TWISTED ADJOINT L-VALUES, DIHEDRAL CONGRUENCE PRIMES AND THE BLOCH-KATO CONJECTURE

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Abstract. We show that a dihedral congruence prime for a normalised Hecke eigenform f in $S_k(\Gamma_0(D), \chi_D)$, where χ_D is a real quadratic character, appears in the denominator of the Bloch-Kato conjectural formula for the value at 1 of the twisted adjoint L-function of f. We then use a formula of Zagier to prove that it appears in the denominator of a suitably normalised $L(1, \mathrm{ad}^0(g) \otimes \chi_D)$ for *some* $g \in S_k(\Gamma_0(D), \chi_D)$.

1. Introduction

Let $F = \mathbb{Q}(\sqrt{D})$ be a real quadratic field, with discriminant $D > 0$. Let $f \in S_k(\Gamma_0(D), \chi_D)$ be a normalised Hecke eigenform, where $k \geq 2$ and χ_D is the Legendre symbol attached to $F/\mathbb{Q}$. Say $f = \sum_{m=1}^{\infty} a_m(f) q^m$. Let K_f be the CM subfield of $\mathbb{C}$ generated by the Hecke eigenvalues of f, with real subfield K_f^+, ring of integers $\mathcal{O}_f$, and let $\mathcal{O}(f)$ be the order in $\mathcal{O}_f$ generated by the $a_m(f)$. Let $f_c = \sum_{m=1}^{\infty} \overline{a_m(f)} q^m$ be the complex conjugate eigenform, and note that f_c is the newform associated to the twisted form f_{χ_D}. (This is because for each prime $q \nmid D$, T_q and $\langle q \rangle^{-1} T_q$ are adjoints for the Peterssen inner product, so $a_q(f)$ is real or purely imaginary according as $\chi_D(q) = 1$ or -1, respectively.) Note that because the conductor of χ_D is D, $S_k(\Gamma_0(D), \chi_D)$ contains no old forms. The following is very easy to prove. For reference, it is a trivial modification of a special case of [BG, Lemma 3.1].

Lemma 1.1. *Let $\mathfrak{P} \mid (p)$ be a prime divisor in K_f. Suppose that $p \nmid [\mathcal{O}_f : \mathcal{O}(f)]$. We have $f \equiv f_c \pmod{\mathfrak{P}}$, i.e. $a_m(f) \equiv \overline{a_m(f)} \pmod{\mathfrak{P}}$ $\forall m$, if and only if $\mathfrak{P}$ is ramified in K_f/K_f^+.*

Congruences between the Hecke eigenvalues of automorphic forms often produce non-zero elements in groups whose orders appear in the Bloch-Kato conjecture on special values of L-functions. When the L-values in question are amenable to analysis or to computation, this can provide an opportunity to test the conjecture, by proving a consequence or computing data that support it. In order to examine the consequences of the congruences, one has to interpret them in terms of Galois representations.

Given $f, \mathfrak{P}$ as above, let

$$\rho_{f,\mathfrak{P}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(K_{f,\mathfrak{P}})$$

be the continuous linear representation attached to f by Deligne [De2]. For every prime $q \nmid Dp$, $\rho_{f,\mathfrak{P}}$ is unramified at q, and if $\mathrm{Frob}_q \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\overline{\mathbb{F}}_q$ as

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$x \mapsto x^q$ then

$$\det(I - \rho_{f,\mathfrak{P}}(\mathrm{Frob}_q^{-1})X) = 1 - a_q(f)X + \chi_D(q)q^{k-1}X^2.$$

Choosing a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant $\mathcal{O}_{f,\mathfrak{P}}$-lattice in the 2-dimensional $K_{f,\mathfrak{P}}$-vector space on which $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts via $\rho_{f,\mathfrak{P}}$, then reducing modulo $\mathfrak{P}$, one obtains a residual representation

$$\overline{\rho}_{f,\mathfrak{P}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_{\mathfrak{P}}),$$

where $\mathbb{F}_{\mathfrak{P}} := \mathcal{O}_{f,\mathfrak{P}}/\mathfrak{P}$.

Proposition 1.2. *Let $F = \mathbb{Q}(\sqrt{D})$ be a real quadratic field, with discriminant $D > 0$. Let $f \in S_k(\Gamma_0(D), \chi_D)$ be a normalised Hecke eigenform, $\mathfrak{P} \mid (p)$ a prime divisor in K_f such that $f \equiv f_c \pmod{\mathfrak{P}}$. Suppose that $p \nmid 2D$, that $\mathfrak{P} \nmid a_f(p)$ (i.e. f is ordinary at $\mathfrak{P}$) and that $\overline{\rho}_{f,\mathfrak{P}}$ is absolutely irreducible. Then*

1. *$\overline{\rho}_{f,\mathfrak{P}} \simeq \overline{\rho}_{f,\mathfrak{P}} \otimes \chi_D$, where χ_D is viewed as a character of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.*
2. *The restriction of $\overline{\rho}_{f,\mathfrak{P}}$ to $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ is reducible.*
3. *The prime p splits in F, say as $\mathfrak{p}\mathfrak{p}^\sigma$. The representation $\overline{\rho}_{f,\mathfrak{P}}$ is induced from a character $\phi_{\mathfrak{p}}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$, coming via class field theory from the character $(\mathcal{O}_F/\mathfrak{p})^\times \to \overline{\mathbb{F}}_p^\times$ that is the $(1-k)$-power of the identity character. Equally it is induced by $\phi_{\mathfrak{p}^c}$, similarly defined but of conductor $\mathfrak{p}^c$ in place of $\mathfrak{p}$.*
4. *$p \mid \mathrm{Norm}_{F/\mathbb{Q}}((\epsilon_+)^{k-1} - 1)$, where ϵ_+ is a generator for the group of totally positive units of $\mathcal{O}_F$.*

Conversely, if $p \nmid 6D$ is a prime that splits, $(p) = \mathfrak{p}\mathfrak{p}^\sigma$ in $\mathcal{O}_F$, and if $\mathfrak{p} \mid ((\epsilon_+)^{k-1} - 1)$ (equivalently $p \mid \mathrm{Norm}_{F/\mathbb{Q}}((\epsilon_+)^{k-1} - 1)$) then there exists a normalised Hecke eigenform $f \in S_k(\Gamma_0(D), \chi_D)$, ordinary at $\mathfrak{P}$, such that $f \equiv f_c \pmod{\mathfrak{P}}$ and $\overline{\rho}_{f,\mathfrak{P}}$ is absolutely irreducible.

A convenient reference for the proof is [BG], where (1)-(4) are covered by Theorem 2.1 and the converse part is covered by Theorem 2.11, both of which are more general statements. I have largely adopted their notation, though their $\rho_{f,\mathfrak{P}}$ is the dual of ours. (4) is a consequence of class field theory, that the character $\phi_{\mathfrak{p}}$ in (3) must kill the totally positive unit ϵ_+. It was proved in the case $k = 2$ by Ohta [O], confirming an experimental observation of Shimura [Sh, before Proposition 7.34]. For general k it is part of Theorem 1 in a paper of Hida [H1]. The converse part was proved by Koike in the case $k = 2$, which was all he needed [K, Proposition 4.1], and in general it is again part of Hida's Theorem 1.

