathbb{Q}_p)$ with Satake parameter χ may be calculated using the Satake isomorphism. In particular, if μ is minuscule, meaning that the orbit of μ under W_G is the set of weights for the irreducible representation θ_μ of $\hat{G}$ with highest weight μ, then the eigenvalue is $p^{\langle \rho_G, \mu\rangle} \mathrm{Tr}(\theta_\mu(t(\chi))) = p^{\langle \rho_G, \mu\rangle} \sum_{w \in W_G} |\chi(w(\mu)(p))|_p$ [Gr, 3.13,6.2]. Similarly for spherical representations of $M(\mathbb{Q}_p)$.

3. Motives and L-functions

Recall that the representation Π of $M(\mathbb{A})$, at an unramified prime p, has a Satake parameter $\chi_p \in {}_{\mathbb{Q}_p}X^*(T) \otimes i\mathbb{R}$, or $t(\chi_p) \in \hat{T}(\mathbb{C}) \subset \hat{M}(\mathbb{C})$. Now let $\tilde{t}(\chi_p) := (t(\chi_p), \mathrm{Frob}_p) \in {}^LM$. The $\hat{M}$-conjugacy class of $\tilde{t}(\chi_p)$ is what we should really be calling the Satake parameter. Given a representation $r : {}^LM \to \mathrm{GL}_d$, we may define a local L-factor

$$L_p(s, \Pi_p, r) := \det(I - r(\tilde{t}(\chi_p))p^{-s})^{-1},$$

then an L-function (in general incomplete)

$$L_\Sigma(s, \Pi, r) := \prod_{p \notin \Sigma} L_p(s, \Pi_p, r),$$

where Σ is a finite set of primes containing all those such that either $G/\mathbb{Q}_p$ (i.e. $F/\mathbb{Q}$) or Π_p is ramified.

In particular, we take for r the adjoint representation of ${}^LM$ on the Lie algebra $\hat{\mathfrak{n}}$ of the unipotent radical of the maximal parabolic $\hat{P}$. Now $\hat{\mathfrak{n}}$ is a direct sum of subspaces on which $\hat{T}$ acts by those positive roots of $\hat{G}$ that are not roots of $\hat{M}$. These are identified with certain coroots $\check{\gamma}$ of G, as γ runs through Φ_N. Recall that G splits over an imaginary quadratic extension $F/\mathbb{Q}$. Let $\mathrm{Gal}(F/\mathbb{Q}) = \{\mathrm{id}\,., c\}$.

Lemma 3.1. (1) *If p splits in F then*

$$L_p(s, \Pi_p, r)^{-1} = \prod_{\gamma\in\Phi_N}(1 - \check{\gamma}(t(\chi_p))p^{-s}) = \prod_{\gamma\in\Phi_N}(1 - |\chi_p(\check{\gamma}(p))|_p\, p^{-s}).$$

(2) *If p is inert in F, let $\Phi_N(\mathrm{fix}) := \{\gamma \in \Phi_N : c(\gamma) = \gamma\}$, $\Phi_N(\mathrm{nfix}) := \Phi_N - \Phi_N(\mathrm{fix})$, and let $\Phi_N(\mathrm{nfix})^{\mathrm{rep}}$ be a set of representatives of the $\mathrm{Gal}(F/\mathbb{Q})$-orbits in $\Phi_N(\mathrm{nfix})$, each orbit being a pair of elements exchanged by c. Then*

$$L_p(s, \Pi_p, r)^{-1} = \prod_{\gamma\in\Phi_N(\mathrm{fix})}(1 - |\chi_p(\check{\gamma}(p))|_p\, p^{-s}) \prod_{\gamma\in\Phi_N(\mathrm{nfix})^{\mathrm{rep}}}(1 - |\chi_p(\check{\gamma}(p))|_p^2\, p^{-2s}).$$

Proof. If p splits then each root space for each $\check{\gamma}$ is an eigenspace for the action of $\tilde{t}(\chi)$, with eigenvalue $\check{\gamma}(t(\chi_p)) = |\chi_p(\check{\gamma}(p))|_p$. If p is inert then we can deal with $\gamma \in \Phi_N(\mathrm{fix})$ similarly. If $\gamma \in \Phi_N(\mathrm{nfix})$ then $r(\tilde{t}(\chi))$ exchanges the root spaces of $\check{\gamma}$ and $c(\check{\gamma})$ (because conjugation by c does). On the 2-dimensional direct sum of these root spaces, it is represented by the matrix $\begin{pmatrix} 0 & \check{\gamma}(t(\chi_p)) \\ \check{\gamma}(t(\chi_p)) & 0 \end{pmatrix}$. Note that $c(\check{\gamma})(t(\chi_p)) = \check{\gamma}(t(\chi_p))$, since $\chi_p \in {}_{\mathbb{Q}}X^*(T)$. □

Actually, r is a direct sum of irreducible representations r_i for some $1 \le i \le m$, where r_i acts on the direct sum $\hat{\mathfrak{n}}_i$ of root spaces for $\Phi_N^i := \{\check{\gamma} \in \Phi_N : \langle\tilde{\alpha}, \check{\gamma}\rangle = i\}$ [Ki, Theorem 6.6]. Then

$$L_\Sigma(s, \Pi, r) = \prod_{i=1}^m L_\Sigma(s, \Pi, r_i).$$

Note that $L_\Sigma(0, \Pi \otimes |s\tilde{\alpha}|, r_i) = L_\Sigma(is, \Pi, r_i)$ (and beware that here i is not $\sqrt{-1}$).

Let $s \in \mathbb{C}$ be chosen so that $\lambda + s\tilde{\alpha}$ is *algebraically integral*, i.e. $\langle\lambda + s\tilde{\alpha}, \check{\gamma}\rangle \in \mathbb{Z}$ for all $\gamma \in \Phi$. By [Cl, Conjecture 4.5] (applied to the conjectural functorial lift of $\Pi\otimes|s\tilde{\alpha}|$ to $\mathrm{GL}_d(\mathbb{A})$), there should exist a motive $\mathcal{M}(r, \Pi\otimes|s\tilde{\alpha}|)$ of rank $d = \dim(\hat{\mathfrak{n}})$, such that if $\rho_{\Pi\otimes|s\tilde{\alpha}|} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_d(E_{\mathfrak{q}})$ is the representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on a $\mathfrak{q}$-adic realisation (where E is a field of definition of the Satake parameters [BG, Definitions 2.2.1, 3.1.3], and $\mathfrak{q}$ is any prime divisor), then $\rho_{\Pi\otimes|s\tilde{\alpha}|}(\mathrm{Frob}_p^{-1})$ is conjugate in $\mathrm{GL}_d(E_{\mathfrak{q}})$ to $r(\tilde{t}(\chi_p + s\tilde{\alpha}))$. In fact, this should be a direct sum $\oplus_{i=1}^m \mathcal{M}(r_i, \Pi \otimes |s\tilde{\alpha}|)$, with $\mathfrak{q}$-adic realisations $\rho_{\Pi\otimes|s\tilde{\alpha}|,i}$.

If $r_\infty : W_{\mathbb{R}} \to {}^LM$ is the Langlands parameter at ∞ (of $\Pi\otimes|s\tilde{\alpha}|$) then, restricting to the subgroup $\mathbb{C}^\times$, of index two in the Weil group $W_{\mathbb{R}}$, $r_\infty(z)$ is conjugate in $\hat{M}(\mathbb{C})$ to $(\lambda + s\tilde{\alpha})(z)(\lambda' + s\tilde{\alpha})(\overline{z})$, where λ' is in the same W_M-orbit as $c(\lambda)$ (and c is the action of complex conjugation). This follows from the discussions in [Bo, §10.5] and [BG, §2.3]. Actually, because Π_∞ is unitary, the image of $W_{\mathbb{R}}$ under its Langlands parameter should be bounded, forcing $\lambda' = -\lambda$ (which is in the W_M-orbit of $c(\lambda)$ also, by Lemma 2.1). Observe also that $\tilde{\alpha}$ is fixed by c and by W_M. Anyway, $r_i \circ r_\infty(z)$ is then conjugate in $\mathrm{GL}_{d_i}(\mathbb{C})$ to $\mathrm{diag}(z^{\langle\lambda+s\tilde{\alpha},\check{\gamma}\rangle}\overline{z}^{\langle-\lambda+s\tilde{\alpha},\check{\gamma}\rangle})_{\gamma\in\Phi_N^i} =$

$\mathrm{diag}(z^{\langle\lambda,\check{\gamma}\rangle+is}\,\overline{z}^{-\langle\lambda,\check{\gamma}\rangle+is})_{\gamma\in\Phi_N^i}$. It follows that the Hodge type of $\mathcal{M}(r_i,\Pi\otimes|s\tilde{\alpha}|)$ should be $\{(-\langle\lambda,\check{\gamma}\rangle-is,\langle\lambda,\check{\gamma}\rangle-is):\gamma\in\Phi_N^i\}$. (For the minus sign, see [De2, 1.1.1.1]. This accords with the fact that making a positive Tate twist reduces the weight.) The complex conjugation F_∞ on the Betti realisation $H_B(\mathcal{M}(r_i,\Pi\otimes|s\tilde{\alpha}|))\otimes\mathbb{C}$ should exchange (p,q) and (q,p), so the next lemma is no surprise.

Lemma 3.2. *If $\gamma\in\Phi_N$ then $\gamma':=w_0^Mc(\gamma)$ is also in Φ_N, and $\langle\lambda,\check{\gamma'}\rangle=-\langle\lambda,\check{\gamma}\rangle$.*

Proof. w_0^M is represented by the conjugation action of some element of M, which preserves N, as does c, so $\gamma'\in\Phi_N$. By Lemma 2.1, $\lambda=-w_0^Mc(\lambda)$. Hence $\langle\lambda,\gamma'\rangle=-\langle w_0^Mc(\lambda),w_0^Mc(\gamma)\rangle=-\langle\lambda,\gamma\rangle$. Now multiply by $\frac{2}{\langle\gamma',\gamma'\rangle}=\frac{2}{\langle\gamma,\gamma\rangle}$. $\square$

In fact, it is easy to see that if $\gamma\in\Phi_N^i$ then $\gamma'\in\Phi_N^i$.

We shall be especially concerned with the Tate twist $\mathcal{M}(r_i,\Pi\otimes|s\tilde{\alpha}|)(1)$. Let $H_B(\mathcal{M}(r_i,\Pi\otimes|s\tilde{\alpha}|)(1))$ and $H_{\mathrm{dR}}(\mathcal{M}(r_i,\Pi\otimes|s\tilde{\alpha}|)(1))$ be the Betti and de Rham realisations, and let $H_B(\mathcal{M}(r_i,\Pi\otimes|s\tilde{\alpha}|)(1))^{\pm}$ be the eigenspaces for the complex conjugation F_∞. As in [De1, 1.7], $\mathcal{M}(r_i,\Pi\otimes|s\tilde{\alpha}|)(1)$ is *critical* if $\dim(H_B(\mathcal{M}(r_i,\Pi\otimes|s\tilde{\alpha}|)(1))^+)=\dim(H_{\mathrm{dR}}(\mathcal{M}(r_i,\Pi\otimes|s\tilde{\alpha}|)(1))/\mathrm{Fil}^0)$. Let $\mathrm{wt}=-2is-2$ be the weight of $\mathcal{M}(r_i,\Pi\otimes|s\tilde{\alpha}|)(1)$ (so $p+q=\mathrm{wt}$ for all (p,q) in the Hodge type), and let $h^{p,q}$ be the dimension of the (p,q)-part $H^{p,q}$ of $H_B(\mathcal{M}(r_i,\Pi\otimes|s\tilde{\alpha}|)(1))\otimes\mathbb{C}$. Note that F_∞ exchanges $H^{p,q}$ and $H^{q,p}$, so $h^{p,q}=h^{q,p}$.

