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TRIPLE PRODUCT L-VALUES AND DIHEDRAL CONGRUENCES FOR CUSP FORMS

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ABSTRACT. Let $p \equiv 3 \pmod 4$ be a prime, and $k = (p+1)/2$. In this paper we prove that a certain trace of normalised, rightmost critical values of triple product L-functions, of cuspidal Hecke eigenforms of level one and weight k, is non-integral at p if and only if the class number $h(\sqrt{-p}) > 1$. We use the Bloch-Kato conjecture to explain this, using "dihedral" congruences, modulo a divisor of p, for cuspidal Hecke eigenforms of level one and weight k (e.g. $p = 23$, $k = 12$, $g = \Delta$). Exploiting the Galois interpretation of such congruences, we may produce global torsion elements which contribute to the denominator of the conjectural formula for some L-value contributing to the trace.

1. Introduction

Let $S_k(\Gamma)$ be the space of cuspidal, elliptic modular forms of integer weight k with respect to $\Gamma = \mathsf{SL}_2(\mathbb{Z})$. Let $f \in S_k(\Gamma)$ be a primitive Hecke eigenform with Satake parameters $\alpha_p(f), \beta_p(f)$, normalized by $\alpha_p(f)\beta_p(f) = p^{k-1}$. We put

$$A_p(f) := \begin{pmatrix} \alpha_p(f) & 0 \\ 0 & \beta_p(f) \end{pmatrix}.$$

The Rankin triple L-function $L(f \otimes g \otimes h, s)$ for primitive Hecke eigenforms $f, g, h \in S_k(\Gamma)$ is given by the infinite product

$$\prod_{p \text{ prime}} \left\{ \mathsf{det}\left(1_8 - A_p(f) \otimes A_p(g) \otimes A_p(h)\, p^{-s}\right)\right\}^{-1} \quad \text{for } \mathrm{Re}(s) \gg 0. \tag{1.1}$$

The space of cusp forms $S_k = S_k(\Gamma)$ of weight k on the upper half-space

$$\mathbb{H} := \{z = x + iy \in \mathbb{C} \mid y > 0\}$$

is a normed space with

$$\| f \| := \sqrt{\int_{\Gamma\backslash\mathbb{H}} f(z)\overline{f(z)}\mathsf{Im}(z)^{k-2}\, dz}. \tag{1.2}$$

[1]This work was partly supported by the *Max-Planck-Institut für Mathematik, Bonn.*
Date: October 5th, 2009.
2000 *Mathematics Subject Classification.* 11F67, 11B68, 11E41, 11F33, 11F80.
Key words and phrases. modular form, L-function, Bloch-Kato conjecture.

Let

$$\widehat{L}(f\otimes g\otimes h,s):=\Gamma_{\mathbb{C}}(s)\,\Gamma_{\mathbb{C}}(s-k+1)^3 L(f\otimes g\otimes h,s)$$

be the completed Rankin triple L-function with $\Gamma_{\mathbb{C}}(s):=2\,(2\pi)^{-s}\Gamma(s)$. It satisfies a functional equation

$$\widehat{L}(f\otimes g\otimes h,3k-2-s)=-\widehat{L}(f\otimes g\otimes h,s).$$

In accord with Deligne's conjecture [De],

$$\widehat{L}(f\otimes g\otimes h,2k-2)^{\mathsf{alg}}:=\frac{\widehat{L}(f\otimes g\otimes h,2k-2)}{\|\, f\,\|^2\|\, g\,\|^2\|\, h\,\|^2}\in K^{\times}. \tag{1.3}$$

Here K is the totally real number field generated by the Hecke eigenvalues of f,g and h. Note that the critical values for $L(f\otimes g\otimes h,s)$ are taken at integers from k to $2k-2$. Our first result is the following:

Theorem 1.1. *Let k be an integer such that S_k is non-trivial and $p:=2k-1$ is a prime. Let $(f_a)_a$ be a primitive Hecke eigenbasis of S_k. Then we have:*

$$\sum_{a,b,c=1}^{\dim S_k}\widehat{L}(f_a\otimes f_b\otimes f_c,2k-2)^{\mathsf{alg}}\in p^{-1}\mathbb{Z}_{(p)}^{\times} \tag{1.4}$$

if and only if the class number $h(\sqrt{-p})$ of the imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$ is larger than one. If $h(\sqrt{-p})=1$ then

$$\sum_{a,b,c=1}^{\dim S_k}\widehat{L}(f_a\otimes f_b\otimes f_c,2k-2)^{\mathsf{alg}}\in \mathbb{Z}_{(p)}.$$

Here $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at p, and $\mathbb{Z}_{(p)}^{\times}$ its unit group.

Remark. We could replace $\widehat{L}$ by L here, since the gamma factors are units at p.

Example. Let Δ be the unique primitive cusp form of weight 12 given by

$$\Delta(z):=e^{2\pi iz}\prod_{n=1}^{\infty}(1-e^{2\pi inz})^{24}.$$

The special value of the Rankin triple L-function $\widehat{L}(\Delta^{\otimes 3},s)$ at the critical value $s=22$ may be deduced from the table at the end of [Mi], which confirms a numerical prediction from [Z]. We have

$$\widehat{L}(\Delta^{\otimes 3},22)^{\mathsf{alg}}=\frac{2^{48}\cdot 3^9\cdot 5^3\cdot 7}{23\cdot 691^2}. \tag{1.5}$$

The prime number 23 appears in the denominator since $h(\sqrt{-23})=3$.