A consequence of $\overline{\rho}_{f,\mathfrak{P}} \simeq \overline{\rho}_{f,\mathfrak{P}} \otimes \chi_D$ is the existence of a non-zero element of $H^0(\mathbb{Q}, \mathrm{ad}^0(\overline{\rho}_{f,\mathfrak{P}}) \otimes \chi_D)$ (Lemma 3.1). As we shall see in §3, this contributes to the denominator of a conjectural formula for the value at $s = 1$ of a "twisted adjoint" L-function $L(s, \mathrm{ad}^0(f) \otimes \chi_D)$, whose Euler factor at any prime $q \nmid D$ is

$$L_q(s, \mathrm{ad}^0(f) \otimes \chi_D) = [(1-(\alpha_q/\beta_q)\chi_D(q)q^{-s})(1-\chi_D(q)q^{-s})(1-(\beta_q/\alpha_q)\chi_D(q)q^{-s})]^{-1},$$

where the Euler factor at q of the Hecke L-function $L(s, f)$ is

$$L_q(s, f) = [(1 - \alpha_q q^{-s})(1 - \beta_q q^{-s})]^{-1}.$$

Since $\alpha_q\beta_q = \chi_D(q)q^{k-1}$, we also have $L(s, \mathrm{ad}^0(f) \otimes \chi_D) = L(s + k - 1, \mathrm{Sym}^2(f))$, so we are equally looking at the value $L(k, \mathrm{Sym}^2(f))$. The conjectural formula is an instance of the Bloch-Kato conjecture. It is in fact a formula for the factorisation of the algebraic number

$$\frac{L(1, \mathrm{ad}^0(f) \otimes \chi_D)}{\Omega},$$

where Ω is a suitably normalised Deligne period [De1], and we are looking at the $\mathfrak{P}$-part. There is also a term in the numerator of the conjectural formula, the order of a certain Selmer group. In Lemma 3.2 we are able to show, under certain hypotheses, that this Selmer group contributes nothing at $\mathfrak{P}$, so we expect that $\mathrm{ord}_{\mathfrak{P}}\left(\frac{L(1,\mathrm{ad}^0(f)\otimes\chi_D)}{\Omega}\right) < 0$.

Zagier [Z] proved a formula for the critical values of $L(s, \mathrm{ad}^0(f) \otimes \chi_D)$, in particular for the algebraic number $\frac{L(1,\mathrm{ad}^0(f)\otimes\chi_D)}{\pi^{k+1}(f,f)}$, where (f, f) is the Peterssen norm. So we need to make use of the relation between Ω and the Petersson norm, between which intervenes a certain cohomological congruence ideal η_f, which is the subject of §2. For the very special type of simple congruence we are looking at, we are able, with the help of a "multiplicity one" theorem of Faltings and Jordan [FJ], to say (under mild hypotheses) exactly what the $\mathfrak{P}$-part of η_f is; see Proposition 2.2. We use this in §3, both in proving triviality of the Selmer group (Lemma 3.2) and in producing a definite prediction that

$$\mathrm{ord}_{\mathfrak{P}}\left(\frac{L(1, \mathrm{ad}^0(f) \otimes \chi_D)}{\pi^{k+1}(f, f)}\right) < -\mathrm{ord}_{\mathfrak{P}}(\mathfrak{d}(K_f/K_f^+)).$$

In §4 we seek to use Zagier's formula to confirm this, but have to settle for showing that it is true for *some* normalised Hecke eigenform $f \in S_k(\Gamma_0(D), \chi_D)$, not necessarily one satisfying $f \equiv f_c \pmod{\mathfrak{P}}$; see Proposition 4.1. (We also need conditions $D \equiv 1 \pmod 4$ and $k > 2$.) Remarkably, the required contribution of $\mathfrak{P}$ to the denominator comes from $((\epsilon_+)^{k-1} - 1)$, after summing a geometric series. The occurrence of divisors of $((\epsilon_+)^{k-1} - 1)$ in the denominator of $\frac{L(1,\mathrm{ad}^0(f)\otimes\chi_D)}{\pi^{k+1}(f,f)}$ was observed in [DHI, 2.2] in numerical examples, which Doi and Ishii computed using Zagier's formula, so presumably they likewise summed this series.

Ghate [G, §10, Remark 4] has an alternative explanation for the appearance of dihedral congruence primes in the denominator of $\frac{L(1,\mathrm{ad}^0(f)\otimes\chi_D)}{\Omega}$. This is based on the fact that, as a congruence prime for f, $\mathfrak{P}$ appears in the numerator of a suitably normalised $L(1, \mathrm{ad}^0(f))$, by a theorem of Hida [H3, Theorem 5.16], but because f and f_c have the same Doi-Naganuma lift $\hat{f} = \hat{f}_c$ (base change to F), $f \equiv f_c \pmod{\mathfrak{P}}$ does not make $\mathfrak{P}$ a congruence prime for $\hat{f}$, so it is not expected to appear in the numerator of a suitably normalised $L(1, \mathrm{ad}^0(\hat{f})) = L(1, \mathrm{ad}^0(f))L(1, \mathrm{ad}^0(f) \otimes \chi_D)$. Thus it is required in the denominator of the second factor (subject to a conjectured period relation making the normalisations compatible).

The viewpoint taken here is that of [DH], where however the situation was a little different. We had a Hecke eigenform $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ with $\overline{\rho}_{f,\mathfrak{P}}$ dihedral, whose existence depended on non-triviality of the class group of $\mathbb{Q}(\sqrt{-p})$, with $p \equiv 3 \pmod 4$), and $k = (p+1)/2$. Again, we confirmed a prediction of the Bloch-Kato conjecture, this time proving that $\mathrm{ord}_{\mathfrak{P}}\left(\frac{L(\mathrm{Sym}^2(g),2k-2)}{\Omega}\right) < 0$ (a rightmost, rather than near-central, critical value) for *some* Hecke eigenform $g \in S_k(\mathrm{SL}_2(\mathbb{Z}))$. (Strictly speaking, since we did not prove triviality of the Selmer group, we had to reverse this logic, predicting and then proving the existence of f after showing that of g, an approach we could have taken here too.)

2. The congruence ideal

In this section we prove a technical result ready for use in the following section. Let $F = \mathbb{Q}(\sqrt{D})$ be a real quadratic field, with discriminant $D > 0$, $f \in S_k(\Gamma_0(D), \chi_D)$ a normalised Hecke eigenform. Let K_f be the CM subfield of $\mathbb{C}$ generated by the Hecke eigenvalues of f, with ring of integers $\mathcal{O}_f$, real subfield K_f^+, with its ring of integers $\mathcal{O}_f^+$. Let $\mathbb{T}$ be the ring generated over $\mathcal{O}_f^+$ by the endomorphisms of $S_k(\Gamma_0(D), \chi_D)$ given by all the Hecke operators T_q for all primes q. Let $\theta_f : \mathbb{T} \to \mathcal{O}_f$ be the homomorphism such that $T(f) = \theta_f(T)f\ \forall T \in \mathbb{T}$. Let S be the set of primes dividing $D(k!)$.