Proposition 3.3. *Let b_i be the smallest non-zero positive value of $\langle\lambda,\check{\gamma}\rangle$, for $\gamma\in\Phi_N^i$.*

1. *If wt is odd, or if wt is even but $h^{\mathrm{wt}/2,\mathrm{wt}/2}=0$, then $\mathcal{M}(r_i,\Pi\otimes|s\tilde{\alpha}|)(1)$ is critical for $0<s\leq\frac{b_i-1}{i}$ (subject also to $\lambda+s\tilde{\alpha}$ being algebraically integral).*
2. *If wt is even and $h^{\mathrm{wt}/2,\mathrm{wt}/2}\neq 0$, suppose that F_∞ acts on $H^{\mathrm{wt}/2,\mathrm{wt}/2}$ by a scalar (necessarily $(-1)^t$ with $t=0$ or 1). Then $\mathcal{M}(r_i,\Pi\otimes|s\tilde{\alpha}|)(1)$ is critical for $0<s\leq\frac{b_i-1}{i}$, subject also to $\lambda+s\tilde{\alpha}$ being algebraically integral, and the extra condition $t=0$. (Note that whenever s goes up by 1, $\mathcal{M}(r_i,\Pi\otimes|s\tilde{\alpha}|)(1)$ gets Tate twisted by i, so t changes by i $\pmod 2$. So the condition $t=0$ amounts to a kind of parity condition on is.)*

This is [BD, Proposition 3.2]. Note that in case (2), if F_∞ did not act by a scalar then there would be no critical values. The proposition describes all the positive s for which $\mathcal{M}(r_i,\Pi\otimes|s\tilde{\alpha}|)(1)$ is critical. We ignore negative s. In terms of L-functions, we are ignoring critical values that are central or left-of-centre. If there is no non-zero value of $\langle\lambda,\check{\gamma}\rangle$ for $\gamma\in\Phi_N^i$ then there is no upper bound on s–we might say that $b_i=\infty$.

$\mathcal{M}(r,\Pi\otimes|s\tilde{\alpha}|)(1)$ is critical for $0<s\leq\min_i\frac{b_i-1}{i}$, subject also to $\lambda+s\tilde{\alpha}$ being algebraically integral, and the simultaneous parity conditions. As discussed at the end of [BD, §3], we may chose $w\in W_G$ such that $w(\lambda+s\tilde{\alpha})$ is dominant and regular (for G) precisely for s satisfying the above inequalities.

4. The main conjecture

Recall the field of definition E. Then $\mathcal{M}(r_i,\Pi\otimes|s\tilde{\alpha}|)(1)$ should have coefficients in E. Let $\mathfrak{q}$ be a prime divisor, dividing a rational prime q such that $q>\mathcal{B}_i$, and such that Π_q is unramified, where $\mathcal{B}_i=\begin{cases}2\max_{\gamma\in\Phi_N^i}\langle\lambda,\check{\gamma}\rangle+1 & \text{if }\max_{\gamma\in\Phi_N^i}\langle\lambda,\check{\gamma}\rangle\neq 0;\\ 2+is & \text{if }\max_{\gamma\in\Phi_N^i}\langle\lambda,\check{\gamma}\rangle=0.\end{cases}$
Let $O_{\mathfrak{q}}$ be the ring of integers of $E_{\mathfrak{q}}$, and $O_{(\mathfrak{q})}$ the localisation at $\mathfrak{q}$ of the ring

of integers O_E of E. Suppose that $s > 0$ and $\mathcal{M}(r, \Pi \otimes |s\tilde{\alpha}|)(1)$ is critical. For $1 \leq i \leq m$, choose an $O_{(\mathfrak{q})}$-lattice $T_{i,B}$ in $H_B(\mathcal{M}(r_i, \Pi \otimes |s\tilde{\alpha}|))$ in such a way that $T_{i,\mathfrak{q}} := T_{i,B} \otimes O_{\mathfrak{q}}$ is a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant lattice in the $\mathfrak{q}$-adic realisation. Then choose an $O_{(\mathfrak{q})}$-lattice $T_{i,\mathrm{dR}}$ in $H_{\mathrm{dR}}(\mathcal{M}(r_i, \Pi \otimes |s\tilde{\alpha}|))$ in such a way that

$$\mathbb{V}(T_{i,\mathrm{dR}} \otimes O_{\mathfrak{q}}) = T_{i,\mathfrak{q}}$$

as $\mathrm{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$-representations, where $\mathbb{V}$ is the version of the Fontaine-Lafaille functor used in [DFG]. This choice ensures that the $\mathfrak{q}$-part of the Tamagawa factor at q is trivial (by [BK, Theorem 4.1(iii)]), thus simplifying the Bloch-Kato conjecture below. The condition $q > \mathcal{B}_i$ ensures that the condition (*) in [BK, Theorem 4.1(iii)] holds.

Let Ω be a Deligne period scaled according to the above choice, i.e. the determinant of the isomorphism

$$H_B(\mathcal{M}(r_i, \Pi \otimes |s\tilde{\alpha}|)(1))^+ \otimes \mathbb{C} \simeq (H_{\mathrm{dR}}(\mathcal{M}(r_i, \Pi \otimes |s\tilde{\alpha}|)(1))/\mathrm{Fil}^0) \otimes \mathbb{C},$$

calculated with respect to bases of $T_{i,B}^+$ and $T_{i,\mathrm{dR}}/\mathrm{Fil}^1$, so well-defined up to $O_{(\mathfrak{q})}^\times$. Let Σ be a finite set of finite primes, containing all p such that G or Π_p is ramified, but not containing q.

In Case (1) below, the formulation of the $\mathfrak{q}$-part of the Bloch-Kato conjecture is based on [DFG, (59)], similarly using the exact sequence in their Lemma 2.1.

Conjecture 4.1 (Bloch-Kato). (1) *If* $\Sigma \neq \emptyset$ *then*

$$\mathrm{ord}_{\mathfrak{q}}\left(\frac{L_\Sigma(1+is, \Pi, r_i)}{\Omega}\right) = \mathrm{ord}_{\mathfrak{q}}\left(\frac{\# H^1_\Sigma(\mathbb{Q}, T^*_{i,\mathfrak{q}} \otimes (E_{\mathfrak{q}}/O_{\mathfrak{q}}))}{\# H^0(\mathbb{Q}, T^*_{i,\mathfrak{q}} \otimes (E_{\mathfrak{q}}/O_{\mathfrak{q}}))}\right).$$

(2) *If* $\Sigma = \emptyset$ *then* $\mathrm{ord}_{\mathfrak{q}}\left(\frac{L(1+is,\Pi,r_i)}{\Omega}\right)$

$$= \mathrm{ord}_{\mathfrak{q}}\left(\frac{\# H^1_\emptyset(\mathbb{Q}, T^*_{i,\mathfrak{q}} \otimes (E_{\mathfrak{q}}/O_{\mathfrak{q}}))}{\# H^0(\mathbb{Q}, T^*_{i,\mathfrak{q}} \otimes (E_{\mathfrak{q}}/O_{\mathfrak{q}})) \# H^0(\mathbb{Q}, T_{i,\mathfrak{q}}(1) \otimes (E_{\mathfrak{q}}/O_{\mathfrak{q}}))}\right).$$

Here, $T^*_{i,\mathfrak{q}} = \mathrm{Hom}_{O_{\mathfrak{q}}}(T_{i,\mathfrak{q}}, O_{\mathfrak{q}})$, with the dual action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and $\# A$ denotes the Fitting ideal of a finite $O_{\mathfrak{q}}$-module A. On the right hand side, in the numerator is a Bloch-Kato Selmer group with local conditions (unramified at $p \neq q$, crystalline at $p = q$) only at $p \notin \Sigma$. The ratio of the two sides is independent of the choice of Σ as above.

Recall $T'_\mu \in \mathcal{H} = \mathcal{H}(G(\mathbb{Q}_p), G(\mathbb{Z}_p))$, the characteristic function of the double coset $G(\mathbb{Z}_p)\mu(p)G(\mathbb{Z}_p)$, where $\mu \in X_*(T_\#)$. If $w(\lambda + s\tilde{\alpha})$ is self-dual, let $a(\mu) := \langle w(\lambda + s\tilde{\alpha}) - \rho_G, \mu\rangle$, and let $T_\mu := p^{a(\mu)}T'_\mu$. The Hecke eigenvalue for T_μ on the spherical representation $\Pi_p \otimes |s\tilde{\alpha}|_p$, or on $\tilde{\Pi}_p$ below, should be an algebraic integer. If p splits in F and μ is minuscule then this Hecke eigenvalue is $p^{\langle w(\lambda+s\tilde{\alpha}),\mu\rangle}\mathrm{Tr}(\theta_\mu(t(\chi)))$, where χ, or $t(\chi)$, is the Satake parameter. (Recall the end of §2, and that θ_μ is the irreducible representation of $\hat{G}$ with highest weight μ.) In what follows, we enlarge the field E to be a common field of definition for the Hecke eigenvalues of T_μ (for all $\mu \in X_*(T_\#)$) on the $\Pi_p \otimes |s\tilde{\alpha}|_p$ and the $\tilde{\Pi}_p$ (for all unramified p), and replace $\mathfrak{q}$ by any divisor in this possibly larger field. Let $T_\mu(\mathrm{Ind}_P^G(\Pi_p \otimes |s\tilde{\alpha}|_p))$ and $T_\mu(\tilde{\Pi}_p)$ denote the Hecke eigenvalues.

Conjecture 4.2. *Choose* $s > 1$ *such that* $\mathcal{M}(r, \Pi \otimes |s\tilde{\alpha}|)(1)$ *is critical. (Recall that* Π *is an irreducible, unitary, cuspidal automorphic representation of* $M(\mathbb{A})$*.)*

Suppose that $\lambda + s\tilde{\alpha}$ is conjugate self-dual, i.e. W_G-equivalent to the complex conjugate of its negative. Now fixing i, with $\mathfrak{q}$ and Σ as above (in particular, $q > \mathcal{B}_i$), suppose that $\mathrm{ord}_{\mathfrak{q}}\left(\frac{L_\Sigma(1+is,\Pi,r_i)}{\Omega}\right) > 0$. *Suppose also that the irreducible components of the $\mathfrak{q}$-adic representation $\rho_{\Pi\otimes|s\tilde{\alpha}|,i}$ remain irreducible mod $\mathfrak{q}$. Then there exists an irreducible, cuspidal, tempered, automorphic representation $\tilde{\Pi}$ of $G(\mathbb{A})$ such that*

(1) *$\tilde{\Pi}_\infty$ has infinitesimal character $\lambda + s\tilde{\alpha}$ (up to W_G), i.e. the same as* $\mathrm{Ind}_P^G(\Pi \otimes |s\tilde{\alpha}|)$.
(2) *At any finite $p \notin \Sigma$, $\tilde{\Pi}_p$ is unramified, and for all $\mu \in X_*(T_\#)$,*
$$T_\mu(\tilde{\Pi}_p) \equiv T_\mu(\mathrm{Ind}_P^G(\Pi_p \otimes |s\tilde{\alpha}|_p)) \pmod{\mathfrak{q}}.$$

The conjugate self-duality condition is necessary so that there is a possibility of $\lambda + s\tilde{\alpha}$ being the infinitesimal character of a unitary representation. In all the examples below, the condition is automatically satisfied.

5. $U(n,n)$ and Hermitian modular forms

For any integer $n \geq 1$, let
$$G = U(n,n) = \{A \in \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_{2n} \mid AJ\,{}^t\overline{A} = J\},$$
where $J = \begin{bmatrix} 0_n & -I_n \\ I_n & 0_n \end{bmatrix}$, be the unitary group associated to an Hermitian form of signature (n,n). (If $\mathrm{tr}_{F/\mathbb{Q}}(\eta) = 0$ then ηJ is an Hermitian matrix.)

The algebraic group $G/\mathbb{Q}$ has a maximal (but non-split) torus T, with $T(\mathbb{Q}) = \{\mathrm{diag}(a_1,\ldots,a_n,a_1^*,\ldots,a_n^*) : a_1,\ldots,a_n \in F^\times\}$, where $a^* := \overline{a}^{-1}$, and $T_d(\mathbb{Q}) = \{\mathrm{diag}(a_1,\ldots,a_n,a_1^{-1},\ldots,a_n^{-1}) : a_1,\ldots,a_n \in \mathbb{Q}\}$. In the complexification $G(\mathbb{C}) \simeq \mathrm{GL}_{2n}(\mathbb{C})$, $T(\mathbb{C})$ becomes the standard diagonal torus with characters e_i, for $1 \leq i \leq 2n$, such that $e_i(\mathrm{diag}(t_1,\ldots,t_{2n})) := t_i$. In a little more detail, $F \otimes \mathbb{C} \simeq \mathbb{C} \times \mathbb{C}$, with F embedded via $z \mapsto (z,\overline{z})$, and $T(\mathbb{C}) =$
$$\{\mathrm{diag}((t_1,t_{n+1}^{-1}),(t_2,t_{n+2}^{-1}),\ldots,(t_n,t_{2n}^{-1}),(t_{n+1},t_1^{-1}),\ldots,(t_{2n},t_n^{-1})) : t_1,\ldots,t_{2n} \in \mathbb{C}^\times\},$$
$G(\mathbb{C}) = \{(A, J\,{}^tA^{-1}J^{-1}) : A \in \mathrm{GL}_{2n}(\mathbb{C})\}$, which we can identify with $\mathrm{GL}_{2n}(\mathbb{C})$ by looking only at the first entry. The natural action of $\mathrm{Gal}(F/\mathbb{Q})$ (with complex conjugation swapping the entries of an ordered pair) induces an action on the character group of $T(\mathbb{C})$, with the non-trivial element switching e_i with $-e_{n+i}$, for $1 \leq i \leq n$.