In §2 we accomplish one main goal of this paper, namely to prove Theorem 1.1. The proof uses pullback formulas for Siegel-Eisenstein series of degrees 2 and 3, and explicit formulas for the Fourier coefficients of Siegel-Eisenstein series of degrees $1, 2$ and 3, together with Garrett's integral representation for the rightmost critical value $L(f \otimes g \otimes h, 2k-2)$. The p in the denominator comes from the appearance of the Bernoulli number B_{2k-2} in formulas for Fourier coefficients. The class number enters through a congruence with a Bernoulli number (Lemma 2.1 below), apparently due to Carlitz [C]. When the class number is 1, this causes the p to be cancelled. A peculiar consequence of this congruence is that the class number can be expressed in terms of triple product or symmetric square L-values. As an aside, this is observed in §3.
Our goal in the remainder of the paper is to link Theorem 1.1 with what the Bloch-Kato conjecture on special values of motivic L-functions says in our special case. In §5 we state the Bloch-Kato conjecture in the case of critical values of $L(f \otimes g \otimes h, s)$. After working out the relation between the Deligne period and $\| f \|^2 \| g \|^2 \| h \|^2$, the product of Petersson norms, we find that the conjecture reads

$$\frac{L(f \otimes g \otimes h, t)}{(2\pi i)^u\, i^{3-3k} \| f \|^2 \| g \|^2 \| h \|^2} = \frac{\prod_{\ell \leq \infty} c_\ell(t)\ \#\text{Ш}(t)}{\#H^0(\mathbb{Q}, A(t)) \#H^0(\mathbb{Q}, A(3k-2-t))\, c(f)c(g)c(h)},$$

where $u = 4t + 3 - 3k$. The various terms, all of which depend on f, g and h, are defined in §5. If the Bloch-Kato conjecture is true, and if $h(\sqrt{-p}) > 1$, then Theorem 1.1 implies that, in the case $t = k$ (paired with $2k-2$ by the functional equation) there should be some f, g and h such that the right-hand-side makes a contribution to the trace that is non-integral at (a divisor of) p. Given that $\#\text{Ш}(k)$ is an integer, there should then be some f, g and h such that the contribution of the other factors on the right-hand-side is non-integral at p.
In §4 we recall from [DH] the significance of the condition $h(\sqrt{-p}) > 1$. Using class field theory it allows us to construct certain Galois representations, which can be identified with the $(\text{mod}\, \mathfrak{p})$ representations arising from some cuspidal Hecke eigenforms of level 1 and weight k. These are the forms we shall take for f, g and h. In §7 we show that $\text{ord}_{\mathfrak{p}}(\prod_{\ell \leq \infty} c_\ell(k)) \leq 0$, and in §6 we show that the global torsion factor $\#H^0(\mathbb{Q}, A(2k-2))$ makes a non-trivial contribution to the $\mathfrak{p}$-part of the denominator. The analysis of the $\mathfrak{p}$-part of $c_p(k)$ presents a technical challenge, since $p = 2k-1$ is smaller than the (degree) length $3k-2$ of the Hodge filtration of the triple product premotivic structure. This is why we are only able to produce a bound.
This paper may be viewed as extending our work in [DH], from symmetric square L-functions to triple product L-functions. If $n \equiv 2 \pmod 4$ then $L(\text{Sym}^n f, s)$ has a critical value (its rightmost) at $s = \frac{n+2}{4}(2k-2)$, and the construction of global torsion from [DH] easily generalises. If $n \equiv 3 \pmod 4$ then any n-fold tensor product L-function (for weight k cuspidal Hecke eigenforms) has a critical value (again, the rightmost) at $s = \frac{n+1}{4}(2k-2)$, and the related construction of global torsion in this

paper also generalises. For other $n \geq 1$ the L-function in question does not have any critical value at a multiple of $(2k-2)$. It is only for $n=2$ and 3 that we know the pullback formulas needed to evaluate the critical value. In any case, it seems that our luck might run out beyond $n=3$ for analysing Tamagawa factors.

2. Proof of Theorem 1.1

Let $\mathbb{H}_n$ be the Siegel upper half-space and $\Gamma_n := \mathsf{Sp}(2n)(\mathbb{Z})$ the Siegel modular group of degree n. Then we put, for $\gamma := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_n$,

$$\gamma(z) := (az+b)(cz+d)^{-1}, \quad j(\gamma, z) := \mathsf{det}(cz+d).$$

We denote by $M_k^n(\Gamma_n)$ the space of Siegel modular forms of degree n and weight k. By $S_k^n(\Gamma_n)$ we denote the subspace of cusp forms. To simplify notation we omit the index n when it is equal to 1. Examples of Siegel modular forms are given by Eisenstein series of Klingen type. Let r be an integer $0 \leq r < n$ and

$$\Gamma_{n,r} := \left\{ \begin{pmatrix} * & * \\ 0_{n+r,n-r} & * \end{pmatrix} \in \Gamma_n \right\}.$$