We consider the premotivic structure (with coefficients in $\mathbb{Q}$) $M(D, \chi_D)_!$ constructed in [DFG1, §1.4.2]. (See [DFG1, §§1.1.1,1.1.2] for generalities on premotivic and S-integral premotivic structures.) It has realisations $M(D, \chi_D)_{!,B}$, $M(D, \chi_D)_{!,\mathrm{dR}}$, $M(D, \chi_D)_{!,\ell}$ and $M(D, \chi_D)_{!,\ell\text{-crys}}$ (for each prime $\ell \notin S$). (Actually, even for $\ell \in S$ we have $M(D, \chi_D)_{!,\ell}$, but strictly speaking it is not part of the structure.) The first two are $\mathbb{Q}$-vector spaces, subspaces of the first singular and algebraic de Rham cohomologies of the modular curve $X_1(N)$ with coefficients in a local system depending on k. The second two are $\mathbb{Q}_\ell$-vector spaces, coming from ℓ-adic (étale) and crystalline cohomology. This gives various extra structures and comparison isomorphisms. For instance, $M(D, \chi_D)_{!,\ell}$ has a continuous linear action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and $M(D, \chi_D)_{!,\mathrm{dR}}$ has a filtration, with $\mathrm{Fil}^{k-1} M(D, \chi_D)_{!,\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C} \simeq S_k(\Gamma_0(D), \chi_D)$. In this sense, $M(D, \chi_D)_!$ is the premotivic structure associated to $S_k(\Gamma_0(D), \chi_D)$. By [DFG1, §1.5.3, Proposition 1.3], there is a Poincaré duality isomorphism

$$\hat{\delta}_! : M(D, \chi_D)_! \to \mathrm{Hom}_{\mathbb{Q}}(M(D, \chi_D)_!, M_{\chi_D}(1-k)),$$

where $M_{\chi_D}(1-k)$ is a Tate twist of a rank-1 premotivic structure attached to the Dirichlet character χ_D. This duality isomorphism is compatible with natural actions of $\mathbb{T}$, and the associated perfect pairing is alternating. Thus

$$[,] : \wedge^2 M(D, \chi_D)_! \simeq M_{\chi_D}(1-k).$$

We have also an S-integral premotivic structure $\mathcal{M}(D, \chi_D)_!$. Among its realisations, $\mathcal{M}(D, \chi_D)_{!,B}$ is a $\mathbb{Z}$-lattice in $M(D, \chi_D)_{!,B}$, $\mathcal{M}(D, \chi_D)_{!,\mathrm{dR}}$ is a $\mathbb{Z}_S$-lattice in $M(D, \chi_D)_{!,\mathrm{dR}}$, $\mathcal{M}(D, \chi_D)_{!,\ell}$ is a $\mathbb{Z}_\ell$-lattice in $M(D, \chi_D)_{!,\ell}$, preserved by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and $\mathcal{M}(D, \chi_D)_{!,\ell\text{-crys}}$ is a $\mathbb{Z}_\ell$-lattice in $M(D, \chi_D)_{!,\ell\text{-crys}}$.

We also have a premotivic structure M_f with coefficients in K_f [DFG1, §1.6.2]. It is a substructure of $M(D, \chi_D)_! \otimes_{\mathbb{Q}} K_f$, with $\mathrm{Fil}^{k-1} M_f = K_f f$. For any prime divisor λ of K_f, the λ-adic realisation $M_{f,\lambda}$ is a 2-dimensional $K_{f,\lambda}$-vector space with continuous $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action. This is the Galois representation attached to f. The S-integral premotivic structure $\mathcal{M}_f$ has $\mathrm{Fil}^{k-1}\mathcal{M}_{f,\mathrm{dR}} = \mathcal{O}_{f,S} f$. The $\mathcal{O}_{f,\lambda}$-lattice $\mathcal{M}_{f,\lambda}$ in $M_{f,\lambda}$ is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant, and $\overline{\mathcal{M}}_{f,\lambda} := \mathcal{M}_{f,\lambda}/\lambda\mathcal{M}_{f,\lambda}$ is the residual representation.

The isomorphism $\hat{\delta}_! : M(D, \chi_D)_! \to \mathrm{Hom}_{\mathbb{Q}}(M(D, \chi_D)_!, M_{\chi_D}(1-k))$ restricts (after extension of scalars) to an isomorphism

$$\hat{\delta}_f : M_f \to \mathrm{Hom}_{K_f}(M_f, M_{\chi_D}(1-k) \otimes K_f),$$

i.e. $[,]: \wedge^2_{K_f} M_f \simeq M_{\chi_D}(1-k)\otimes K_f$. However, although the duality pairing gives $\hat{\delta}_!: \mathcal{M}(D,\chi_D)_! \to \mathrm{Hom}_{\mathbb{Z}_S}(\mathcal{M}(D,\chi_D)_!, \mathcal{M}_{\chi_D}(1-k)\otimes \mathcal{O}_{f,S})$, it does not restrict to $[,]: \wedge^2_{\mathcal{O}_{f,S}}\mathcal{M}_f \simeq \mathcal{M}_{\chi_D}(1-k)\otimes\mathcal{O}_{f,S}$, rather

$$[,]: \wedge^2_{\mathcal{O}_{f,S}}\mathcal{M}_f \simeq \eta_f\mathcal{M}_{\chi_D}(1-k)\otimes\mathcal{O}_{f,S},$$

for some integral ideal η_f, as noted in [DFG2, §2]; see also [DFG1, §1.7.3].

Definition 2.1. *This η_f is the* congruence ideal *for f.*

Proposition 2.2. *Fix $\mathfrak{P}$ a prime divisor in K_f, with $\mathfrak{P} \nmid D(k!)[\mathcal{O}_f : \theta_f(\mathbb{T})]$. Suppose that given $g \in S_k(\Gamma_0(D),\chi_D)$ a normalised Hecke eigenform, we have a congruence $\theta_g(T) \equiv \theta_f(T) \pmod{\mathfrak{P}}$ $\forall T \in \mathbb{T}$, if and only if $g = f$ or $g = f_c$, the complex conjugate eigenform. Then*

$$\mathrm{ord}_{\mathfrak{P}}(\eta_f) = 1.$$

Note that $\theta_f(\mathbb{T})$ is the same thing as $\mathcal{O}(f)$. To prove the proposition, we need two lemmas. Recall the $\mathcal{O}_f^+$-algebra $\mathbb{T}$, and the homomorphism $\theta_f: \mathbb{T}\to\mathcal{O}_f$. Let $\overline{\theta_f}: \mathbb{T}\to\mathbb{F}_{\mathfrak{P}}$ be the composition of θ_f with the reduction map $\mathcal{O}_f \to \mathcal{O}_f/\mathfrak{P} =: \mathbb{F}_{\mathfrak{P}}$, and let $\mathfrak{m} := \ker\overline{\theta_f}$, $\mathbb{T}_{\mathfrak{m}}$ the local completion at $\mathfrak{m}$. We may define a premotivic structure $M_{[f]}$, with coefficients in K_f^+, associated with the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-orbit $[f] := \{f, f_c\}$, as the kernel of the appropriate ideal of $\mathbb{T}$ acting on $M(D,\chi_D)_!\otimes_{\mathbb{Q}} K_f^+$, so that $\mathrm{Fil}^{k-1}M_{[f]}\otimes_{K_f^+}\mathbb{C} \simeq \mathbb{C}f\oplus\mathbb{C}f_c$, and similarly an S-integral premotivic structure $\mathcal{M}_{[f]}$ (with coefficients in $O^+_{f,S}$). Let $\mathfrak{P}$ be as in the proposition, with $\mathfrak{P}^+$ the divisor below it in K_f^+.