We may choose a system of simple positive roots $e_1 - e_2, \ldots, e_{n-1} - e_n, e_n - e_{2n}, e_{n+2} - e_{n+1}, \ldots, e_{2n} - e_{2n-1}$. Then $\mathrm{Gal}(F/\mathbb{Q})$ fixes $e_n - e_{2n}$ and switches $e_i - e_{i+1}$ with $e_{n+i+1} - e_{n+i}$, for $1 \leq i < n$. The set of positive roots is then $\Phi^+ = \{e_i - e_j, e_{n+j} - e_{n+i} : 1 \leq i < j \leq n\} \cup \{e_i - e_j : 1 \leq i \leq n,\ n+1 \leq j \leq 2n\}$. The half-sum of positive roots is given by $\rho = \frac{1}{2}((2n-1)(e_1 - e_{n+1}) + (2n-3)(e_2 - e_{n+2}) + \cdots + (e_n - e_{2n}))$.

The absolute Weyl group $W_G \simeq S_{2n}$, the symmetric group on $2n$ letters, permuting the t_i, while the relative Weyl group ${}_{\mathbb{Q}}W$ is a subgroup isomorphic to $S_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$, where S_n permutes the t_{n+i} the same way it permutes the t_i, for $1 \leq i \leq n$, and $(\mathbb{Z}/2\mathbb{Z})^n$ is generated by involutions that swap t_i and t_{n+i} (hence a_i and a_i^{-1} in the entries of an element of $T_d(\mathbb{Q})$). The product of all these involutions is the long element w_0^G. It follows that $c(\lambda) = -w_0^G\lambda$ for any $\lambda \in X^*(T) \otimes \mathbb{R}$, so the conjugate self-dual condition of Conjecture 4.2 is automatically satisfied. The restriction to $X^*(T/S) \otimes \mathbb{R}$ of the standard inner product on $X^*(T) \otimes \mathbb{R}$, with orthonormal basis $\{e_1, \dots, e_{2n}\}$, is W_G-invariant.

Let $\mathfrak{g}$ be the Lie algebra of $G(\mathbb{R})$ and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition, into $\pm$ eigenspaces for the Cartan involution $\theta : A \mapsto -\overline{A}^t$. Then $\mathfrak{k}$ is the Lie algebra of a maximal compact subgroup K_∞, which is isomorphic to $U(n) \times U(n)$, via $\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \mapsto (A - iB, A + iB)$. Let $\mathfrak{g}_\mathbb{C}$ be $\mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$, and let $\mathfrak{t}_\mathbb{C}$ be the Lie algebra of the maximal torus $T(\mathbb{C}) \simeq (\mathbb{C}^\times)^n$. The derivatives of the characters e_i of $T(\mathbb{C})$ give us basis elements for $\mathfrak{t}_\mathbb{C}^*$, with the same names. There is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{k}$ (and of $\mathfrak{g}$) and a basis $\{e_1', \dots, e_{2n}'\}$ for $i\mathfrak{h}^*$ (where $\mathfrak{h}^* = \mathrm{Hom}_\mathbb{R}(\mathfrak{h}, \mathbb{R})$) such that the representations of G with highest weights $\sum a_i e_i$ and $\sum a_i e_i'$ are isomorphic. In a little more detail, conjugation by $\begin{pmatrix} I_n & -iI_n \\ I_n & iI_n \end{pmatrix}$ turns the elements $\mathrm{diag}(i, 0, \dots, 0), \mathrm{diag}(0, i, 0, \dots, 0), \dots, \mathrm{diag}(0, \dots, 0, i)$ of $\mathfrak{t}_\mathbb{C}$ into elements of $\mathfrak{k}$ which, under the isomorphism $\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \mapsto (A - iB, A + iB)$, correspond to the elements $(\mathrm{diag}(i, 0, \dots, 0), \mathrm{diag}(0, \dots, 0)), \dots, (\mathrm{diag}(0, \dots, 0), \mathrm{diag}(0, \dots, 0, i))$ (respectively) of $\mathfrak{u}(n) \times \mathfrak{u}(n)$. Thus we may identify elements of $i\mathfrak{h}^*$ with elements of $X^*(T) \otimes \mathbb{R}$, in such a way that, when algebraically integral, they are the highest weights of the same irreducible finite-dimensional representations of $G(\mathbb{R})$. Recalling the relationship between highest weights and infinitesimal characters, we see that corresponding elements of $i\mathfrak{h}^*$ and $X^*(T) \otimes \mathbb{R}$ label the same character of $Z(\mathfrak{g}_\mathbb{C})$, via Harish-Chandra isomorphisms with $U(\mathfrak{h}_\mathbb{C})^W$ and $U(\mathfrak{t}_\mathbb{C})^W$.

Let Π be a unitary, irreducible, cuspidal automorphic representation of $G(\mathbb{A})$, trivial on $S(\mathbb{R})$. Suppose that Π_∞ has infinitesimal character $\lambda \in X^*(T) \otimes \mathbb{R}$, chosen dominant in its W_G-orbit. In fact, let's suppose that Π_∞ is holomorphic discrete series with Harish-Chandra parameter λ. See [Ma] or [Da] for a convenient concise summary. The compact positive roots turn out to be $\{e_i - e_j, e_{n+j} - e_{n+i} : 1 \leq i < j \leq n\}$, while the non-compact positive roots are $\{e_i - e_j : 1 \leq i \leq n,\ n+1 \leq j \leq 2n\}$. Consequently, the half-sum of compact positive roots is

$$\rho_c = \frac{1}{2}((n-1)(e_1 + e_{2n}) + (n-3)(e_2 + e_{2n-1}) + \cdots - (n-1)(e_n + e_{n+1})),$$

and the half-sum of non-compact positive roots is

$$\rho_n = \frac{n}{2}(e_1 + e_2 + \cdots + e_n - e_{n+1} - e_{n+2} - \cdots - e_{2n}).$$

When restricted to K_∞, Π_∞ decomposes as an infinite direct sum of irreducible representations, each occurring with finite multiplicity, and the lowest highest weight of such a "K_∞-type" is

$$\lambda + \rho_n - \rho_c = \lambda + \frac{1}{2}\left((e_1 - e_{n+1}) + 3(e_2 - e_{n+2}) + \ldots + (n-1)(e_n - e_{2n})\right)$$

[Da, Theorem 4.4].

Let $\theta : \mathrm{GL}_n \times \mathrm{GL}_n \to V$ be the irreducible complex representation whose restriction to K_∞ is isomorphic to this lowest K_∞-type. Let K_f be an open compact subgroup of $G(\mathbb{A}_f)$ (omitting the real component) such that the representation Π_f of $G(\mathbb{A}_f)$ has a K_f-fixed vector. Realising Π in a space of cuspidal, complex-valued functions on $G(\mathbb{A})$, with the action of $G(\mathbb{A})$ by right translation, each element of $v \in V$ (when tensored with the aforesaid K_f-fixed vector) may be identified with a complex-valued function Φ_v on $G(\mathbb{A})$. Hence we have a map $G(\mathbb{A})\times V \to \mathbb{C}$, which may also be viewed as a function $\Phi : G(\mathbb{A}) \to V^*$. This function is, of course, right K_f-invariant and left $G(\mathbb{Q})$-invariant. If $g \in G(\mathbb{A})$, $k_\infty \in K_\infty$ and $v \in V$ then

$$\Phi(gk_\infty)(v) = \Phi_v(gk_\infty) = \Phi_{k_\infty v}(g) = \Phi(g)(k_\infty v) = (\theta^*(k_\infty^{-1})\Phi(g))(v),$$

where $\theta^* : \mathrm{GL}_n \times \mathrm{GL}_n \to V^*$ is the dual representation. Hence $\Phi(gk_\infty) = \theta^*(k_\infty^{-1})(\Phi(g))$.

Let $\mathfrak{H}_n = \{Z \in M_n(\mathbb{C}) : -i(Z - {}^t\overline{Z}) > 0\}$ be the Hermitian upper-half space, acted on by $G(\mathbb{R})$ via $\begin{pmatrix} A & B \\ C & D\end{pmatrix}(Z) = (AZ+B)(CZ+D)^{-1}$. The same formula gives an action of $G(\mathbb{R})$ on $\overline{\mathfrak{H}_n} := \{\overline{Z} : Z \in \mathfrak{H}_n\}$. For $g \in G(\mathbb{R})$ and $Z \in \mathfrak{H}_n$, define automorphy factors $j_1(g,Z) = {}^t(\overline{C}\,{}^tZ + \overline{D})^{-1}$, $j_2(g,Z) = (CZ+D)$, the first one the transpose inverse of Shimura's [Sh, (3.16)]. Both are in $\mathrm{GL}_n(\mathbb{C})$, and we may define $j(g,Z) := (j_1(g,Z), j_2(g,Z)) \in \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$.

Lemma 5.1. *For all $Z \in \mathfrak{H}_n$ and $g, h \in G(\mathbb{R})$, $j_i(gh,Z) = j_i(g,h(Z))j_i(h,Z)$, for $i = 1,2$, i.e. $j(gh,Z) = j(g,h(Z))j(h,Z)$.*

Proof. An elementary calculation proves it for j_2, and similarly that $j_2(gh, {}^t\overline{Z}) = j_2(g, h({}^t\overline{Z}))j_2(h, {}^t\overline{Z})$. Complex conjugating, then taking transpose inverse, proves it for j_1. □

Note that if $g = \begin{pmatrix} A & -B \\ B & A\end{pmatrix} \in K_\infty$, and if $Z = iI_n$, then $j_2(g,Z) = (A+iB)$, while $I = (A-iB)\overline{{}^t(A-iB)} = (A-iB){}^t(\overline{A}+i\overline{B})$ implies that $j_1(g,Z) = (A-iB)$.

Define $F : \mathfrak{H}_n \to V^*$ by

$$F(Z) := \theta^*(j(g,iI_n))\Phi(g,1),$$

where $g \in G(\mathbb{R})$ is such that $g(iI_n) = Z$, and $(g,1) \in G(\mathbb{R}) \times G(\mathbb{A}_f) = G(\mathbb{A})$. If $k_\infty \in K_\infty$ then $gk_\infty(iI_n) = g(iI_n) = Z$, in fact K_∞ is the stabiliser in $G(\mathbb{R})$ of iI_n, so we must check that F is well-defined.

$$\theta^*(j(gk_\infty, iI_n))\Phi(gk_\infty,1) = \theta^*(j(g,iI_n)j(k_\infty,iI_n))\Phi(gk_\infty,1)$$

(using Lemma 5.1)

$$= \theta^*((j(g,iI_n)j(k_\infty,iI_n)))\theta^*(k_\infty^{-1})\Phi(g,1).$$

Now if $k_\infty = \begin{pmatrix} A & -B \\ B & A\end{pmatrix}$, $j(k_\infty, iI_n) = (A-iB, A+iB)$, the element of $U(n)\times U(n)$ with which k_∞ is identified. So k_∞ and $j(k_\infty, iI_n)$ are essentially the same thing, hence

$$\theta^*(j(gk_\infty, iI_n))\Phi(gk_\infty,1) = \theta^*(j(g,iI_n))\Phi(g,1),$$

so F is indeed well-defined.

We can decompose $\mathfrak{p}$ as a direct sum $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ of subspaces spanned by positive and negative (respectively) non-compact roots. Then F is killed by $\mathfrak{p}^-$ (because it belongs to the *lowest* K_∞-type). But it happens that $\mathfrak{p}^- = \left\{\begin{pmatrix} A & -iA \\ -iA & -A\end{pmatrix} : A \text{ real}\right\}$, and from this it follows that f is holomorphic, as in [AS, 3.2].