Let $k > n+r+1$ and $f \in S_k^r(\Gamma_r)$. Then we recall the definition of the Klingen type Eisenstein series $E_k^{n,r}(f)$ attached to f (if $r=0$ we take always $f=1$).

$$E_k^{n,r}(f,z) := \sum_{\gamma \in \Gamma_{n,r} \backslash \Gamma_n} f(\gamma(z)_*)\, j(\gamma, z)^{-k}, \tag{2.1}$$

where, if $z = \left(\begin{smallmatrix} w_1 & w_2 \\ w_3 & w_4 \end{smallmatrix}\right)$ then $z_* := w_4 \in \mathbb{H}_r$. If $r=0$ then we obtain the classical Siegel type Eisenstein series $E_k^n(z)$. Let Φ be the so-called Siegel Φ operator. It is a linear map from M_k^n to M_k^{n-1}. For $n \geq 2$, $\Phi(E_k^{n,r})) = E_k^{n-1,r}$ and $\ker(\Phi) = S_k^n$. We fix, for every partition $n_1 + \ldots + n_l$ of n, the diagonal embedding of $\mathbb{H}_{n_1} \times \ldots \times \mathbb{H}_{n_l}$ into $\mathbb{H}_n$:

$$(z_1, \ldots, z_l) \mapsto \begin{pmatrix} z_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & z_l \end{pmatrix}.$$

For $F \in M_k^n$ we have for the generalized Witt map

$$F|_{\mathbb{H}_{n_1} \times \cdots \times \mathbb{H}_{n_l}} \in M_k^{n_1} \otimes \cdots \otimes M_k^{n_l}.$$

Garrett [Ga1] obtained the following non-trivial spectral decomposition.

$$E_k^{n+m}|_{\mathbb{H}_n \times \mathbb{H}_m} = E_k^n \otimes E_k^m + \sum_{1 \leq r \leq \mathsf{min}(n,m)} \sum_{j=1}^{\mathsf{dim} S_k^r} c_j^{(r)}\, E_k^{n,r}(f_j^{(r)}) \otimes E_k^{m,r}(f_j^{(r)}). \tag{2.2}$$

Here $f_j^{(r)}$ runs through a Hecke eigenbasis of S_k^r and $c_j^{(r)}$ are some non-vanishing algebraic numbers. Here we want to note that $c_j^{(r)}$ only depends on $f_j^{(r)}$ (and not on n, m). Letting $(f_a)_a$ be a primitive Hecke eigenbasis of S_k, we have

$$E_k^3|_{\mathbb{H}_2\times\mathbb{H}} = E_k^2 \otimes E_k + \sum_{a=1}^{\dim S_k} c_a E_k^{2,1}(f_a) \otimes f_a. \tag{2.3}$$

As in Section 2 and Theorem 2.3 of [He], we obtain

$$E_k^{2,1}(f)|_{\mathbb{H}\times\mathbb{H}} = f \otimes E_k + E_k \otimes f + \sum_{a,b=1}^{\dim S_k} l_{a,b}(f)\, f_a \otimes f_b. \tag{2.4}$$

Here $l_{a,b}(f) \in \mathbb{C}$ a priori could be zero. Since we also have a formula for $E_k^2|_{\mathbb{H}\times\mathbb{H}}$, and already know that $E_k^3|_{\mathbb{H}\times\mathbb{H}\times\mathbb{H}}$ is symmetric in all three variables, we have:

$$\begin{aligned} E_k^3|_{\mathbb{H}\times\mathbb{H}\times\mathbb{H}} \;=\; & E_k^{\otimes 3} + \sum_{a=1}^{\dim S_k} c_a\, E_k \otimes f_a \otimes f_a \\ & + \sum_{b=1}^{\dim S_k} c_b\, f_b \otimes E_k \otimes f_b + \sum_{c=1}^{\dim S_k} c_c\, f_c \otimes f_c \otimes E_k \\ & + \sum_{a,b,c=1}^{\dim S_k} l_{a,b,c}\, f_a \otimes f_b \otimes f_c. \end{aligned} \tag{2.5}$$

Let B_k be the k-th Bernoulli number. Then, though the complete decomposition (2.5) is not in [Ga2], given (2.5) it is a direct consequence of the main theorem (1.3) of [Ga2] (with the correct power of 2), that

$$l_{a,b,c} = -\,2^{3-3k} \cdot \frac{k\cdot(2k-2)}{B_k\, B_{2k-2}}\, \frac{\widehat{L}(f_a \otimes f_b \otimes f_c, 2k-2)}{\parallel f_a \parallel^2 \parallel f_b \parallel^2 \parallel f_c \parallel^2}\,. \tag{2.6}$$

It is well-known that the Fourier coefficients $A_k^n(T)$ of E_k^n (T half-integral positive semi-definite matrix) are rational with bounded denominator. Please also note that $A_k^n(0) = 1$ and

$$A_k^{n-1}(N) = A_k^n \left(\begin{smallmatrix} N & 0 \\ 0 & 0 \end{smallmatrix} \right).$$