Lemma 2.3. *$\mathcal{M}_{[f],\mathfrak{P}^+} \simeq \mathbb{T}^2_{\mathfrak{m}}$ as a $\mathbb{T}_{\mathfrak{m}}$-module.*

Proof. First note that, because of the congruence $\theta_{f_c}(T) \equiv \theta_f(T) \pmod{\mathfrak{P}}$ $\forall T\in\mathbb{T}$, $\mathcal{M}_{[f],\mathfrak{P}^+}$ is a $\mathbb{T}_{\mathfrak{m}}$-module. One may prove, just as in the proof of [FJ, Theorem 2.1], the "multiplicity one" formula

$$\overline{\mathcal{M}}_{[f],\mathfrak{P}^+}[\mathfrak{m}] \simeq (\mathbb{T}/\mathfrak{m})^2.$$

The lemma then follows by a standard application of Nakayama's Lemma. □

Lemma 2.4. *Suppose that $\mathfrak{P}$ is ramified in K_f/K_f^+. Consider the map ψ, K_f^+-linear in the first factor, K_f-linear in the second factor, given by*

$$\psi: K_f\otimes_{K_f^+}K_f \simeq K_f^2: \qquad \alpha\otimes\beta\mapsto(\alpha\beta,\overline{\alpha}\beta).$$

Then

$$\psi(\mathcal{O}_{f,\mathfrak{P}}\otimes_{\mathcal{O}^+_{f,\mathfrak{P}^+}}\mathcal{O}_{f,\mathfrak{P}}) = \{(z,w)\in\mathcal{O}^2_{f,\mathfrak{P}}:\ z\equiv w \pmod{\mathfrak{P}}\}.$$

Proof. Choosing an $\mathcal{O}^+_{f,\mathfrak{P}^+}$-basis $\{1,\pi\}$ for $\mathcal{O}_{f,\mathfrak{P}}$, where π is a uniformiser for $\mathfrak{P}$ and π^2 a uniformiser for $\mathfrak{P}^+$, every element of $\mathcal{O}_{f,\mathfrak{P}}\otimes_{\mathcal{O}^+_{f,\mathfrak{P}^+}}\mathcal{O}_{f,\mathfrak{P}}$ is of the form $1\otimes(a+b\pi)+\pi\otimes(c+d\pi)$, with $a,b,c,d\in\mathcal{O}^+_{f,\mathfrak{P}^+}$. Now

$$\psi(1\otimes(a+b\pi)+\pi\otimes(c+d\pi)) = (a+d\pi^2+(b+c)\pi, a-d\pi^2+(b-c)\pi) = (x+y\pi, u+v\pi),$$

where

$$a=\frac{x+u}{2},\ b=\frac{y+v}{2},\ c=\frac{y-v}{2},\ d=\frac{x-u}{2\pi^2}.$$

The condition $x\equiv u \pmod{\pi^2}$ is equivalent to $z\equiv w \pmod{\mathfrak{P}}$. □

Proof of Proposition 2.2. By Lemma 1.1, $\mathfrak{P}$ is ramified in K_f/K_f^+. By Lemma 2.3, $\mathcal{M}_{[f],\mathfrak{P}^+} \simeq \mathbb{T}_{\mathfrak{m}}^2$ as a $\mathbb{T}_{\mathfrak{m}}$-module. Since $p \nmid [\mathcal{O}_f : \theta_f(\mathbb{T})]$, θ_f induces an isomorphism between $\mathbb{T}_{\mathfrak{m}}$ and $\mathcal{O}_{f,\mathfrak{P}}$, hence $\mathcal{M}_{[f],\mathfrak{P}^+} \simeq (\mathcal{O}_{f,\mathfrak{P}})^2$. Inside $\mathcal{M}_{[f]} \otimes_{\mathcal{O}_f^+} \mathcal{O}_f$, the substructure $\mathcal{M}_f$ is defined by the condition that the action of any $T \in \mathbb{T}$ via the first factor matches the action of $\theta_f(T)$ via the second factor. For $\mathcal{M}_{f_c}$ we just replace θ_f by θ_{f_c}. Identifying $\mathcal{M}_{[f],\mathfrak{P}^+} \otimes_{\mathcal{O}^+_{f,\mathfrak{P}^+}} \mathcal{O}_{f,\mathfrak{P}}$ with $\mathcal{O}^2_{f,\mathfrak{P}} \otimes_{\mathcal{O}^+_{f,\mathfrak{P}^+}} \mathcal{O}_{f,\mathfrak{P}}$, and applying Lemma 2.4, we find that

$$\mathcal{M}_{f,\mathfrak{P}} \simeq \{(z_1, 0, z_2, 0) \in \mathcal{O}^4_{f,\mathfrak{P}} : \ z_1, z_2 \in \mathfrak{P}\}$$

and

$$\mathcal{M}_{f_c,\mathfrak{P}} \simeq \{(0, w_1, 0, w_2) \in \mathcal{O}^4_{f,\mathfrak{P}} : \ w_1, w_2 \in \mathfrak{P}\}.$$

Hence

$$[\mathcal{M}_{[f],\mathfrak{P}^+} \otimes_{\mathcal{O}^+_{f,\mathfrak{P}^+}} \mathcal{O}_{f,\mathfrak{P}} : \mathcal{M}_{f,\mathfrak{P}} \oplus \mathcal{M}_{f_c,\mathfrak{P}}] = \mathfrak{P}^2.$$

This $\mathcal{M}_{f,\mathfrak{P}} \oplus \mathcal{M}_{f_c,\mathfrak{P}}$ is an orthogonal direct sum for the pairing $[,]$. Recall that

$$[,] : \wedge^2_{\mathcal{O}_{f,\mathfrak{P}}} \mathcal{M}_{f,\mathfrak{P}} \simeq \eta_f \mathcal{M}_{\chi_D}(1-k) \otimes \mathcal{O}_{f,\mathfrak{P}},$$

and similarly $[,] : \wedge^2_{\mathcal{O}_{f,\mathfrak{P}}} \mathcal{M}_{f_c,\mathfrak{P}} \simeq \eta_{f_c} \mathcal{M}_{\chi_D}(1-k) \otimes \mathcal{O}_{f,\mathfrak{P}}$. But the condition that $\theta_g(T) \equiv \theta_f(T) \pmod{\mathfrak{P}}\ \forall T \in \mathbb{T}$, only for $g = f$ or $g = f_c$, implies that

$$[,] : \wedge^2_{\mathcal{O}^+_{f,\mathfrak{P}^+}} \mathcal{M}_{[f],\mathfrak{P}^+} \simeq \mathcal{M}_{\chi_D}(1-k) \otimes \mathcal{O}^+_{f,\mathfrak{P}^+}.$$