Let $\Gamma = G(\mathbb{Q}) \cap K_f$, and take $\gamma \in \Gamma$. Then

$$F(\gamma(Z)) = \theta^*(j(\gamma g, iI_n))\Phi(\gamma g, 1) = \theta^*(j(\gamma g, iI_n))\Phi(g, \gamma^{-1})$$

(since $\gamma \in G(\mathbb{Q})$)

$$= \theta^*(j(\gamma g, iI_n))\Phi(g,1)$$

(since $\gamma \in K_f$)

$$= \theta^*(j(\gamma, Z)j(g, iI_n))\Phi(g,1)$$

(by Lemma 5.1)

$$= \theta^*(j(\gamma, Z))F(Z).$$

We have found that $F : \mathfrak{H}_n \to V^*$ is a holomorphic function such that

$$F(\gamma(Z)) = \theta^*(j(g,Z))(f(Z)) \ \ \forall \gamma \in \Gamma,\, Z \in \mathfrak{H}_n.$$

We might say that F is an Hermitian modular form (in general vector-valued) of weight θ^* for Γ. It will in fact be a cusp form, in the sense of [Kl2, §3.1].

If we suppose now that the class number of O_F is 1, and that K_f is such that $\det K_p = (O_{K,p})^\times$ for all p, then, as in [Kl2, Proposition 3.8], $G(\mathbb{A}) = G(\mathbb{Q})G(\mathbb{R})K_f$, where $G(\mathbb{Q})$ is embedded diagonally in $G(\mathbb{A})$. In this case, Φ can be recovered from F by $\Phi(\gamma g k) = \Phi(g,1) = \theta^*(j(g, iI_n)^{-1})F(g(iI_n))$, where $\gamma \in G(\mathbb{Q}), g \in G(\mathbb{R})$ and $k \in K_f$. But in general one needs a finite set of Hermitian modular forms for different Γ, as explained in [Kl2, §3.2].

6. A functorial lift from $\mathrm{SO}(n+1,n)$ to $U(n,n)$

Let

$$H = \mathrm{SO}(n+1,n) = \{g \in M_{2n+1} :\ {}^t g\tilde{J}g = \tilde{J},\, \det(g) = 1\},$$

with $\tilde{J} = \begin{pmatrix} 0_n & 0 & I_n \\ 0 & 2 & 0 \\ I_n & 0 & 0_n \end{pmatrix}$. It has a maximal torus T comprising elements of the form $\mathrm{diag}(t_1, \ldots, t_n, 1, t_1^{-1}, \ldots, t_n^{-1})$, mapped to t_i by characters $\tilde{e}_i$, for $1 \leq i \leq n$, which span $X^*(T)$. The cocharacter group $X_*(T)$ is spanned by $\{\tilde{f}_1, \ldots, \tilde{f}_n\}$, where $\tilde{f}_1 : t \mapsto \mathrm{diag}(t, 1, \ldots, 1, 1, t^{-1}, 1, \ldots, 1)$, etc. so $\langle \tilde{e}_i, \tilde{f}_j \rangle = \delta_{ij}$. We can order the roots so that $\Phi_H^+ = \{\tilde{e}_i - \tilde{e}_j : i < j\} \cup \{\tilde{e}_i : 1 \leq i \leq n\} \cup \{\tilde{e}_i + \tilde{e}_j : i < j\}$, and $\Delta_H = \{\tilde{e}_1 - \tilde{e}_2, \tilde{e}_2 - \tilde{e}_3, \ldots, \tilde{e}_{n-1} - \tilde{e}_n, \tilde{e}_n\}$. The simple coroots (in order) are $\{\tilde{f}_1 - \tilde{f}_2, \ldots, \tilde{f}_{n-1} - \tilde{f}_n, 2\tilde{f}_n\}$. Note that for any root β with coroot $\check{\beta}$, we have $\langle \beta, \check{\beta} \rangle = 2$.

Now let $H' = \mathrm{Sp}_n = \{g \in M_{2n} :\ {}^t gJg = J\}$, where, as in the previous section, $J = \begin{bmatrix} 0_n & -I_n \\ I_n & 0_n \end{bmatrix}$. It has a maximal torus T' comprising elements of the form $\mathrm{diag}(t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1})$, mapped to t_i by characters e'_i, for $1 \leq i \leq n$, which span $X^*(T')$. The cocharacter group $X_*(T')$ is spanned by $\{f'_1, \ldots, f'_n\}$, where $f'_1 : t \mapsto \mathrm{diag}(t, 1, \ldots, 1, t^{-1}, 1, \ldots, 1)$, etc. so $\langle e'_i, f'_j \rangle = \delta_{ij}$. We can order the roots so that $\Phi_{H'}^+ = \{e'_i - e'_j : i < j\} \cup \{2e'_i : 1 \leq i \leq n\} \cup \{e'_i + e'_j : i < j\}$, and $\Delta_{H'} = \{e'_1 - e'_2, e'_2 - e'_3, \ldots, e'_{n-1} - e'_n, 2e'_n\}$. The simple coroots (in order) are $\{f'_1 - f'_2, \ldots, f'_{n-1} - f'_n, f'_n\}$.

We see then that the root systems of H and H' are dual to each other, so $H' = \hat{H}$, the Langlands dual of H. The isomorphisms $X^*(T) \simeq X_*(T')$ and $X^*(T') \simeq X_*(T)$ are such that $\tilde{e}_i \leftrightarrow f'_i$ and $e'_i \leftrightarrow \tilde{f}_i$, respectively. Normally we would write ${}^L H = \hat{H}(\mathbb{C})$ when the group H is split, but for the purposes of this section we use the version ${}^L H = \hat{H}(\mathbb{C}) \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, with the trivial action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\hat{H}(\mathbb{C})$ (by conjugation). Similarly, if $G = U(n,n)$ then ${}^L G = \hat{G}(\mathbb{C}) \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) =$

$\mathrm{GL}_{2n}(\mathbb{C}) \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, with the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\mathrm{GL}_{2n}(\mathbb{C})$ factoring through $\mathrm{Gal}(F/\mathbb{Q})$: if $c \in \mathrm{Gal}(F/\mathbb{Q})$ is complex conjugation and $A \in \mathrm{GL}_{2n}(\mathbb{C})$ then

$$cAc^{-1} = J({}^tA^{-1})J^{-1}.$$

This can be justified using the precise definition of L-groups in [Bo, I.2]. In n-by-n blocks,

$$J\begin{bmatrix} A & B \\ C & D \end{bmatrix} J^{-1} = \begin{bmatrix} D & -C \\ -B & A \end{bmatrix},$$

which implies that the given action of $\mathrm{Gal}(F/\mathbb{Q})$ preserves T, and the orderings of roots and coroots implied by our choice of Δ_G in the previous section, and gives the correct action on $X^*(T)$.

There is an L-homomorphism (commuting with the projections to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$)

$$\theta : {}^LH \to {}^LG, \ \ (A, \sigma) \mapsto (A, \sigma).$$

The reason this works is that $A \in \mathrm{Sp}_n(\mathbb{C}) \implies J({}^tA^{-1})J^{-1} = A$, so the action of c on A is trivial on the right as well as on the left. Noting that G is quasi split, Langlands functoriality now predicts that any automorphic representation π of H "lifts" to an automorphic representation Π of G. If π_∞ has infinitesimal character $\sum_{i=1}^n a_i\tilde{e}_i$ then Π_∞ has infinitesimal character $\sum_{i=1}^n a_i(e_i - e_{n+i})$, and if π_p is unramified with Satake parameter $\sum_{i=1}^n b_i\tilde{e}_i$ (and also p is unramified in $F/\mathbb{Q}$), then Π_p is unramified with Satake parameter $\sum_{i=1}^n b_i(e_i - e_{n+i})$.

In the special case $n = 1$, $\mathrm{SO}(2,1) \simeq \mathrm{PGL}_2$. This arises from the conjugation action of PGL_2 on the 3-dimensional space of trace-0 matrices, preserving the quadratic form given by the determinant. If $A = \begin{pmatrix} x_2 & x_1 \\ x_3 & -x_2 \end{pmatrix}$ is such a trace-0 matrix, then $-2\det A = x_1x_3 + 2x_2^2 + x_3x_1$, the quadratic form associated with $\tilde{J}$. Under the isomorphism, $\mathrm{diag}(t_1, t_2) \in \mathrm{PGL}_2$ maps to $\mathrm{diag}(t_1t_2^{-1}, 1, t_2t_1^{-1}) \in \mathrm{SO}(2,1)$, as one readily checks by calculating the conjugation action on A. Hence the characters $a\tilde{e}_1$ of (the maximal torus of) $\mathrm{SO}(2,1)$ and $a(e_1 - e_2)$ of PGL_2 correspond. (The fairly obvious notation for characters of PGL_2 and PGSp_2 is as in [BD].)

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(1,1)$ then $a\overline{d} - b\overline{c} = 1$ and $[a : b : c : d] = [\overline{a} : \overline{b} : \overline{c} : \overline{d}]$, so there is an exact sequence

$$0 \longrightarrow U(1) \longrightarrow U(1,1) \longrightarrow \mathrm{PGL}_2.$$

(The right-hand map is not surjective, since anything in the image must have determinant equal to a norm, modulo squares.) Considering the relationship between infinitesimal characters of π_∞ and Π_∞, and between Satake parameters of π_p and Π_p, we see that for $n = 1$ the lift is obtained by simply pulling back from $\mathrm{PGL}_2(\mathbb{A})$ to $U(1,1)(\mathbb{A})$.

In the special case $n = 2$, $\mathrm{SO}(3,2) \simeq \mathrm{PGSp}_2$. This arises from the conjugation action of PGSp_2 on the 5-dimensional space of matrices $A = \begin{pmatrix} x_3 & x_2 & 0 & -x_1 \\ x_5 & -x_3 & x_1 & 0 \\ 0 & x_4 & x_3 & x_5 \\ -x_4 & 0 & x_2 & -x_3 \end{pmatrix}$ such that $AJ = J\,{}^tA$, preserving the quadratic form $(1/2)\mathrm{Tr}(A^2) = x_1x_4 + x_2x_5 + 2x_3^2 + x_4x_1 + x_5x_2$, which is that associated with $\tilde{J}$. Under the isomorphism, $\mathrm{diag}(t_1, t_2, t_0t_1^{-1}, t_0t_2^{-1}) \in \mathrm{PGSp}_2$ maps to $\mathrm{diag}(t_1t_2t_0^{-1}, t_1t_2^{-1}, 1, t_0t_1^{-1}t_2^{-1}, t_2t_1^{-1}) \in \mathrm{SO}(3,2)$, and the characters $ae_1 + be_2 - \frac{1}{2}(a+b)e_0$ of PGSp_2 and $\frac{a+b}{2}\tilde{e}_1 + \frac{a-b}{2}\tilde{e}_2$

of $\mathrm{SO}(3,2)$ correspond. In particular, looking at the infinitesimal character of Π_∞ when Π is generated by a Siegel modular form of weight $\mathrm{Sym}^j \det^k$, $(j+k-1)e_1 + (k-2)e_2 - \frac{j+2k-3}{2}e_0$ corresponds to $\frac{j+2k-3}{2}\tilde{e}_1 + \frac{j+1}{2}\tilde{e}_2$.

Similarly, there should exist a functorial lift from Sp_n to $U(n+1,n)$. Details are omitted, since we shall not need it for the examples in this paper.

7. $G = U(2,2)$, $M \simeq \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_2)$

In this section, $G = U(2,2)$, $\Delta_G = \{e_1 - e_2, e_2 - e_4, e_4 - e_3\}$, and we choose $\alpha = e_2 - e_4$, which is fixed by $\mathrm{Gal}(F/\mathbb{Q})$, so $\Delta_\alpha = \{e_2 - e_4\}$ and $\Delta_M = \{e_1 - e_2, e_4 - e_3\}$. The Levi subgroup $M \simeq \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$, with $A \mapsto \begin{pmatrix} A & 0 \\ 0 & {}^t\overline{A}^{-1} \end{pmatrix}$, and P is the Siegel parabolic subgroup of G. We have $\Phi_N = \{e_2 - e_4, e_1 - e_3, e_2 - e_3, e_1 - e_4\}$, $\rho_P = (e_1 + e_2 - e_3 - e_4)$, $\langle \rho_P, \check{\alpha}\rangle = 2$, $\tilde{\alpha} = \frac{1}{2}(e_1 + e_2 - e_3 - e_4)$.