This follows from the already mentioned properties of the Siegel Φ-operator. We need some further notation. We denote by $a_k(n)$ the n-th Fourier coefficient of the elliptic Eisenstein series E_k^1. Let $A_k^3[n, m, r]$ denote the finite sum of all Fourier coefficients of E_k^3 with the entries n, m, r in the diagonal. Then a matrix T involved in this sum can have rank $1, 2$ or 3. In the case $n = m = r = 1$, we denote the related decomposition by

$$A_k^3(1,1,1) = A_1 + A_2 + A_3.$$

From formula (2.5) we obtain

$$A_1 + A_2 + A_3 = a_k(1)^3 + 3\,a_k(1)\sum_a c_a + \sum_{a,b,c} l_{a,b,c}.$$

In the case of degree $n = 2$ we have, from (2.2),

$$A_k^2\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) + 2\,A_k^2\begin{pmatrix}1&1/2\\1/2&1\end{pmatrix} + 2A_k^2\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right) = a_k(1)^2 + \sum_a c_a. \tag{2.7}$$

This can be simplified to

$$\sum_a c_a = A_k^2\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) + 2\,A_k^2\begin{pmatrix}1&1/2\\1/2&1\end{pmatrix} + 2a_k(1) - a_k(1)^2 \tag{2.8}$$

using the identity

$$\begin{pmatrix}1&0\\-1&1\end{pmatrix}\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}1&-1\\0&1\end{pmatrix} = \begin{pmatrix}1&0\\0&0\end{pmatrix}. \tag{2.9}$$

A straightforward calulation also shows that $A_1 = 4a_k(1)$. This leads to

$$\sum_{a,b,c} l_{a,b,c} = A + B \tag{2.10}$$

$$A = A_2 + A_3 - 3a_k(1)\left(A_k^2\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) + 2\,A_k^2\begin{pmatrix}1&1/2\\1/2&1\end{pmatrix}\right) \tag{2.11}$$

$$B = 4a_k(1) - 6a_k(1)^2 + 2a_k(1)^3. \tag{2.12}$$

Next we recall some facts used in Section 2.3 of [DH].

Lemma 2.1. *Suppose that $p := 2k-1$ is prime. Then*

$$\mathrm{ord}_p\, B_k = 0 \quad \text{and} \quad \mathrm{ord}_p\, B_{2k-2} = -1. \tag{2.13}$$

(From this it follows that $\mathrm{ord}_p a_k(1) = 0$ and $\mathrm{ord}_p a_{2k-2}(1) = 1$.) Further we have

$$a_k(1) \equiv 2/h(\sqrt{-p}) \pmod p. \tag{2.14}$$

The first part is just an instance of the v.Staudt-Clausen theorem. The congruence, which is of crucial importance since it introduces the class number, is from (5.2) of [C].

Lemma 2.2. *Let $p := 2k-1$ be a prime and S_k be non-trivial. Then $p|A$.*

Proof. We show more. We prove that $p|A_2, A_3$ and $p|(A - (A_2 + A_3)$. First we start with the divisibility of A_3. It follows from Remark 5.4 in [Bö] that $2a_k(1)a_{2k-2}(1)$ is the greatest common divisor of all $A_k^3(T)$, $T > 0$. Hence for every $T > 0$ an $m \in \mathbb{Z}$ exists, such that

$$A_k^3(T) = m2a_k(1)a_{2k-2}(1),$$

from which it follows from Lemma 2.1 that $p \mid A_k^3(T)$. Hence $p \mid A_3$.

Next we determine all possible singular T. We say that two matrices T, S of suitable sizes n, m are equivalent if $A_k^n(T) = A_k^m(S)$. Let

$$T = \begin{pmatrix} 1 & a/2 & b/2 \\ a/2 & 1 & c/2 \\ b/2 & c/2 & 1 \end{pmatrix} \quad (a, b, c \in \mathbb{Z},\ 0 \le |a|, |b|, |c| \le 2). \tag{2.15}$$

Let $T \ge 0$ as above. Then T is singular if and only if

$$a^2 + b^2 + c^2 - abc = 4. \tag{2.16}$$

If $abc = 0$ then two of the variables have to be zero and the third one equal to ± 2. For all 6 cases the value of the corresponding Fourier coefficient is equal to $A_k^2\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Now we assume $abc \ne 0$. Hence we have to look at the cases:

$$a^2 + b^2 + c^2 = \begin{cases} 3 & I \\ 6 & II \\ 9 & III \\ 12 & IV. \end{cases} \tag{2.17}$$

Let us consider each case by turn.

In case III), $abc = 5$, which is impossible given $0 \le |a|, |b|, |c| \le 2$.

In case IV), $abc = 8$. There are 4 possible T, which are all equivalent to 1. Since they have rank 1, they make no contribution to A_2.

In case II), $abc = 2$. There are 12 possible T, which are all equivalent to

$$\begin{pmatrix} 1 & 1 & 1/2 \\ 1 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix}. \tag{2.18}$$

But this matrix is equivalent to $\left(\begin{smallmatrix} 1 & 1/2 \\ 1/2 & 1 \end{smallmatrix}\right)$.