It follows (using also symmetry between f and f_c) that $\mathrm{ord}_{\mathfrak{P}}(\eta_f) = \mathrm{ord}_{\mathfrak{P}}(\eta_{f_c}) = 1$. □

3. The Bloch-Kato conjecture

As before, let $f \in S_k(\Gamma_0(D), \chi_D)$ be a normalised Hecke eigenform, K_f the CM subfield of $\mathbb{C}$ generated by the Hecke eigenvalues of f. We saw the premotivic structure M_f, with coefficients in K_f, and the S-integral premotivic structure $\mathcal{M}_f$, where S is the set of primes dividing $D(k!)$. Following [DFG1, §1.7.1], we consider the adjoint premotivic structure $A_f = \mathrm{ad}^0(M_f)$, the kernel of the trace morphism $\mathrm{Hom}_{K_f}(M_f, M_f) \to K_f$, and the associated S-integral premotivic structure $\mathcal{A}_f$. We will need also $A_{f,\chi_D} := A_f \otimes M_{\chi_D}$ and $\mathcal{A}_{f,\chi_D} := \mathcal{A}_f \otimes \mathcal{M}_{\chi_D}$. We can recover the Hecke L-function $L(s,f) = \sum_{m=1}^{\infty} a_f(m) m^{-s}$ in the following way. For each finite prime q, choose any $\ell \neq q$, and $\lambda \mid \ell$ in K_f. Let $F_p(X) = \det(I - \rho|_{V^{I_q}}(\mathrm{Frob}_q^{-1})X)$, where $V = M_{f,\lambda}$. Then $L(s,f) = \prod_q L_q(s,f)$, where $L_q(s,f)^{-1} = F_q(q^{-s})$. We may also define an adjoint L-function $L(s, \mathrm{ad}^0(f))$, and a twisted adjoint L-function $L(s, \mathrm{ad}^0(f) \otimes \chi_D)$, by using $V = A_{f,\lambda}$ and $V = A_{f,\chi_D,\lambda}$, respectively. The Euler factors at "bad" primes $q \mid D$ are as follows:

$$L_q(s,f) = (1 - a_f(q) q^{-s})^{-1}, \text{ with } a_f(q)\overline{a_f(q)} = q^{k-1};$$
$$L_q(s, \mathrm{ad}^0(f)) = (1 - q^{-s})^{-1};$$
$$L_q(s, \mathrm{ad}^0(f) \otimes \chi_D) = ((1 - a_f(q)^2 q^{1-k-s})(1 - \overline{a_f(q)}^2 q^{1-k-s}))^{-1}.$$

Our $L(s, \mathrm{ad}^0(f) \otimes \chi_D)$ is the same as Zagier's $D_f(s+k-1)$ in [Z, §6], but note that in Ghate's $L(s, \mathrm{Ad}(f))$ and $L(s, \mathrm{Ad}(f) \otimes \chi_D)$ [G, §5], Euler factors at primes $q \mid D$ are omitted. Since the dual of M_f is $M_f \otimes M_{\chi_D}(1-k)$, $L(s, \mathrm{ad}^0(f) \otimes \chi_D)$ can also be described as $L(s+k-1, \mathrm{Sym}^2(f))$, i.e. $D_f(s) = L(s, \mathrm{Sym}^2(f))$.

Lemma 3.1. *Let $f \in S_k(\Gamma_0(D), \chi_D)$ be a normalised Hecke eigenform, $\mathfrak{P}$ a prime divisor in K_f such that $f \equiv f_c \pmod{\mathfrak{P}}$ and $\overline{\rho}_{f,\mathfrak{P}}$ is absolutely irreducible. Then $H^0(\mathbb{Q}, \mathcal{A}_{f,\chi_D,\mathfrak{P}}/A_{f,\chi_D,\mathfrak{P}})$ is non-trivial.*

Proof. By (1) of Proposition 1.2, $\overline{\rho}_{f,\mathfrak{P}} \simeq \overline{\rho}_{f,\mathfrak{P}} \otimes \chi_D$. (For just this part, we do not need the additional conditions of the proposition.) At any $q \mid D$, the space of $\overline{\rho}_{f,\mathfrak{P}}$ has an unramified line and a line on which I_q acts via χ_D, by a Theorem of Langlands and Carayol [H2, Theorem 4.2.7(3)(a)]. Tensoring with χ_D swaps those lines, so an isomorphism from $\overline{\rho}_{f,\mathfrak{P}}$ to $\overline{\rho}_{f,\mathfrak{P}} \otimes \chi_D$ must have trace zero, and gives us a non-zero element of $\mathfrak{P}$-torsion in $H^0(\mathbb{Q}, \mathcal{A}_{f,\chi_D}/A_{f,\chi_D})$. □

This is the key Galois-theoretical consequence of the congruence $f \equiv f_c \pmod{\mathfrak{P}}$, since the order of $H^0(\mathbb{Q}, \mathcal{A}_{f,\chi_D}/A_{f,\chi_D})$ appears in the denominator of the conjectural formula for $L(1, \mathrm{ad}^0(f) \otimes \chi_D)$ given by the Bloch-Kato conjecture. We must prepare ourselves to look at other terms in the formula.

Given a field F and a continuous $\mathrm{Gal}(\overline{F}/F)$-module M, $H^1(F, M)$ will mean for us $H^1_{\mathrm{cont.}}(F, M)$ (the quotient of continuous cocycles by continuous coboundaries). Given a finite-dimensional continuous representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over $\mathbb{Q}_p$, unramified outside a finite set of primes, following Bloch and Kato [BK] we define

$$H^1_f(\mathbb{Q}_q, V) := \begin{cases} H^1_{\mathrm{ur}}(\mathbb{Q}_q, V) & q \neq p \\ \ker(H^1(\mathbb{Q}_q, V) \to H^1(\mathbb{Q}_q, V \otimes B_{\mathrm{crys}})) & q = p \end{cases},$$

where I_q is the inertia subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$, B_{crys} is Fontaine's ring, as defined in [BK, §1], and

$$H^1_{\mathrm{ur}}(\mathbb{Q}_q, M) := \ker(H^1(\mathbb{Q}_q, M) \to H^1(I_q, M)).$$

Now let $T \subset V$ be a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable $\mathbb{Z}_p$-lattice, and $W := V/T$. Further define

$$H^1_f(\mathbb{Q}_q, W) := \mathrm{im}(H^1_f(\mathbb{Q}_q, V) \to H^1(\mathbb{Q}_q, W),$$

and for any finite set of primes Σ not containing p let $H^1_\Sigma(\mathbb{Q}, W)$ be the subgroup of elements of $H^1(\mathbb{Q}, W)$ whose images in $H^1(\mathbb{Q}_q, W)$ lie in $H^1_f(\mathbb{Q}_q, W)$, for all (finite) primes $q \notin \Sigma$. As noted in [DFG1, §2.1] if V is unramified at q (with $q \neq p$) then $H^1_f(\mathbb{Q}_q, W) = H^1_{\mathrm{ur}}(\mathbb{Q}_q, W)$.