Let Π be a unitary, irreducible, cuspidal automorphic representation of $M(\mathbb{A})$, which can also be thought of as $\mathrm{GL}_2(\mathbb{A}_F)$. If Π comes from a Bianchi modular form f of weight k' then Π_∞ (as a representation of $\mathrm{GL}_2(\mathbb{C})$) is principal series, (unitarily parabolically) induced from the character sending $\mathrm{diag}(z_1, z_2)$ to $(z_1/\overline{z_1})^{(k'-1)/2}(\overline{z_2}/z_2)^{(k'-1)/2}$ (cf. [U, §3.1]). It follows that the infinitesimal character for Π_∞ (as a representation of $(\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2)(\mathbb{R})$) is $\lambda = \frac{k'-1}{2}(e_1 - e_2 + e_3 - e_4)$.

Suppose now, for simplicity, that Π has trivial central character. If p is inert and Π_p is unramified then the Hecke eigenvalue for $T(p)$ on f is $a_p(f) = p^{k'-1}(\alpha_p^2 + \alpha_p^{-2})$ for some α_p^2 such that $|\alpha_p^2| = 1$. (The reason for the square will become clear later.) Embedding M in G, Π_p is induced from the character

$$\chi_p : \mathrm{diag}(p, 1, p^{-1}, 1) \mapsto \alpha_p^2, \;\; \mathrm{diag}(1, p, 1, p^{-1}) \mapsto \alpha_p^{-2},$$

which is

$$\chi_p = -\log_p(\alpha_p)(e_1 - e_2 - e_3 + e_4) \in {}_{\mathbb{Q}_p}X^*(T) \otimes \mathbb{R}.$$

If $p = \mathfrak{p}\overline{\mathfrak{p}}$ splits and Π_p (i.e. $\Pi_\mathfrak{p}$ and $\Pi_{\overline{\mathfrak{p}}}$, if we view Π as a representation of $\mathrm{GL}_2(\mathbb{A}_F)$) is unramified, then the Hecke eigenvalues for $T(\mathfrak{p})$ and $T(\overline{\mathfrak{p}})$ are $a_\mathfrak{p}(f) = p^{(k'-1)/2}(\alpha_\mathfrak{p} + \alpha_\mathfrak{p}^{-1})$ and $a_{\overline{\mathfrak{p}}}(f) = p^{(k'-1)/2}(\alpha_{\overline{\mathfrak{p}}} + \alpha_{\overline{\mathfrak{p}}}^{-1})$, respectively, with $|\alpha_\mathfrak{p}| = |\alpha_{\overline{\mathfrak{p}}}| = 1$. Embedding $M(\mathbb{Q}_p) = \mathrm{GL}_2(\mathbb{Q}_p \times \mathbb{Q}_p)$ in $G(\mathbb{Q}_p) \subset \mathrm{GL}_4(F \otimes \mathbb{Q}_p) \simeq \mathrm{GL}_4(\mathbb{Q}_p \times \mathbb{Q}_p)$, Π_p is induced from the character

$$\chi_p : \mathrm{diag}((p,1),(1,1),(1,p^{-1}),(1,1)) \mapsto \alpha_\mathfrak{p}, \;\; \mathrm{diag}((1,1),(p,1),(1,1),(1,p^{-1})) \mapsto \alpha_\mathfrak{p}^{-1},$$

$$\mathrm{diag}((1,p),(1,1),(p^{-1},1),(1,1)) \mapsto \alpha_{\overline{\mathfrak{p}}}, \;\; \mathrm{diag}((1,1),(1,p),(1,1),(p^{-1},1)) \mapsto \alpha_{\overline{\mathfrak{p}}}^{-1},$$

which is

$$\chi_p = -\log_p(\alpha_\mathfrak{p})(e_1 - e_2) + \log_p(\alpha_{\overline{\mathfrak{p}}})(e_3 - e_4).$$

In the table below, we list together pairs of roots in the same $\mathrm{Gal}(F/\mathbb{Q})$-orbit.

$\gamma \in \Phi_N$	$\langle \lambda + s\tilde{\alpha}, \check{\gamma}\rangle$	$\lvert\chi_p(\check{\gamma}(p))\rvert_p$ (p inert)	$\lvert\chi_p(\check{\gamma}(p))\rvert_p$ (p split)
$e_2 - e_4$	s	α_p^{-2}	$\alpha_\mathfrak{p}^{-1}\alpha_{\overline{\mathfrak{p}}}$
$e_1 - e_3$	s	α_p^2	$\alpha_\mathfrak{p}\alpha_{\overline{\mathfrak{p}}}^{-1}$
$\{e_2 - e_3, e_1 - e_4\}$	$\{\pm(k'-1) + s\}$	1	$\{\alpha_\mathfrak{p}^{-1}\alpha_{\overline{\mathfrak{p}}}^{-1}, \alpha_\mathfrak{p}\alpha_{\overline{\mathfrak{p}}}\}$

Using the table, $m=1$ in $r=\oplus_{i=1}^m r_i$, and using also Lemma 3.1, $L_p(s,\Pi_p,r)$

$$= \begin{cases} (1-\alpha_p^2 p^{-s})(1-\alpha_p^{-2}p^{-s})(1-p^{-2s}) & \text{if } p \text{ is inert;} \\ (1-\alpha_{\mathfrak{p}}\alpha_{\overline{\mathfrak{p}}}p^{-s})(1-\alpha_{\mathfrak{p}}\alpha_{\overline{\mathfrak{p}}}^{-1}p^{-s})(1-\alpha_{\mathfrak{p}}^{-1}\alpha_{\overline{\mathfrak{p}}}p^{-s})(1-\alpha_{\mathfrak{p}}^{-1}\alpha_{\overline{\mathfrak{p}}}^{-1}p^{-s}) & \text{if } p \text{ is split,} \end{cases}$$

and $L_\Sigma(s,\Pi,r)$ is the Asai L-function [Gh, §3.2]. We need $s\in\mathbb{Z}$ for $\lambda+s\tilde{\alpha}$ to be algebraically integral. (Look at the second column of the table.)

$$\lambda+s\tilde{\alpha} = \left(\frac{k'-1+s}{2}\right)e_1+\left(\frac{s-(k'-1)}{2}\right)e_2+\left(\frac{k'-1-s}{2}\right)e_3+\left(\frac{-(k'-1)-s}{2}\right)e_4.$$

If we choose $w=(432)\in W$ then

$$w(\lambda+s\tilde{\alpha}) = \left(\frac{k'-1+s}{2}\right)e_1+\left(\frac{k'-1-s}{2}\right)e_2-\left(\frac{k'-1-s}{2}\right)e_4-\left(\frac{k'-1+s}{2}\right)e_3,$$

which is dominant and regular for $0<s<k'-1$. Looking at [Gh, §4.1], letting his $n=k'-2, v_i=v_c=0$, we see that for $s+1=k'-1$, F_∞ should act on the 2-dimensional $H^{k'-1,k'-1}$ in $H_B(\mathcal{M}(r_i,\Pi\otimes|s\tilde{\alpha}|)(1))$ as $+1$. So we have a parity condition $s\equiv k'-2 \pmod 2$. Overall then, $s\equiv k' \pmod 2$, with $0<s<k'-1$, and we shall exclude $s=1$.

In the case that p splits, consider $\mu=f_1\in X_*(T_\#)=X_*(T)$, so $\mu(p)=\mathrm{diag}((p,1),(1,1),(1,p^{-1}),(1,1))$, where $\langle e_i,f_j\rangle=\delta_{ij}$. As a character of $\hat{T}$, f_1 is the highest weight of the standard representation of $\hat{G}\simeq \mathrm{GL}_4$, with weights $\{f_1,f_2,f_4,f_3\}$. Using $\chi_p+s\tilde{\alpha}=-\log_p(\alpha_{\mathfrak{p}})(e_1-e_2)+\log_p(\alpha_{\overline{\mathfrak{p}}})(e_3-e_4)+\frac{s}{2}(e_1+e_2-e_3-e_4)$, we find

μ	$\lvert(\chi_p+s\tilde{\alpha})(\mu(p))\rvert_p$
f_1	$\alpha_{\mathfrak{p}}p^{-s/2}$
f_2	$\alpha_{\mathfrak{p}}^{-1}p^{-s/2}$
f_3	$\alpha_{\overline{\mathfrak{p}}}p^{s/2}$
f_4	$\alpha_{\overline{\mathfrak{p}}}^{-1}p^{s/2}$

The trace is $p^{-s/2}((\alpha_{\mathfrak{p}}+\alpha_{\mathfrak{p}}^{-1})+p^{s/2}(\alpha_{\overline{\mathfrak{p}}}+\alpha_{\overline{\mathfrak{p}}}^{-1})$. Multiplying by $p^{\langle w(\lambda+s\tilde{\alpha}),f_1\rangle}=p^{(k'-1+s)/2}$, we find that

$$T_{f_1}(\mathrm{Ind}_P^G(\Pi_p\otimes|s\tilde{\alpha}|_p))=a_{\mathfrak{p}}(f)+p^s a_{\overline{\mathfrak{p}}}(f).$$

In the case that p is inert, consider $\mu=f_1-f_3\in X_*(T_\#)=X_*(T_d)$, so $\mu(p)=\mathrm{diag}(p,1,p^{-1},1)$. As in [Kl2, (4.4)] or [Kl1, (5.7)] (but scaling everything by $\mathrm{diag}(1/p,1/p,1/p,1/p)$, and with K_p on the other side), there is a coset decomposition $K_p\mu(p)K_p=\sqcup_{j=1}^t b_jK_p$, with each $b_j\in B$, the Borel subgroup. In fact, letting $B=TU$ with U the unipotent radical of B, we have

$$b_j\in\begin{cases} \mathrm{diag}(1/p,1,p,1)U & \text{for 1 value of } j; \\ \mathrm{diag}(1,1/p,1,p)U & \text{for } p^2 \text{ values of } j; \\ \mathrm{diag}(1,p,1,1/p)U & \text{for } p^4 \text{ values of } j; \\ \mathrm{diag}(p,1,1/p,1)U & \text{for } p^6 \text{ values of } j; \\ \mathrm{diag}(1,1,1,1)U & \text{for } p^3-p^2+p-1 \text{ values of } j. \end{cases}$$

Recall that $\rho_G=(1/2)(3e_1+e_2-e_4-3e_3)$. Hence $\chi_p+s\tilde{\alpha}+\rho_G=-\log_p(\alpha_p)(e_1-e_2-e_3+e_4)+(s/2)(e_1+e_2-e_3-e_4)+(1/2)(3e_1+e_2-e_4-3e_3)$, which we should apply to each diagonal element, according to [Ca, IV.(33),(35),(39)], getting

$$T'_{f_1-f_3}(\mathrm{Ind}_P^G(\Pi_p\otimes|s\tilde{\alpha}|_p))$$
$$=\alpha_p^{-2}p^sp^3+p^2(\alpha_p^2p^sp^1)+p^4(\alpha_p^{-2}p^{-s}p^{-1})+p^6(\alpha_p^2p^{-s}p^{-3})+(p^3-p^2+p-1).$$

Multiplying by $p^{\langle w(\lambda+s\tilde{\alpha})-\rho_G,f_1-f_3\rangle}=p^{k'-1+s-3}$, we find that

$$T_{f_1-f_3}(\mathrm{Ind}_P^G(\Pi_p\otimes|s\tilde{\alpha}|_p))=a_p(f)(1+p^{2s})+p^{k'-1+s-3}(p^3-p^2+p-1).$$

If $q>2k'$ and $\mathrm{ord}_{\mathfrak{q}}\left(\frac{L_\Sigma(1+s,\Pi,r)}{\Omega}\right)>0$ then Conjecture 4.2 predicts the existence of an irreducible, tempered, cuspidal, automorphic representation $\tilde{\Pi}$ of $G(\mathbb{A})$, satisfying congruences mod $\mathfrak{q}$ of Hecke eigenvalues, with $\Pi\otimes|s\tilde{\alpha}|$. The infinitesimal character of $\tilde{\Pi}_\infty$ would be $w(\lambda+s\tilde{\alpha})$. Unless we are in a case where $\tilde{\Pi}$ is some kind of endoscopic lift, we would expect to be able to choose $\tilde{\Pi}$ (subject to the congruences, which depend only on $\tilde{\Pi}|_{G(\mathbb{A}_f)}$) in such a way that $\tilde{\Pi}_\infty$ is holomorphic discrete series. Then, as in §5, the lowest K_∞-type of $\tilde{\Pi}_\infty$ has highest weight $\Lambda=w(\lambda+s\tilde{\alpha})+\rho_n-\rho_c$. In our case, $\rho_n-\rho_c=\frac{1}{2}(e_1-e_3)+\frac{3}{2}(e_2-e_4)$, so

$$\Lambda=\left(\frac{k'+s}{2}\right)e_1+\left(\frac{k'+2-s}{2}\right)e_2-\left(\frac{k'+2-s}{2}\right)e_4-\left(\frac{k'+s}{2}\right)e_3.$$

This would be the highest weight of the dual of the representation of $\mathrm{GL}_2\times\mathrm{GL}_2$ in which the vector-valued Hermitian modular form F (see §5) takes its values.