In case I), $abc = -1$. There are 4 possible T, which are equivalent to

$$\begin{pmatrix} 1 & -1/2 & 1/2 \\ -1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix}. \tag{2.19}$$

Since

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1/2 & 1/2 \\ -1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1/2 \\ 0 & 1/2 & 1 \end{pmatrix},$$

all 4 possible T are equivalent to $\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$. Finally we obtain the quite useful formula

$$A_2 = 6A_k^2 \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) + 16A_k^2 \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}.$$

Now we recall from [DH], Section 2.3, that $p|A_k^2 \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$ and $p|A_k^2 \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. This proves finally the lemma. ☐

Lemma 2.3. *Let $p := 2k - 1$ be a prime and S_k be non-trivial. Then we have*

$$B \in \begin{cases} \mathbb{Z}_{(p)}^\times & \text{for } h(\sqrt{-p}) > 2 \\ p\,\mathbb{Z}_{(p)} & \text{for } h(\sqrt{-p}) \in \{1, 2\} \end{cases}. \tag{2.20}$$

Proof. Recall that $B = 4a_k(1) - 6a_k(1)^2 + 2a_k(1)^3$. Since $2, a_k(1) \in \mathbb{Z}_{(p)}^\times$ we have $B \equiv 0 \pmod p$ if and only if

$$a_k(1)^2 - 3a_k(1) + 2 \equiv 0 \pmod p. \tag{2.21}$$

This congruence has two solutions: $a_k(1) \equiv 1 \pmod p$ and $a_k(1) \equiv 2 \pmod p$. By (2.14), this is the same as to say that $h(\sqrt{-p}) \equiv 1 \pmod p$ or $h(\sqrt{-p}) \equiv 2 \pmod p$. Now use the fact that $1 \leq h(\sqrt{-p}) < \sqrt{p}$, to turn congruence into equality. ☐

Remark. It is well-known that the class number for a prime discriminant is always odd. Nevertheless we have included the case $h(\sqrt{-p}) = 2$ for maybe possible generalization to Hilbert modular forms.

Now putting together Lemmas 2.2 and 2.3, (2.10), (2.6) and (2.13), we obtain Theorem 1.1.

3. Recovering class numbers from special values

In the last section we discovered certain properties of the rational integer

$$\mathcal{L}_k := \sum_{a,b,c=1}^{\dim S_k(\Gamma)} l_{a,b,c}. \tag{3.1}$$

It seems to be quite interesting that for almost all primes $p \equiv 3 \pmod 4$ we can extract the class number $h(\sqrt{-p})$ from $\mathcal{L}_k$ for $k = \frac{p+1}{2}$. Let $(f_i)_i$ be a primitive Hecke eigenbasis of $S_k(\Gamma)$ then we have:

$$\mathcal{L}_k = -2^{2-3k} a_k(1) \frac{\sum_{a,b,c=1}^{\dim S_k} \widehat{L}(f_a \otimes f_b \otimes f_c, 2k-2)^{\mathsf{alg}}}{\zeta(3-2k)}. \tag{3.2}$$

A careful analysis of the arguments in the last section shows that

$$\mathcal{L}_k \equiv 4a_k(1) - 6a_k(1)^2 + 2a_k(1)^3 \pmod p. \tag{3.3}$$

This leads to the formula

$$\frac{\sum_{a,b,c=1}^{\dim S_k} \widehat{L}(f_a \otimes f_b \otimes f_c, 2k-2)^{\mathsf{alg}}}{\zeta(3-2k)} \equiv -2^{3k-1}\big(a_k(1)-1\big)\big(a_k(1)-2\big) \pmod{p}.$$

Carlitz's congruence (2.14) turns this into a quadratic congruence for $1/h(\sqrt{-p})$, leading to the following, given that $h(\sqrt{-p}) < p$. (If $a \in \mathbb{Z}_p$ and a is congruent to a quadratic residue $(\bmod\, p)$, then we denote by $\sqrt{a}$ the square root in $\mathbb{Z}_p$ with least remainder on division by p, and for $b \in \mathbb{Z}_p$ we denote by $[b]_p$ the remainder of b upon division by p. Here, remainders are taken between 0 and $p-1$.)

Proposition 3.1. *Let p be a prime, $p \equiv 3 \pmod 4$, and let $k = \frac{p+1}{2}$. Let $(f_i)_i$ be a primitive Hecke eigenbasis of $S_k(\Gamma)$. Then there exists an $\varepsilon = \pm 1$ such that*

$$h(\sqrt{-p}) = \left[\frac{4}{3 - \varepsilon\sqrt{1 - 2^{3-3k}\frac{\sum_{a,b,c=1}^{\dim S_k} \widehat{L}(f_a\otimes f_b\otimes f_c, 2k-2)^{\mathsf{alg}}}{\zeta(3-2k)}}}\right]_p . \tag{3.4}$$

Using (2.9) in [DH], we obtain in a similar, but somewhat simpler manner, the following.

Proposition 3.2.

$$h(\sqrt{-p}) = \left[\frac{1}{1 + \frac{(k-2)!}{((k/2)-1)!}\sum_{i=1}^{\dim S_k(\Gamma)} \frac{\widehat{L}(\mathsf{Sym}^2(f_i),1)^{\mathsf{alg}}}{\zeta(3-2k)}}\right]_p . \tag{3.5}$$

4. Dihedral congruences for cusp forms of level one

In the following theorem, p is not necessarily equal to $2k-1$.