Lemma 3.2. *Let $f \in S_k(\Gamma_0(D), \chi_D)$ be a normalised Hecke eigenform, $\mathfrak{P} \mid p$ a prime divisor in K_f with $p \nmid D(2k-1)(2k-3)(k!)[\mathcal{O}_f : \theta_f(\mathbb{T})]$. Suppose that given $g \in S_k(\Gamma_0(D), \chi_D)$ a normalised Hecke eigenform, we have a congruence $\theta_g(T) \equiv \theta_f(T) \pmod{\mathfrak{P}}$ $\forall T \in \mathbb{T}$, if and only if $g = f$ or $g = f_c$, the complex conjugate eigenform, and that $\rho_{f,\mathfrak{P}} \not\equiv \rho_{f_c,\mathfrak{P}} \pmod{\mathfrak{P}^2}$. Suppose that $\overline{\rho}_{f,\mathfrak{P}}$ is absolutely irreducible and that $\mathfrak{P} \nmid a_f(p)$. Let Σ be the set of primes dividing D. Suppose that for all primes $q \mid D$, $q \not\equiv 1 \pmod{p}$. Then $H^1_\Sigma(\mathbb{Q}, \mathcal{A}_{f,\chi_D,\mathfrak{P}}/A_{f,\chi_D,\mathfrak{P}})$ is trivial.*

Proof. If it were non-trivial, there would be a non-zero element of order $\mathfrak{P}$, which is necessarily in the image of $H^1(\mathbb{Q}, \mathrm{ad}^0(\overline{\rho}_{f,\mathfrak{P}}) \otimes \chi_D)$, say coming from an element α. By Lemma 3.1 we have a non-zero element of order $\mathfrak{P}$ in $H^0(\mathbb{Q}, \mathcal{A}_{f,\chi_D,\mathfrak{P}}/A_{f,\chi_D,\mathfrak{P}})$. By the condition $\rho_{f,\mathfrak{P}} \not\equiv \rho_{f_c,\mathfrak{P}} \pmod{\mathfrak{P}^2}$, there is no element of order $\mathfrak{P}^2$ in $H^0(\mathbb{Q}, \mathcal{A}_{f,\chi_D,\mathfrak{P}}/A_{f,\chi_D,\mathfrak{P}})$. Hence our element of order $\mathfrak{P}$ in $H^0(\mathbb{Q}, \mathcal{A}_{f,\chi_D,\mathfrak{P}}/A_{f,\chi_D,\mathfrak{P}})$ maps to a non-zero element β of $H^1(\mathbb{Q}, \mathrm{ad}^0(\overline{\rho}_{f,\mathfrak{P}}) \otimes \chi_D)$. Since β maps to 0 in $H^1(\mathbb{Q}, \mathcal{A}_{f,\chi_D,\mathfrak{P}}/A_{f,\chi_D,\mathfrak{P}})$ (by exactness), while α maps to a non-zero element, α and β must be linearly independent elements of $H^1(\mathbb{Q}, \mathrm{ad}^0(\overline{\rho}_{f,\mathfrak{P}}) \otimes \chi_D)$. Since

$H^0(\mathbb{Q}, \mathbb{F}_{\mathfrak{P}} \otimes \chi_D)$ is trivial, $H^1(\mathbb{Q}, \mathrm{ad}^0(\overline{\rho}_{f,\mathfrak{P}}) \otimes \chi_D)$ injects into $H^1(\mathbb{Q}, \mathrm{ad}(\overline{\rho}_{f,\mathfrak{P}}) \otimes \chi_D)$. Composing with the isomorphism $\overline{\rho}_{f,\mathfrak{P}} \otimes \chi_D \simeq \overline{\rho}_{f,\mathfrak{P}}$, we obtain independent non-zero elements α', β' of $H^1(\mathbb{Q}, \mathrm{ad}(\overline{\rho}_{f,\mathfrak{P}}))$.

Actually, viewing $\rho_{f_c,\mathfrak{P}}$ as representing a deformation of $\overline{\rho}_{f,\mathfrak{P}}$, we have obtained β' by the standard construction in disguise: if (using bases compatible with $\overline{\rho}_{f_c,\mathfrak{P}} \simeq \overline{\rho}_{f,\mathfrak{P}}$), $\rho_{f_c,\mathfrak{P}}(g) \equiv \rho_{f,\mathfrak{P}}(g)(I + \pi c(g)) \pmod{\mathfrak{P}^2}$, where π is a uniformiser at $\mathfrak{P}$, then the cocycle $g \mapsto c(g)$ represents β'. Since $\rho_{f,\mathfrak{P}}$ and $\rho_{f,\mathfrak{P}} \otimes \chi_D$ have the same determinant, β' actually lives in (the image of) $H^1(\mathbb{Q}, \mathrm{ad}^0(\overline{\rho}_{f,\mathfrak{P}}))$.

Since α' comes from $H^1_{\Sigma}(\mathbb{Q}, \mathcal{A}_{f,\chi_D,\mathfrak{P}}/A_{f,\chi_D,\mathfrak{P}})$, its image in $H^1(\mathbb{Q}, \mathbb{F}_{\mathfrak{P}})$ by the trace map, composed with any linear map $\mathbb{F}_{\mathfrak{P}} \to \mathbb{F}_p$, produces either 0 or an element of $\mathrm{Hom}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathbb{F}_p)$ whose kernel has fixed field a degree p extension of $\mathbb{Q}$, unramified for any $q \nmid D$. (That it is unramified at p is addressed by [BK, Example 3.9].) Such an extension does not exist, given our assumptions that $p \nmid D$ and $q \not\equiv 1 \pmod p$ for all $q \mid D$. Hence the image of α' in $H^1(\mathbb{Q}, \mathbb{F}_{\mathfrak{P}})$ is 0, so α' also lives in $H^1(\mathbb{Q}, \mathrm{ad}^0(\overline{\rho}_{f,\mathfrak{P}}))$.

By Proposition 1.2, $\rho_{f,\mathfrak{P}}$ is dihedral, from which it easily follows that $H^0(\mathbb{Q}, \mathcal{A}_{f,\mathfrak{P}}/A_{f,\mathfrak{P}})$ is trivial. Hence α', β' map to independent non-zero elements α'', β'' of $H^1(\mathbb{Q}, \mathcal{A}_{f,\mathfrak{P}}/A_{f,\mathfrak{P}})$. Using [DFG1, Proposition 2.2] we see that α'' (having come from $H^1_{\Sigma}(\mathbb{Q}, \mathcal{A}_{f,\chi_D,\mathfrak{P}}/A_{f,\chi_D,\mathfrak{P}})$) satisfies the local conditions to lie in $H^1_{\Sigma}(\mathbb{Q}, \mathcal{A}_{f,\mathfrak{P}}/A_{f,\mathfrak{P}})$. So does β'', since $\rho_{f_c,\mathfrak{P}}$ is unramified at $q \nmid pD$ and crystalline at p.