A special case is when Π is the base change to $\mathrm{GL}_2(\mathbb{A}_F)$ of the automorphic representation Π' of $\mathrm{GL}_2(\mathbb{A})$ attached to a cuspidal Hecke eigenform f'. We insist that f' is not a CM form, i.e. that Π' is not automorphically induced from $\mathrm{GL}_1(\mathbb{A}_F)$, so that Π is then cuspidal [La]. If p splits in F then $a_{\mathfrak{p}}(f)=a_{\bar{\mathfrak{p}}}(f)=a_p(f')$, while if p is inert in F then $a_p(f')=p^{(k'-1)/2}(\alpha_p+\psi(p)\alpha_p^{-1})$, for some α_p such that $a_p(f)=p^{k'-1}(\alpha_p^2+\alpha_p^{-2})$. Here ψ is the central character of Π', thought of interchangeably as a Dirichlet character or the associated character of $\mathrm{GL}_1(\mathbb{A})$ (which is trivial on $\mathrm{GL}_1(\mathbb{R})$). Given our simplifying assumption that the central character of Π is trivial, ψ is either trivial or the quadratic character associated to $F/\mathbb{Q}$. Anyway,

$$L_\Sigma(s,\Pi,r)=L_\Sigma(\mathrm{Sym}^2f',s+(k'-1))\zeta_\Sigma(s),$$

and for an inert prime p we have

$$T_{f_1-f_3}(\mathrm{Ind}_P^G(\Pi_p\otimes|s\tilde{\alpha}|_p))=(a_p(f')^2-\psi(p)2p^{k'-1})(1+p^{2s})+p^{k'-1+s-3}(p^3-p^2+p-1),$$

while for a split prime p,

$$T_{f_1}(\mathrm{Ind}_P^G(\Pi_p\otimes|s\tilde{\alpha}|_p))=a_p(f')(1+p^s).$$

First consider the case that ψ is the quadratic character. Then k' and s are odd.

We have a scrap of numerical evidence for Conjecture 4.2. Let $F=\mathbb{Q}(\sqrt{-3})$. Then $S_9(\Gamma_0(3),\psi)$ is two-dimensional, spanned by a normalised newform f' with coefficients in $\mathbb{Q}(\sqrt{-14})$, and its conjugate.

$$f'=q+6\sqrt{-14}\,q^2+(45-18\sqrt{-14})\,q^3-248\,q^4-60\sqrt{-14}\,q^5+\dots.$$

Further along, $a_9(f')=-2511-1620\sqrt{-14}$ and $a_{18}(f')=136080-15066\sqrt{-14}$. Plugging these coefficients into [Ka2, Theorem 4.3], one finds that

$$\frac{-i\Gamma(8)\Gamma(14)\Gamma(7)}{\Gamma(3)}\frac{L(\mathrm{Sym}^2f',14)}{2^{26}\pi^{20}\langle f,f\rangle}=\frac{9992960}{6561}-\frac{23680}{45927}\sqrt{-14},$$

which has norm $\frac{2^{15}\cdot 5^3\cdot 19\cdot 37^2}{3^8\cdot 7}$. We can take $\mathfrak{q}$ then to be an appropriate divisor of $q = 19$ (split in $\mathbb{Q}(\sqrt{-14})$) or 37 (inert in $\mathbb{Q}(\sqrt{-14})$). For this example, $k' = 9$ and $s = 5$, so $w(\lambda + s\tilde{\alpha}) = (13/2)(e_1 - e_3) + (3/2)(e_2 - e_4)$. In [Du1], we supposed that $\tilde{\Pi}$ should be a functorial lift from the inner form $U(4)$ (compact at infinity), so looked for vectors in automorphic representations of $U(4)(\mathbb{A})$, as algebraic modular forms with values in the representation of highest weight $w(\lambda + s\tilde{\alpha}) - \rho_G = 5(e_1 - e_3) + (e_2 - e_4)$, with level structure such that $\tilde{\Pi}$ would be unramified away from 3. We found a 2-dimensional space of forms, spanned by Hecke eigenforms, with eigenvalues for $T_{f_1 - f_3}$ (at the inert prime $p = 2$) apparently equal to 17424 and -10656. (These are 2^{10} times the numbers given in [Du1].) For $p = 2$, $(a_p(f')^2 + 2p^{k'-1})(1 + p^{2s}) + p^{k'-1+s-3}(p^3 - p^2 + p - 1) = 13320$. The differences between this and the two Hecke eigenvalues are $2^3 \cdot 3^3 \cdot 19$ and $-2^3 \cdot 3^4 \cdot 37$, lending support to the conjecture.

In the excluded case $s = 1$ (still with ψ non-trivial and k' odd, with Π a base-change lift), $\Lambda = \frac{k'+1}{2}(e_1 + e_2 - e_3 - e_4)$, which is the highest weight of a scalar-valued representation of $\mathrm{GL}_2 \times \mathrm{GL}_2$, whose dual would give rise to an automorphy factor $\det(\overline{C}\,{}^tZ + \overline{D})^{(k'+1)/2} \det(CZ + D)^{(k'+1)/2}$, which by [Sh, (3.23)] is the same as $\det(g)^{-(k'+1)/2} \det(CZ + D)^{k'+1}$, which matches that in [Kl2, Definitions 3.1,5.1]. Note also that $L(\mathrm{Sym}^2 f', (k'-1) + s + 1)$ is the critical value closest to (the right of) the centre. There is a CAP (cuspidal associated to parabolic) representation of $G(\mathbb{A})$, the Maass lift, isomorphic to $\Pi \otimes |s\tilde{\alpha}|$ locally at unramified finite primes, and the substitute for Conjecture 4.2 is that there should be congruences mod $\mathfrak{q}$ of Hecke eigenvalues between this Maass lift and $\tilde{\Pi}$, a non-CAP cuspidal automorphic representation. Such congruences have been proven by Klosin [Kl1, Kl2], subject to certain conditions.

Turning now to the case that ψ is trivial, k' is even, Π a base-change lift, if the case $G = \mathrm{GSp}_2$, $M \simeq \mathrm{GL}_2 \times \mathrm{GL}_1$, P the Klingen parabolic of [BD, Conjecture 4.2] is true (see [BD, §6]) then, identifying PGSp_2 with $\mathrm{SO}(3,2)$, the representation $\tilde{\Pi}$ of $\mathrm{SO}(3,2)(\mathbb{A})$ satisfying that conjecture would have functorial lift to $U(2,2)(\mathbb{A})$ (recall §6) satisfying this conjecture, though there the factor $\zeta_\Sigma(s)$ in $L_\Sigma(s, \Pi, r_1)$ is missing. Note in particular that in [BD, §6], $s = k - 2$, so the $a_p(f)(1 + p^s)$ we see here is the same as the $a_p(f)(1 + p^{k-2})$ there. Moreover, in the coefficients of the infinitesimal character, with $k' = j + k$ and $s = k - 2$, the $\frac{k'-1+s}{2}$ and $\frac{k'-1-s}{2}$ we see here match the $\frac{j+2k-3}{2}$ and $\frac{j+1}{2}$ at the end of §6.

8. $G = U(2,2)$, $M \simeq \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_1) \times U(1,1)$

In this section, $G = U(2,2)$, $\Delta_G = \{e_1 - e_2, e_2 - e_4, e_4 - e_3\}$, and we choose $\alpha = e_1 - e_2$, so $\Delta_\alpha = \{e_1 - e_2, e_4 - e_3\}$ and $\Delta_M = \{e_2 - e_4\}$. The Levi subgroup $M \simeq \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_1) \times U(1,1)$, with $\left(e, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \mapsto \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & \overline{e}^{-1} & 0 \\ 0 & c & 0 & d \end{pmatrix}$, and P is the Klingen parabolic subgroup of G. We have $\Phi_N = \{e_1 - e_2, e_4 - e_3, e_1 - e_4, e_2 - e_3, e_1 - e_3\}$, $\rho_P = \frac{3}{2}(e_1 - e_3)$, $\langle \rho_P, \check{\alpha} \rangle = 3/2$, $\tilde{\alpha} = e_1 - e_3$.

Let Π' be a unitary, irreducible, cuspidal automorphic representation of $U(1,1)(\mathbb{A})$, and let $\Pi = 1 \times \Pi'$ on $M(\mathbb{A})$. Suppose for simplicity that Π' has trivial central character, and is a functorial lift from $\mathrm{SO}(2,1) \simeq \mathrm{PGL}_2$, let's say coming from a cuspidal Hecke eigenform f of weight k' and with Hecke eigenvalue for $T(p)$ equal to $a_p(f) = p^{(k'-1)/2}(\alpha_p + \alpha_p^{-1})$ (for any prime p unramified for f and for $F/\mathbb{Q}$). Then for such a prime, Π_p has

$$\chi_p = -\log_p(\alpha_p)(e_2 - e_4),$$

and Π_∞ has infinitesimal character

$$\lambda = \frac{k'-1}{2}(e_2 - e_4).$$

In the table below, we list together pairs of roots in the same $\mathrm{Gal}(F/\mathbb{Q})$-orbit.

$\gamma \in \Phi_N$	$\langle \lambda + s\tilde{\alpha}, \check{\gamma} \rangle$	$\lvert\chi_p(\check{\gamma}(p))\rvert_p$
$\{e_1 - e_2, e_4 - e_3\}$	$-\frac{k'-1}{2} + s$	α_p^{-1}
$\{e_1 - e_4, e_2 - e_3\}$	$\frac{k'-1}{2} + s$	α_p
$e_1 - e_3$	$2s$	1

Using the table, $m = 2$ in $r = \oplus_{i=1}^m r_i$, and using also Lemma 3.1, $L_\Sigma(s, \Pi, r_1) = L_\Sigma(f, s + \frac{k'-1}{2})L_\Sigma(f \otimes \psi, s + \frac{k'-1}{2})$, where ψ is the quadratic character for $F/\mathbb{Q}$, and $L_\Sigma(s, \Pi, r_2) = \zeta_\Sigma(s)$. Since k' is even, we need $s \in \frac{1}{2} + \mathbb{Z}$ for $\lambda + s\tilde{\alpha}$ to be algebraically integral. (Look at the second column of the table.)

$$\lambda + s\tilde{\alpha} = se_1 + \left(\frac{k'-1}{2}\right)e_2 - se_3 - \left(\frac{k'-1}{2}\right)e_4.$$

If we choose $w = (12)(34) \in {}_{\mathbb{Q}}W$ then

$$w(\lambda + s\tilde{\alpha}) = \left(\frac{k'-1}{2}\right)e_1 + se_2 - se_4 - \left(\frac{k'-1}{2}\right)e_3,$$

which is dominant and regular for $0 < s < \frac{k'-1}{2}$. We shall exclude $s = 1/2$.