Theorem 4.1 (Deligne)**.** *Let $g = \sum_{n=1}^{\infty} a_n q^n$ be a normalised newform of weight $k \geq 2$ and character ϵ, for $\Gamma_1(N)$. Let $K = \mathbb{Q}(\{a_n\})$, and let $\mathfrak{p} \mid p$ be some prime of the ring of integers O_K, with completions $K_{\mathfrak{p}}$ and $O_{\mathfrak{p}}$. There exists a continuous representation*

$$\rho_g = \rho_{g,\mathfrak{p}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(K_{\mathfrak{p}}),$$

unramified outside pN, such that if $\ell \nmid pN$ is a prime, and Frob_ℓ is an arithmetic Frobenius element, then

$$\mathrm{Tr}(\rho_g(\mathrm{Frob}_\ell^{-1})) = a_\ell,\ \det(\rho_g(\mathrm{Frob}_\ell^{-1})) = \epsilon(\ell)\ell^{k-1}.$$

One can conjugate so that ρ_g takes values in $\mathrm{GL}_2(O_{\mathfrak{p}})$, then reduce $\pmod{\mathfrak{p}}$ to get a continuous representation $\overline{\rho}_g = \overline{\rho}_{g,\mathfrak{p}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$, which, if it is irreducible, is independent of the choice of invariant $O_{\mathfrak{p}}$-lattice. The following was proved in §3 of

[DH], and may be regarded as an instance of Serre's conjecture [Se1] (Khare's theorem [Kh]).

Proposition 4.2. *Let* $p \equiv 3 \pmod 4$ *be a prime, and let* $k := (p+1)/2$. *Suppose that* $h(\sqrt{-p}) > 1$. *Let* $F := \mathbb{Q}(\sqrt{-p})$, *and let* H *be the Hilbert class field of* F. *Let* $\tau : \mathrm{Gal}(H/F) \to \overline{\mathbb{F}}_p^{\times}$ *be any non-trivial character. Then there exist a normalised, cuspidal Hecke eigenform* $g = g_\tau = \sum_{n=1}^{\infty} a_n\, q^n$ *for* $\mathrm{SL}_2(\mathbb{Z})$, *of weight* k, *and a prime* $\mathfrak{p} \mid p$ *of* $\mathbb{Q}(\{a_n\})$ *such that* $\overline{\rho}_{g,\mathfrak{p}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ *has dihedral image, factoring through* $\mathrm{Gal}(H/\mathbb{Q})$, *and its restriction to* $\mathrm{Gal}(H/F)$ *is equivalent to the sum of the characters* τ *and* τ^{-1}.

Note that $h(\sqrt{-p})$, the order of $\mathrm{Gal}(H/F)$, is odd, so τ has odd order. Note also that $g_\tau = g_{\tau^{-1}}$. The representation $\overline{\rho}_{g,\mathfrak{p}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ was constructed by lifting from $\mathrm{Gal}(H/\mathbb{Q})$ the representation induced from the character τ of $\mathrm{Gal}(H/F)$.
Using $a_\ell \equiv \mathrm{Tr}(\rho_g(\mathrm{Frob}_\ell^{-1})) \pmod{\mathfrak{p}}$, one easily checks that $a_\ell \equiv 0 \pmod{\mathfrak{p}}$ for all primes ℓ with $\left(\frac{\ell}{p}\right) = -1$. The case $k = 12, p = 23, g = \Delta$ of this congruence was discovered by Wilton [Wilt], and studied further by Swinnerton-Dyer and Serre [SD, Se2, Se3] in the context of Galois representations. Swinnerton-Dyer also observed the congruence in the case $k = 16, p = 31$, and its absence in the case $k = 22, p = 43$, where $h(\sqrt{-p}) = 1$.

Lemma 4.3. $\overline{\rho}_{g,\mathfrak{p}}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ *is a direct sum of the trivial character and the quadratic character* χ_{-p}, *and* $a_p \equiv 1 \pmod{\mathfrak{p}}$.

Proof. First, since the principal ideal $(\sqrt{-p})$ of O_F splits completely in the Hilbert class field H, $\overline{\rho}_{g,\mathfrak{p}}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is induced from the trivial character of $\mathrm{Gal}(\mathbb{Q}_p(\sqrt{-p})/\mathbb{Q}_p)$, so is indeed a direct sum of the trivial character and the quadratic character χ_{-p}. This implies that $a_p \not\equiv 0 \pmod{\mathfrak{p}}$, since if $a_p \equiv 0 \pmod{\mathfrak{p}}$ then a theorem of Fontaine (proved in [E] as Theorem 2.6) shows that $\overline{\rho}_{g,\mathfrak{p}}|_{I_p}$ would be the sum of the $(k-1)$-powers of the two fundamental characters of level two, which it isn't. Now we know that $a_p \not\equiv 0 \pmod{\mathfrak{p}}$ (i.e. that g is ordinary at $\mathfrak{p}$), we may apply a theorem of Deligne (proved in §12 of [Gr]), which implies that $\overline{\rho}_{g,\mathfrak{p}}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is reducible, with a unique unramified composition factor taking Frob_p^{-1} to a_p. Since the trivial character is a composition factor of $\overline{\rho}_{g,\mathfrak{p}}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$, we get $a_p \equiv 1 \pmod{\mathfrak{p}}$. □

Since $a_p \not\equiv 0 \pmod{\mathfrak{p}}$ and $\overline{\rho}_{g,\mathfrak{p}}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ splits, the main theorem of [Gr] implies that g has a "companion form" of weight $k' = p + 1 - k$. In our case, since $p = 2k - 1$, $k' = k$, and in fact g is its own companion. (The case $k = 12, p = 23, g = \Delta$ is used as an example in §17 of [Gr].)