We have now that $\mathfrak{P}^2$ divides the Fitting ideal $\mathrm{Fitt}_{\mathcal{O}_{f,\mathfrak{P}}}(H^1_{\Sigma}(\mathbb{Q}, \mathcal{A}_{f,\mathfrak{P}}/A_{f,\mathfrak{P}}))$. Since $p \nmid D(2k-1)(2k-3)(k!)$, the restriction of $\overline{\rho}_{f,\mathfrak{P}}$ to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p}))$ is absolutely irreducible, by [DFG1, Lemma 2.5]. Then by [DFG1, Theorem 3.7, Proposition 1.4(c)],

$$\mathrm{Fitt}_{\mathcal{O}_{f,\mathfrak{P}}}(H^1_{\Sigma}(\mathbb{Q}, \mathcal{A}_{f,\mathfrak{P}}/A_{f,\mathfrak{P}})) = \eta_f \prod_{q \mid D} L_q(1, \mathrm{ad}^0(f))^{-1} = \eta_f \prod_{q \mid D} (1 - q^{-1}).$$

By our assumption that $q \not\equiv 1 \pmod p$ for all $q \mid D$, we have $\mathrm{Fitt}_{\mathcal{O}_{f,\mathfrak{P}}}(H^1_{\Sigma}(\mathbb{Q}, \mathcal{A}_{f,\mathfrak{P}}/A_{f,\mathfrak{P}})) = \eta_f$, but by Proposition 2.2, $\mathrm{ord}_{\mathfrak{P}}(\eta_f) = 1$, contradicting $\mathfrak{P}^2 \mid \mathrm{Fitt}_{\mathcal{O}_{f,\mathfrak{P}}}(H^1_{\Sigma}(\mathbb{Q}, \mathcal{A}_{f,\mathfrak{P}}/A_{f,\mathfrak{P}}))$. □

Since $\Sigma \neq \emptyset$, the $\mathfrak{P}$-part of the Bloch-Kato conjecture, applied to the critical value $L(1, \mathrm{ad}^0(f) \otimes \chi_D)$, may be formulated as follows, following [DFG1, (59)], and using the exact sequence in their Lemma 2.1.

$$\mathrm{ord}_{\mathfrak{P}}\left(\frac{L(1, \mathrm{ad}^0(f) \otimes \chi_D)}{\Omega}\right) = \mathrm{ord}_{\mathfrak{P}}\left(\frac{\mathrm{Fitt}_{\mathcal{O}_{f,\mathcal{P}}} H^1_{\Sigma}(\mathbb{Q}, \mathcal{A}_{f,\chi_D,\mathfrak{P}}/A_{f,\chi_D,\mathfrak{P}})}{\mathrm{Fitt}_{\mathcal{O}_{f,\mathcal{P}}} H^0(\mathbb{Q}, \mathcal{A}_{f,\chi_D,\mathfrak{P}}/A_{f,\chi_D,\mathfrak{P}})}\right),$$

where Ω is a certain Deligne period normalised by the integral structure $\mathcal{A}_f$. (We are retaining the condition $p \nmid D(2k-1)(2k-3)(k!)$, hence as in [DFG1, Proposition 2.16] the Tamagawa factor is trivial, so does not appear.) By Lemmas 3.1 and 3.2 we then predict that (subject to the conditions of Lemma 3.2)

$$\mathrm{ord}_{\mathfrak{P}}\left(\frac{L(1, \mathrm{ad}^0(f) \otimes \chi_D)}{\Omega}\right) < 0.$$

As in [Du, §5], up to $\mathfrak{P}$-units (where our Ω is the $(2\pi i)^{2k}\Omega$ there),

$$\Omega = \pi^{k+1}(f, f)\eta_f^{-1}.$$

(For the type of argument leading to the relation between the Petersson norm (f, f), periods $\Omega^{\pm}$ of M_f, and η_f, as in [Du, (4)], a good additional reference is [H3, (5.18)].

The $\langle\zeta_+,\zeta_-\rangle$ in [H3, Theorem 5.16] is our η_f.) So the Bloch-Kato conjecture leads to the prediction that (subject to the conditions of Lemma 3.2)

$$\mathrm{ord}_{\mathfrak{P}}\left(\frac{L(1,\mathrm{ad}^0(f)\otimes\chi_D)}{\pi^{k+1}(f,f)}\right) < -\mathrm{ord}_{\mathfrak{P}}(\eta_f).$$

Using Proposition 2.2 we may reformulate this again as

$$\mathrm{ord}_{\mathfrak{P}}\left(\frac{L(1,\mathrm{ad}^0(f)\otimes\chi_D)}{\pi^{k+1}(f,f)}\right) < -\mathrm{ord}_{\mathfrak{P}}(\mathfrak{d}(K_f/K_f^+)),$$

where of course $\mathrm{ord}_{\mathfrak{P}}(\mathfrak{d}(K_f/K_f^+)) = 1$ because $\mathfrak{P}$ is ramified in this quadratic extension. (Here, $\mathfrak{d}(K_f/K_f^+)$ denotes the relative different.) In the following section we shall prove something slightly weaker, that if $p \mid \mathrm{Norm}_{F/\mathbb{Q}}((\epsilon_+)^{k-1}-1)$ then $\mathrm{ord}_{\mathfrak{P}}\left(\frac{L(1,\mathrm{ad}^0(g)\otimes\chi_D)}{\pi^{k+1}(g,g)}\right) < -\mathrm{ord}_{\mathfrak{P}}(\mathfrak{d}(K_g/K_g^+))$ for *some* normalised Hecke eigenform $g\in S_k(\Gamma_0(D),\chi_D)$. Of course we expect it to be f satisfying $f\equiv f_c \pmod{\mathfrak{P}}$, with $\mathfrak{P}$ is ramified in K_f/K_f^+, but we cannot eliminate the possibility that it is only some other g, with $\mathrm{ord}_{\mathfrak{P}}(\mathfrak{d}(K_g/K_g^+)) = 0$. Note that if $\deg(\mathfrak{P}^+) > 1$ then applying a non-trivial element of its decomposition group to the pair f, f_c will produce another pair g, g_c congruent to each other mod $\mathfrak{P}$, for whom we should also see $\mathfrak{P}$ in the denominator.

One might question the condition that $\theta_g(T)\equiv\theta_f(T) \pmod{\mathfrak{P}}\ \forall T\in\mathbb{T}$, if and only if $g=f$ or $g=f_c$. How strong is this? In twelve out of the thirteen numerical examples in [DHI, Table 1], the normalised Hecke eigenforms in $S_k(\Gamma_0(D),\chi_D)$ form a single Galois orbit. Assuming also that $p\nmid[\mathcal{O}_f:\theta_f(\mathbb{T})]$, if the condition failed then an automorphism taking f to g (in addition to one taking f to f_c) would be in the inertia group for $\mathfrak{P}$, so p would be ramified in $K_f^+/\mathbb{Q}$, which seems unlikely. Such a p would be listed in both the second and third columns of the table, for a given row, but in none of those twelve examples does this happen.