In the case that p splits, consider $\mu = f_1 \in X_*(T_\#) = X_*(T)$, so $\mu(p) = \mathrm{diag}((p,1),(1,1),(1,p^{-1}),(1,1))$, where $\langle e_i, f_j \rangle = \delta_{ij}$. As a character of $\hat{T}$, f_1 is the highest weight of the standard representation of $\hat{G} \simeq \mathrm{GL}_4$, with weights $\{f_1, f_2, f_4, f_3\}$. Using $\chi_p + s\tilde{\alpha} = -\log_p(\alpha_p)(e_2 - e_4) + s(e_1 - e_3)$, we find

μ	$\lvert(\chi_p + s\tilde{\alpha})(\mu(p))\rvert_p$
f_1	p^{-s}
f_2	α_p
f_3	p^s
f_4	α_p^{-1}

The trace is $(\alpha_p + \alpha_p^{-1}) + p^s + p^{-s}$. Multiplying by $p^{\langle w(\lambda+s\tilde{\alpha}), f_1 \rangle} = p^{(k'-1)/2}$, we find that

$$T_{f_1}(\mathrm{Ind}_P^G(\Pi_p \otimes |s\tilde{\alpha}|_p)) = a_p(f) + p^{\frac{k'-1}{2}-s} + p^{\frac{k'-1}{2}+s}.$$

In the case that p is inert, consider $\mu = f_1 - f_3 \in X_*(T_\#) = X_*(T_d)$, so $\mu(p) = \mathrm{diag}(p, 1, p^{-1}, 1)$. Proceeding as in the previous section, but this time with $\chi_p + s\tilde{\alpha} + \rho_G = -\log_p(\alpha_p)(e_2 - e_4) + s(e_1 - e_3) + (1/2)(3e_1 + e_2 - e_4 - 3e_3)$, we find

$$T'_{f_1 - f_3}(\mathrm{Ind}_P^G(\Pi_p \otimes |s\tilde{\alpha}|_p))$$
$$= p^{2s}p^3 + p^2(\alpha_p^{-2}p^1) + p^4(\alpha_p^2 p^{-1}) + p^6(p^{-2s}p^{-3}) + (p^3 - p^2 + p - 1).$$

Multiplying by $p^{\langle w(\lambda+s\tilde{\alpha})-\rho_G, f_1-f_3\rangle} = p^{k'-1-3}$, we find that

$$T_{f_1-f_3}(\mathrm{Ind}_P^G(\Pi_p \otimes |s\tilde{\alpha}|_p)) = (a_p(f)^2 - 2p^{k'-1}) + p^{k'-4}(p^3 - p^2 + p - 1).$$

If $q > k'$ and $\mathrm{ord}_{\mathfrak{q}}\left(\frac{L_\Sigma(1+s,\Pi,r_1)}{\Omega}\right) > 0$ then Conjecture 4.2 predicts the existence of an irreducible, tempered, cuspidal, automorphic representation $\tilde{\Pi}$ of $G(\mathbb{A})$, satisfying congruences mod $\mathfrak{q}$ of Hecke eigenvalues, with $\Pi \otimes |s\tilde{\alpha}|$. The infinitesimal character of $\tilde{\Pi}_\infty$ would be $w(\lambda + s\tilde{\alpha})$. Unless we are in a case where $\tilde{\Pi}$ is some kind of endoscopic lift, we would expect to be able to choose $\tilde{\Pi}$ (subject to the congruences, which depend only on $\tilde{\Pi}|_{G(\mathbb{A}_f)}$) in such a way that $\tilde{\Pi}_\infty$ is holomorphic discrete series. Then, as in §5, the lowest K_∞-type of $\tilde{\Pi}_\infty$ has highest weight $\Lambda = w(\lambda + s\tilde{\alpha}) + \rho_n - \rho_c$. In our case, $\rho_n - \rho_c = \frac{1}{2}(e_1 - e_3) + \frac{3}{2}(e_2 - e_4)$, so

$$\Lambda = \frac{k'}{2}(e_1 - e_3) + (s + 3/2)(e_2 - e_4).$$

If the case $G = \mathrm{GSp}_2$, $M \simeq \mathrm{GL}_2 \times \mathrm{GL}_1$, P the Siegel parabolic (and $i = 1$) of [BD, Conjecture 4.2](i.e. Harder's conjecture) is true (see [BD, §7]) then, identifying PGSp_2 with $\mathrm{SO}(3,2)$, the representation $\tilde{\Pi}$ of $\mathrm{SO}(3,2)(\mathbb{A})$ satisfying that conjecture would have functorial lift to $U(2,2)(\mathbb{A})$ (recall §6) satisfying this conjecture. Note in particular that in [BD, §7], $k' = j + 2k - 2$ and $s = \frac{j+1}{2}$, so the $a_p(f)+p^{(k'-1)/2-s}+p^{(k'-1)/2+s}$ we see here is the same as the $a_p(f)+p^{k-2}+p^{j+k-1}$ there. Moreover, in the coefficients of the infinitesimal character, the $\frac{k'-1}{2}$ and s we see here match the $\frac{j+2k-3}{2}$ and $\frac{j+1}{2}$ at the end of §6. The L-function $L_\Sigma(f, \frac{k'-1}{2}+s)$ we see here matches $L_\Sigma(s, \Pi, r_1)$ in [BD, §7]. The conjecture there applies equally well to the quadratic twist $f \otimes \psi$, thus accounting for the factor $L_\Sigma(f \otimes \psi, \frac{k'-1}{2}+s)$ too.

Since congruences between Hermitian Klingen-Eisenstein series and cusp forms might be provable using differential operators and pullback formulas, c.f. [KM, SU], this perhaps provides a means of attacking Harder's conjecture. But one would need to show that the cuspidal automorphic representation $\tilde{\Pi}$ of $U(2,2)(\mathbb{A})$, satisfying the congruence here, is in the image of the functorial lift from $\mathrm{SO}(3,2)$.

The case $G = \mathrm{GSp}_2$, $M \simeq \mathrm{GL}_2 \times \mathrm{GL}_1$, P the Siegel parabolic, $i = 2$ of [BD, Conjecture 4.2] *is* true, as explained in [BD, §7], and if the representation $\tilde{\Pi}$ of $\mathrm{SO}(3,2)(\mathbb{A})$ satisfies that conjecture, its (conjectured) functorial lift to $U(2,2)(\mathbb{A})$ satisfies the case $i = 2$ of the conjecture here.

In the representation $\Pi = 1 \times \Pi'$ of $(\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_1) \times U(1,1))(\mathbb{A})$, we could replace the 1 by the representation of $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_1)(\mathbb{A})$ coming from a Hecke character. Then $L_\Sigma(s, \Pi, r_1)$ would be the tensor product L-function of f and a CM-form, and $L_\Sigma(s, \Pi, r_2)$ would be a Dirichlet L-function. Details are omitted.

9. $G = U(2,1)$, $M \simeq \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_1) \times U(1)$

For any integer $n \geq 1$, let

$$G = U(n+1, n) = \{A \in \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_{2n+1} \mid AJ\,{}^t\overline{A} = J\},$$

where $J = \begin{bmatrix} 0_n & 0 & I_n \\ 0 & 1 & 0 \\ I_n & 0 & 0_n \end{bmatrix}$, be the unitary group associated to an Hermitian form of signature $(n+1, n)$.

The algebraic group $G/\mathbb{Q}$ has a maximal (non-split) torus T, with $T(\mathbb{Q}) = \{\mathrm{diag}(a_1, \ldots, a_n, a_{n+1}, a_1^*, \ldots, a_n^*) : a_1, \ldots, a_n, a_{n+1} \in F^\times, a_{n+1} = a_{n+1}^*\}$, where $a^* := \overline{a}^{-1}$, and $T_d(\mathbb{Q}) = \{\mathrm{diag}(a_1, \ldots, a_n, 1, a_1^{-1}, \ldots, a_n^{-1}) : a_1, \ldots, a_n \in \mathbb{Q}^\times\}$. In the complexification $G(\mathbb{C}) \simeq \mathrm{GL}_{2n+1}(\mathbb{C})$, $T(\mathbb{C})$ becomes the standard diagonal torus with characters $e_i(\mathrm{diag}(t_1, \ldots, t_{2n+1})) := t_i$ for $1 \leq i \leq 2n+1$. In a little more detail, $F \otimes \mathbb{C} \simeq \mathbb{C} \times \mathbb{C}$, with F embedded via $z \mapsto (z, \overline{z})$, and $T(\mathbb{C}) =$

$$\{\mathrm{diag}((t_1, t_{n+2}^{-1}), (t_2, t_{n+3}^{-1}), \ldots, (t_n, t_{2n+1}^{-1}), (t_{n+1}, t_{n+1}^{-1}), (t_{n+2}, t_1^{-1}), \ldots, (t_{2n+1}, t_n^{-1}))$$

$$: t_1, \ldots, t_{2n+1} \in \mathbb{C}^\times\},$$

$G(\mathbb{C}) = \{(A, J\,{}^tA^{-1}J^{-1}) : A \in \mathrm{GL}_{2n+1}(\mathbb{C})\}$, which we can identify with $\mathrm{GL}_{2n+1}(\mathbb{C})$ by looking only at the first entry. The natural action of $\mathrm{Gal}(F/\mathbb{Q})$ (with complex conjugation swapping the entries of an ordered pair) induces an action on the character group of $T(\mathbb{C})$, with the non-trivial element switching e_i with $-e_{n+1+i}$, for $1 \leq i \leq n$, and e_{n+1} with $-e_{n+1}$.

We may choose a system of simple positive roots $e_1 - e_2, \ldots, e_n - e_{n+1}, e_{n+1} - e_{2n+1}, e_{2n+1} - e_{2n}, \ldots, e_{n+3} - e_{n+2}$. Then $\mathrm{Gal}(F/\mathbb{Q})$ fixes $e_n - e_{2n}$ and switches $e_i - e_{i+1}$ with $e_{n+2+i} - e_{n+1+i}$, for $1 \leq i < n$, and $e_n - e_{n+1}$ with $e_{n+1} - e_{2n+1}$. The set of positive roots is then $\Phi^+ = \{e_i - e_j, e_{n+1+j} - e_{n+1+i} : 1 \leq i < j \leq n\} \cup \{e_i - e_j : 1 \leq i \leq n,\ n+2 \leq j \leq 2n+1\} \cup \{e_i - e_{n+1}, e_{n+1} - e_{n+1+i} : 1 \leq i \leq n\}$. The half-sum of positive roots is given by $\rho = n(e_1 - e_{n+2}) + (n-1)(e_2 - e_{n+3}) + \cdots + (e_n - e_{2n+1}))$.

In this section, $n = 1$, so $G = U(2,1)$, $\Delta_G = \{e_1 - e_2, e_2 - e_3\}$, and we choose $\alpha = e_1 - e_2$, so $\Delta_\alpha = \{e_1 - e_2, e_2 - e_3\}$ and $\Delta_M = \emptyset$. The Levi subgroup $M \simeq \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_1 \times U(1)$, with $(a, b) \mapsto \mathrm{diag}(a, b, \overline{a}^{-1})$, and P is the upper-triangular Borel subgroup of G. We have $\Phi_N = \{e_1 - e_2, e_1 - e_3, e_2 - e_3\}$, $\rho_P = e_1 - e_3$, $\langle \rho_P, \check{\alpha} \rangle = 1$, $\tilde{\alpha} = e_1 - e_3$.

Let $\psi_1, \psi_2 : \mathbb{A}_F^\times / F^\times \to \mathbb{C}^\times$ be Hecke characters such that $\psi_{i,\infty}(z) = z^{-m_i}$ (with $m_i \in \mathbb{Z}$), thought of as characters of $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_1)(\mathbb{A})$, with ψ_2 further restricted to the subgroup $U(1)(\mathbb{A})$. Let Π be the unitary, irreducible, cuspidal automorphic representation $(\psi_1 |\cdot|^{m_1}) \times \psi_2$ of $M(\mathbb{A}) \simeq (\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_1) \times U(1))(\mathbb{A})$. The infinitesimal character for Π_∞ is

$$\lambda = -m_1 e_1 - m_2 e_2 + m_1((1/2)(e_1 - e_2)) = -\frac{m_1}{2}(e_1 + e_3) - m_2 e_2.$$

Suppose that Π_p is unramified. Given a prime ideal $\mathfrak{p} \mid (p)$, let π be a uniformiser at $\mathfrak{p}$, and embed $F_\mathfrak{p}$ in $\mathbb{A}_F^\times$ in the usual way. Let $\psi_i(\mathfrak{p}) := \psi_i(\pi)$, i.e. $\psi_i((1, \ldots, 1, \pi, 1, \ldots, 1))$. If p is inert, let $\alpha_p = \psi_1((p))/p^{m_1}$, so $|\alpha_p| = 1$. Then

$$\chi_p = -\frac{1}{2}\log_p(\alpha_p)(e_1 - e_3) \in {}_{\mathbb{Q}_p}X^*(T) \otimes \mathbb{R}.$$

If $p = \mathfrak{p}\overline{\mathfrak{p}}$ splits, let $\alpha_\mathfrak{p} = \psi_1(\mathfrak{p})/p^{m_1/2}$, $\alpha_{\overline{\mathfrak{p}}} = \psi_1(\overline{\mathfrak{p}})/p^{m_1/2}$, $\beta_\mathfrak{p} = \psi_2(\mathfrak{p})/p^{m_2/2}$ and $\beta_{\overline{\mathfrak{p}}} = \psi_2(\overline{\mathfrak{p}})/p^{m_2/2}$ (in which the factors $p^{-m_2/2}$ are arguably superfluous). Then the Satake parameter is

$$\chi_p = -[\log_p(\alpha_\mathfrak{p})e_1 + (\log_p(\beta_\mathfrak{p}) - \log_p(\beta_{\overline{\mathfrak{p}}}))e_2 - \log_p(\alpha_{\overline{\mathfrak{p}}})e_3].$$

In the table below, we list together a pair of roots in the same $\mathrm{Gal}(F/\mathbb{Q})$-orbit.