5. The Bloch-Kato conjecture

Let $\sum_{n=1}^{\infty} a_n(f)q^n = f \in S_k(\Gamma)$ (necessarily for some even $k \geq 12$) be a normalised Hecke eigenform, K a number field containing $\mathbb{Q}(\{a_n(f)\})$. Attached to f is a "premotivic structure" M_f over $\mathbb{Q}$ with coefficients in K. Thus there are 2-dimensional K-vector spaces $M_{f,B}$ and $M_{f,\mathrm{dR}}$ (the Betti and de Rham realisations) and, for each finite prime $\mathfrak{q}$ of O_K, a 2-dimensional $K_{\mathfrak{q}}$-vector space $M_{f,\mathfrak{q}}$, the $\mathfrak{q}$-adic realisation. These come with various structures and comparison isomorphisms, such as $M_{f,B} \otimes_K K_{\mathfrak{q}} \simeq M_{f,\mathfrak{q}}$. See 1.1.1 of [DFG1] for the precise definition of a premotivic structure, and 1.6.2 of [DFG1] for the construction of M_f. The $\mathfrak{q}$-adic realisation $M_{f,\mathfrak{q}}$ realises the representation $\rho_{f,\mathfrak{q}}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. For each prime number ℓ, the restriction to $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ may be used to define a local L-factor, and the Euler product is precisely $L(f,s)$. If $f,g,h \in S_k(\Gamma)$ are normalised Hecke eigenforms, let $M_{f,g,h} := M_f \otimes M_g \otimes M_h$. Then similarly from $M_{f,g,h}$ one obtains $L(f \otimes g \otimes h, s)$. Sometimes we omit the subscripts: $M := M_{f,g,h}$, with similar conventions for other things below.

On $M_{f,B}$ there is an action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, and the eigenspaces $M_{f,B}^{\pm}$ are 1-dimensional. On $M_{f,\mathrm{dR}}$ there is a decreasing filtration, with F^j a 1-dimensional space precisely for $1 \leq j \leq k-1$. The de Rham isomorphism $M_{f,B} \otimes_K \mathbb{C} \simeq M_{f,\mathrm{dR}} \otimes_K \mathbb{C}$ induces isomorphisms between $M_{f,B}^{\pm} \otimes \mathbb{C}$ and $(M_{f,\mathrm{dR}}/F) \otimes \mathbb{C}$, where $F := F^1 = \ldots = F^{k-1}$. Define $\Omega_f^{\pm}$ to be the determinants of these isomorphisms. These depend on the choice of K-bases for $M_{f,B}^{\pm}$ and $M_{f,\mathrm{dR}}/F$, so should be viewed as elements of $\mathbb{C}^\times/K^\times$. Note that if we consider the twist $M_f(j)$ (with $1 \leq j \leq k-1$), then $(M_f(j))_B = (2\pi i)^j M_{f,B}$, so $(M_f(j))_B^+ = (2\pi i)^j M_{f,B}^{(-1)^j}$ and the Deligne period of $M_f(j)$, as the determinant of the isomorphism from $(M_f(j))_B^+ \otimes_K \mathbb{C}$ to $(M_f(j)_{\mathrm{dR}}/F^0 M_f(j)_{\mathrm{dR}}) \otimes_K \mathbb{C} = (M_{f,\mathrm{dR}}/F^j M_{f,\mathrm{dR}}) \otimes_K \mathbb{C}$, is $(2\pi i)^j \Omega_f^{(-1)^j}$.

The eigenspace $M_{f,g,h,B}^+$ is 4-dimensional. On $M_{f,g,h,\mathrm{dR}}$ there is a decreasing filtration, with F^t a 4-dimensional space precisely for $k \leq t \leq 2k-2$. The de Rham isomorphism $M_{f,g,h,B} \otimes_K \mathbb{C} \simeq M_{f,g,h,\mathrm{dR}} \otimes_K \mathbb{C}$ induces isomorphisms between $M_{f,g,h,B}^{\pm} \otimes \mathbb{C}$ and $(M_{f,g,h,\mathrm{dR}}/F') \otimes \mathbb{C}$, where $F' := F^k = \ldots = F^{2k-2}$. Define $\Omega_{f,g,h}^{\pm} \in \mathbb{C}^\times/K^\times$ to be the determinants of these isomorphisms, with respect to K-bases. The Deligne period of $M_{f,g,h}(t)$ is $(2\pi i)^{4t}\,\Omega_{f,g,h}^{(-1)^t}$.