4. The denominator of the twisted adjoint L-value

Proposition 4.1. *Let $F=\mathbb{Q}(\sqrt{D})$ be a real quadratic field, with discriminant $D>0$, $D\equiv 1 \pmod 4$. Fixing an even $k>2$, let ϵ_+ be a generator for the group of totally positive units of $\mathcal{O}_F$, and let $\mathfrak{p}$ be any prime divisor of $(\epsilon_+)^{k-1}-1$ in $\mathcal{O}_F$, with $p\nmid D(k!)$, where $\mathfrak{p}$ divides a rational prime p. There exists a normalised Hecke eigenform $f\in S_k(\Gamma_0(D),\chi_D)$, such that*

$$\mathrm{ord}_{\mathfrak{P}}\left(\frac{L(1,\mathrm{ad}^0(f)\otimes\chi_D)}{\pi^{k+1}(f,f)}\right) < -\mathrm{ord}_{\mathfrak{P}}(\mathfrak{d}(K_f/K_f^+)),$$

where K_f is the subfield of $\mathbb{C}$ generated by the Hecke eigenvalues of f (real subfield K_f^+), $\mathfrak{P}$ is any divisor of p in K_f (except we require $\mathfrak{P}\mid\mathfrak{p}$ if $F\subset K_f$), and $\mathfrak{d}(K_f/K_f^+)$ is the relative different.

Proof. By work of Zagier [Z, (91),(92)], $L(1, \mathrm{ad}^0(f) \otimes \chi_D) = -\frac{\pi}{4}\frac{(4\pi)^k}{\Gamma(k)}(C_{k,1,D}, f)$, where $C_{k,1,D}(z) :=$

$$\sum_{m=0}^{\infty}\left(\sum_{\substack{t\in\mathbb{Z}\\ t^2\leq 4m\\ t^2\equiv 4m \pmod D}} p_{k,1}(t,m) H\left(\frac{4m-t^2}{D}\right) + \frac{1}{\sqrt{D}}\sum_{\substack{\lambda\in\mathcal{O}_F\\ \lambda>0\\ \lambda\lambda'=m}} \min(\lambda,\lambda')^{k-1}\right) e^{2\pi i m z}.$$

Here $p_{k,1}(t,m)$, the coefficient of x^{k-2} in $(1-tx+mx^2)^{-1}$, is an integer, and $H(n)$, the Hurwitz class number, is integral away from 2 and 3. Also, we are thinking of F as embedded in $\mathbb{R}$ in a fixed way, but λ' means the Galois conjugate of λ, i.e. the result of applying the other embedding. Now if $\epsilon \in F$ is a totally positive unit then $\epsilon' = 1/\epsilon$, so given a factorisation $m = \lambda\lambda'$ appearing in the sum, $m = (\epsilon\lambda)(\epsilon'\lambda')$ is another one. Let ϵ_+ be a generator for the group of totally positive units, chosen with $\epsilon_+ < 1$ and $(\epsilon_+)' > 1$. Replacing λ by an associate if necessary, the associates of λ and λ' contribute factorisations with minimum factors $\lambda, \lambda\epsilon_+, \lambda(\epsilon_+)^2, \lambda(\epsilon_+)^3, \ldots$ and $\lambda'\epsilon_+, \lambda'(\epsilon_+)^2, \lambda'(\epsilon_+)^3, \ldots$. These contribute $(\lambda^{k-1} + (\lambda'\epsilon_+)^{k-1})(1 + (\epsilon_+)^{k-1} + (\epsilon_+)^{2(k-1)} + \ldots) = \frac{(\lambda^{k-1}+(\lambda'\epsilon_+)^{k-1})}{(1-(\epsilon_+)^{k-1})}$ to the sum.

Let $\mathfrak{p}$ be any prime divisor of $(\epsilon_+)^{k-1} - 1$ in $\mathcal{O}_F$. Then $\epsilon_+ \equiv 1 \pmod{\mathfrak{p}}$, so if we choose m to be a prime number inert in F, any factorisation $m = \lambda\lambda'$, say with λ associate to 1 and λ' associate to m, has $\lambda^{k-1} + (\lambda'\epsilon_+)^{k-1} \equiv 1 + m^{k-1} \pmod{\mathfrak{p}}$, and it is easy to see that there are infinitely many ways to choose such inert primes m such that $m^{k-1} \not\equiv -1 \pmod{\mathfrak{p}}$, in which case

$$\mathrm{ord}_{\mathfrak{p}}\left(\sum_{\substack{\lambda\in\mathcal{O}_F\\ \lambda>0\\ \lambda\lambda'=m}} \min(\lambda,\lambda')^{k-1}\right) = -\mathrm{ord}_{\mathfrak{p}}((\epsilon_+)^{k-1} - 1).$$

Let $\mathcal{S}$ be the infinite set of inert primes m such that $m^{k-1} \not\equiv -1 \pmod{\mathfrak{p}}$, simultaneously for all prime divisors $\mathfrak{p}$ of $((\epsilon_+)^{k-1} - 1)$. Letting c_m denote the coefficient of $q^m = e^{2\pi i m z}$ in $C_{k,1,D}$, we see that if $\mathfrak{p}$ is a prime divisor in $\mathcal{O}_F$ of $(\epsilon_+)^{k-1} - 1$, with $\mathfrak{p} \nmid 6D$, then $\mathrm{ord}_{\mathfrak{p}} c_m = -\mathrm{ord}_{\mathfrak{p}}((\epsilon_+)^{k-1} - 1)$, for all $m \in \mathcal{S}$.

Since $k > 2$, $C_{k,1,D}$ is a cusp form, and may be expressed as a linear combination of normalised Hecke eigenforms in $S_k(\Gamma_0(D), \chi_D)$. These eigenforms may be divided into $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-orbits, and the contributions to the linear combination coming from any particular orbit (conjugate pair) may be combined. Let $B_{[g]}$ be the contribution from the orbit of g, so that $C_{k,1,D} = \sum_{[g]} B_{[g]}$. Since $\mathrm{ord}_{\mathfrak{p}} c_m = -\mathrm{ord}_{\mathfrak{p}}((\epsilon_+)^{k-1} - 1) < 0$, for all $m \in \mathcal{S}$, there must exist a normalised eigenform f such that if $B_{[f]} = \sum_{m=1}^{\infty} b_m q^m$ then $\mathrm{ord}_{\mathfrak{P}} b_m < 0$, for infinitely many m (some subset of $\mathcal{S}$). We have $B_{[f]} = \mathrm{tr}_{K_f/K_f^+}(\alpha f)$, for some $\alpha \in K_f$, so if $f = \sum_{m=1}^{\infty} a_m q^m$ then $b_m = \mathrm{tr}_{K_f/K_f^+}(\alpha a_m)$ for all $m \geq 1$. If $\mathrm{ord}_{\mathfrak{P}}(\alpha) \geq -\mathrm{ord}_{\mathfrak{P}}(\mathfrak{d}(K_f/K_f^+))$ then $\mathrm{ord}_{\mathfrak{P}}(\mathrm{tr}_{K_f/K_f^+}(\alpha a_m)) \geq 0$ for all m (since $a_m \in \mathcal{O}_f$), so $\mathrm{ord}_{\mathfrak{P}}(b_m) \geq 0$ for all m, contradicting what we just saw. Hence $\mathrm{ord}_{\mathfrak{P}}(\alpha) < -\mathrm{ord}_{\mathfrak{P}}(\mathfrak{d}(K_f/K_f^+))$, and since $\alpha = \frac{(C_{k,1,D}, f)}{(f,f)}$, we obtain the proposition. □

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