$\gamma \in \Phi_N$	$\langle \lambda + s\tilde{\alpha}, \check{\gamma} \rangle$	$\lvert\chi_p(\check{\gamma}(p))\rvert_p$ (p inert)	$\lvert\chi_p(\check{\gamma}(p))\rvert_p$ (p split)
$\{e_1 - e_2, e_2 - e_3\}$	$\pm\frac{m_1 - 2m_2}{2} + s$	$\alpha_p^{1/2}$	$\{\alpha_\mathfrak{p}\beta_{\overline{\mathfrak{p}}}/\beta_\mathfrak{p}, \alpha_{\overline{\mathfrak{p}}}\beta_\mathfrak{p}/\beta_{\overline{\mathfrak{p}}}\}$
$e_1 - e_3$	$2s$	α_p	$\alpha_\mathfrak{p}\alpha_{\overline{\mathfrak{p}}}$

Using the table, $m = 2$ in $r = \oplus_{i=1}^{m} r_i$, and using also Lemma 3.1, $L(s, \Pi, r_1) = L(s + \frac{m_1}{2}, \psi_1\psi_2^c/\psi_2)$, where $\psi_2^c(a) := \psi_2(c(a))$, while $L(s, \Pi, r_2) = L(s, \theta)$, where θ is a Dirichlet character such that, for any prime p, $\psi_1((p)) = \theta(p)p^{m_1}$.

We need $2s \equiv m_1 \pmod 2$ for $\lambda + s\tilde{\alpha}$ to be algebraically integral. (Look at the second column of the table.)

$$\lambda + s\tilde{\alpha} = \left(s - \frac{m_1}{2}\right) e_1 - m_2 e_2 + \left(-s - \frac{m_1}{2}\right) e_3.$$

If $m_1 > 2m_2$ and we choose $w = (12) \in W$ then

$$w(\lambda + s\tilde{\alpha}) = -m_2 e_1 + \left(s - \frac{m_1}{2}\right) e_2 + \left(-s - \frac{m_1}{2}\right) e_3,$$

which is dominant and regular for $0 < s < (m_1 - 2m_2)/2$.

If $m_1 < 2m_2$ and we choose $w = (23) \in W$ then

$$w(\lambda + s\tilde{\alpha}) = \left(s - \frac{m_1}{2}\right) e_1 + \left(-s - \frac{m_1}{2}\right) e_2 - m_2 e_3,$$

which is dominant and regular for $0 < s < (2m_2 - m_1)/2$.

In the case that p splits, consider $\mu = -f_3 \in X_*(T_\#) = X_*(T)$, so $\mu(p) = \mathrm{diag}((1,p),(1,1),(p^{-1},1))$, where $\langle e_i, f_j\rangle = \delta_{ij}$. As a character of $\hat{T}$, $-f_3$ is the highest weight of the dual of the standard representation of $\hat{G} \simeq \mathrm{GL}_3$, with weights $\{-f_1, -f_2, -f_3\}$. Using $\chi_p + s\tilde{\alpha} = -[\log_p(\alpha_{\mathfrak{p}})e_1 + (\log_p(\beta_{\mathfrak{p}}) - \log_p(\beta_{\overline{\mathfrak{p}}}))e_2 - \log_p(\alpha_{\overline{\mathfrak{p}}})e_3] + s(e_1 - e_3)$, we find

μ	$\lvert(\chi_p + s\tilde{\alpha})(\mu(p))\rvert_p$
$-f_1$	$\alpha_{\mathfrak{p}}^{-1}p^s$
$-f_2$	$\beta_{\overline{\mathfrak{p}}}/\beta_{\mathfrak{p}}$
$-f_3$	$\alpha_{\overline{\mathfrak{p}}}p^{-s}$

The trace is

$$p^{-s}\alpha_{\overline{\mathfrak{p}}} + p^s\alpha_{\mathfrak{p}}^{-1} + \frac{\beta_{\overline{\mathfrak{p}}}}{\beta_{\mathfrak{p}}}.$$

At this point we must exercise caution over the scaling of the Hecke operator, since $w(\lambda + s\tilde{\alpha})$ is not in general self-dual. For a split prime $p = \mathfrak{p}\overline{\mathfrak{p}}$, where for simplicity we assume that $\mathfrak{p}$ is principal, let $\mathfrak{p} = (\pi_{\mathfrak{p}})$, where

$$\begin{cases} \pi_{\mathfrak{p}} \equiv 1 \pmod 3 & \text{if } F = \mathbb{Q}(\sqrt{-3}); \\ \pi_{\mathfrak{p}} \equiv 1 \pmod{2+2i} & \text{if } F = \mathbb{Q}(\sqrt{-1}); \\ \chi_F(\pi_{\mathfrak{p}}) = 1 & \text{otherwise,} \end{cases}$$

where χ_F is the quadratic character associated with $F/\mathbb{Q}$,

and let $\pi_{\overline{\mathfrak{p}}} = \overline{\pi_{\mathfrak{p}}}$.

If $r_\infty : W_{\mathbb{R}} \to {}^L M$ is the Langlands parameter at ∞ of $\Pi \otimes |s\tilde{\alpha}|$ then, restricting to the subgroup $\mathbb{C}^\times$, of index two in the Weil group $W_{\mathbb{R}}$, $r_\infty(z)$ is conjugate to

$$\begin{cases} \mathrm{diag}(z^{-m_2}\overline{z}^{m_2}, z^{s-(m_1/2)}\overline{z}^{(m_1/2)-s}, z^{-s-(m_1/2)}\overline{z}^{s+(m_1/2)}) & \text{if } m_1 > 2m_2; \\ \mathrm{diag}(z^{s-(m_1/2)}\overline{z}^{(m_1/2)-s}, z^{-s-(m_1/2)}\overline{z}^{s+(m_1/2)}, z^{-m_2}\overline{z}^{m_2}) & \text{if } m_1 < 2m_2. \end{cases}$$

We can think of the entries as $z^p\overline{z}^q$ for Hodge component (p,q). To make all the exponents non-negative we would have to multiply by

$$\begin{cases} z^{s+(m_1/2)}\overline{z}^{-m_2} & \text{if } m_1 > 2m_2; \\ z^{m_2}\overline{z}^{s-(m_1/2)} & \text{if } m_1 < 2m_2. \end{cases}$$

Consequently we let

$$T_{-f_3} = \begin{cases} \pi_{\mathfrak{p}}^{s+(m_1/2)}\pi_{\overline{\mathfrak{p}}}^{-m_2}p^{-\langle\rho_G,f_1\rangle}T'_{-f_3} & \text{if } m_1 > 2m_2; \\ \pi_{\mathfrak{p}}^{m_2}\pi_{\overline{\mathfrak{p}}}^{s-(m_1/2)}p^{-\langle\rho_G,f_1\rangle}T'_{-f_3} & \text{if } m_1 < 2m_2. \end{cases}$$

(T_{f_1} and T_{-f_3} could also reasonably be labelled $T_{\mathfrak{p}}$ and $T_{\overline{\mathfrak{p}}}$, respectively.) We have then

$$T_{-f_3}(\mathrm{Ind}_P^G(\Pi_p\otimes|s\tilde{\alpha}|_p)) = \begin{cases} \pi_{\mathfrak{p}}^{s+(m_1/2)}\pi_{\overline{\mathfrak{p}}}^{-m_2}(p^{-s}\alpha_{\overline{\mathfrak{p}}}+p^s\alpha_{\mathfrak{p}}^{-1}+\frac{\beta_{\overline{\mathfrak{p}}}}{\beta_{\mathfrak{p}}}) & \text{if } m_1 > 2m_2; \\ \pi_{\mathfrak{p}}^{m_2}\pi_{\overline{\mathfrak{p}}}^{s-(m_1/2)}(p^{-s}\alpha_{\overline{\mathfrak{p}}}+p^s\alpha_{\mathfrak{p}}^{-1}+\frac{\beta_{\overline{\mathfrak{p}}}}{\beta_{\mathfrak{p}}}) & \text{if } m_1 < 2m_2. \end{cases}$$

Example. Let $F=\mathbb{Q}(\sqrt{-3})$. Take $m_1>0$, $m_2=-m_1$, $\psi_2=\psi_1^{-1}$, where ψ_1 is the unique Hecke character of conductor 3 such that, for split primes $p=\mathfrak{p}\overline{\mathfrak{p}}$, $\psi_1(\mathfrak{p})=\pi_{\mathfrak{p}}^{m_1}$ and $\psi_1(\overline{\mathfrak{p}})=\pi_{\overline{\mathfrak{p}}}^{m_1}$. Then $\alpha_{\mathfrak{p}}=\beta_{\overline{\mathfrak{p}}}=\pi_{\mathfrak{p}}^{m_1}/p^{m_1/2}$ and $\alpha_{\overline{\mathfrak{p}}}=\beta_{\mathfrak{p}}=\alpha_{\mathfrak{p}}^{-1}=\pi_{\overline{\mathfrak{p}}}^{m_1}/p^{m_1/2}$, so

$$\begin{aligned} T_{-f_3}(\mathrm{Ind}_P^G(\Pi_p\otimes|s\tilde{\alpha}|_p)) &= \pi_{\mathfrak{p}}^{s+(m_1/2)}\pi_{\overline{\mathfrak{p}}}^{m_1}\left(p^{-s-(m_1/2)}\pi_{\overline{\mathfrak{p}}}^{m_1}+p^{s+(m_1/2)}\pi_{\mathfrak{p}}^{-m_1}+\frac{\pi_{\mathfrak{p}}^{m_1}}{\pi_{\overline{\mathfrak{p}}}^{m_1}}\right) \\ &= \pi_{\overline{\mathfrak{p}}}^{-s+(3m_1/2)}+p^{2s}\pi_{\overline{\mathfrak{p}}}^{-s+(3m_1/2)}+\pi_{\mathfrak{p}}^{s+(3m_1/2)}. \end{aligned}$$

We recover (for $L(1+s,\Pi,r_1)$) a conjecture arrived at in a somewhat different manner by Harder [H2]. Bergström has found numerical evidence in the cases $(q,m_1,s)=(53,5,9/2)$ and $(271,7,7/2)$, using Hecke eigenvalues computed by him and van der Geer, and he expects to find more examples, which they will report on elsewhere. In Harder's notation, $m_1=\frac{1}{3}(3+2n_1+n_2)$, $s=\frac{1}{2}(1+n_2)$, so $T_{-f_3}(\mathrm{Ind}_P^G(\Pi_p\otimes|s\tilde{\alpha}|_p))$ becomes $\pi_{\overline{\mathfrak{p}}}^{1+n_1}+p^{1+n_2}\pi_{\overline{\mathfrak{p}}}^{1+n_1}+\pi_{\mathfrak{p}}^{2+n_1+n_2}$, which is the same as his expression (for T_{f_1} at least, and correcting the multiplication at the bottom of p.5). His approach is via the cohomology of local systems on arithmetic quotients of locally symmetric spaces, and its restriction to the boundary [H1], [BD, §13]. The local system is determined by a representation of G, in whose highest weight n_1 and n_2 are the coefficients of the fundamental dominant weights. Note also that $L(1+s+\frac{m_1}{2},\psi_1\psi_2^c/\psi_2)=L(1+s+\frac{3m_1}{2},\psi_1^3)$ becomes his $L(n_1+n_2+3,\psi_1^3)$, with $\psi_1^3(\mathfrak{p})=\pi_{\mathfrak{p}}^{2n_1+n_2+3}$, so ψ_1^3 is $\psi_{2n_1+n_2+3}$ in Harder's notation. He specifies that $\tilde{\Pi}$ should have a vector fixed by a particular open compact subgroup of $G(\mathbb{Q}_3)$, whereas our conjecture only gives that it should be unramified away from 3.

In this example, the Dirichlet character $\theta=\chi_F^{m_1}$, so has the same parity as m_1. Consequently, if $2s\equiv m_1 \pmod 2$ as above, then $L(1+2s,\Pi,r_2)=L(1+2s,\theta)$ is not a critical value. But the conjecture appears to work for $L(1+s,\Pi,r_1)$, so we shall ignore the precise wording of Conjecture 4.2.

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