We shall choose an O_K-submodule $\mathfrak{M}_{f,B}$, generating $M_{f,B}$ over K, but not necessarily free, and likewise an $O_K[1/S]$-submodule $\mathfrak{M}_{f,\mathrm{dR}}$, generating $M_{f,\mathrm{dR}}$ over K, where S is the set of primes less than or equal to k. We take these as in 1.6.2 of [DFG1]. They are part of the "S-integral premotivic structure" $\mathfrak{M}_f$ associated to f. Actually, it will be convenient to enlarge S so that $O_K[1/S]$ is a principal ideal domain, then replace $\mathfrak{M}_{f,B}$ and $\mathfrak{M}_{f,\mathrm{dR}}$ by their tensor products with the new $O_K[1/S]$. These will now be free, as will be any submodules and quotients. Choosing bases, and using these to calculate the above determinants, we pin down the values of $\Omega^{\pm}$ (up to S-units). Setting $\mathfrak{M}_{f,g,h,B} := \mathfrak{M}_{f,B} \otimes \mathfrak{M}_{g,B} \otimes \mathfrak{M}_{h,B}$ and $\mathfrak{M}_{f,g,h,\mathrm{dR}} := \mathfrak{M}_{f,\mathrm{dR}} \otimes \mathfrak{M}_{g,\mathrm{dR}} \otimes \mathfrak{M}_{h,\mathrm{dR}}$, similarly we pin down $\Omega_{f,g,h}$ (up to S-units). This is not a problem, as we can ensure that S does not contain any prime we are interested in (specifically $p = 2k-1$ if it is prime).

Lemma 5.1. *$\Omega^{\pm}_{f,g,h} = 2(2\pi i)^{3-3k}\Omega_f^+\Omega_f^-\Omega_g^+\Omega_g^-\Omega_h^+\Omega_h^-$ (up to S-units).*

A more general period is calculated, up to $K^{\times}$, in [Bl].

Proof. Let e_f^+ and e_f^- be generators of $\mathfrak{M}^+_{f,B}$ and $\mathfrak{M}^-_{f,B}$ respectively. Let $\{x_f, y_f\}$ be an $O_K[1/S]$-basis for $\mathfrak{M}_{f,\mathrm{dR}}$, with y_f generating the submodule F. Under the isomorphism $M_{f,B} \otimes_K \mathbb{C} \simeq M_{f,\mathrm{dR}} \otimes_K \mathbb{C}$ we have

$$e_f^+ \mapsto \Omega_f^+ x_f + \eta_f^+ y_f,\ e_f^- \mapsto \Omega_f^- x_f + \eta_f^- y_f,$$

for some η_f^+, η_f^-. Likewise for g and h.
To calculate $\Omega^+_{f,g,h}$ we use the basis $\{e_f^+e_g^+e_h^+, e_f^+e_g^-e_h^-, e_f^-e_g^+e_h^-, e_f^-e_g^-e_h^+\}$ for $M^+_{f,g,h,B}$ and the basis of $(M_{f,g,h,\mathrm{dR}}/F')$ represented by $\{x_fx_gx_h, y_fx_gx_h, x_fy_gx_h, x_fx_gy_h\}$. Hence

$$\Omega^+_{f,g,h} = \begin{vmatrix} \Omega_f^+\Omega_g^+\Omega_h^+ & \eta_f^+\Omega_g^+\Omega_h^+ & \Omega_f^+\eta_g^+\Omega_h^+ & \Omega_f^+\Omega_g^+\eta_h^+ \\ \Omega_f^+\Omega_g^-\Omega_h^- & \eta_f^+\Omega_g^-\Omega_h^- & \Omega_f^+\eta_g^-\Omega_h^- & \Omega_f^+\Omega_g^-\eta_h^- \\ \Omega_f^-\Omega_g^+\Omega_h^- & \eta_f^-\Omega_g^+\Omega_h^- & \Omega_f^-\eta_g^+\Omega_h^- & \Omega_f^-\Omega_g^+\eta_h^- \\ \Omega_f^-\Omega_g^-\Omega_h^+ & \eta_f^-\Omega_g^-\Omega_h^+ & \Omega_f^-\eta_g^-\Omega_h^+ & \Omega_f^-\Omega_g^-\eta_h^+ \end{vmatrix}$$

$$= (\Omega_f^+\Omega_f^-\Omega_g^+\Omega_g^-\Omega_h^+\Omega_h^-)^2 \begin{vmatrix} 1 & \eta_f^+/\Omega_f^+ & \eta_g^+/\Omega_g^+ & \eta_h^+/\Omega_h^+ \\ 1 & \eta_f^+/\Omega_f^+ & \eta_g^-/\Omega_g^- & \eta_h^-/\Omega_h^- \\ 1 & \eta_f^-/\Omega_f^- & \eta_g^+/\Omega_g^+ & \eta_h^-/\Omega_h^- \\ 1 & \eta_f^-/\Omega_f^- & \eta_g^-/\Omega_g^- & \eta_h^+/\Omega_h^+ \end{vmatrix}.$$

Subtracting the second row from the first, and the fourth row from the third, then expanding down the first column, we obtain

$$\Omega^+_{f,g,h} = -2\delta_f\delta_g\delta_h(\Omega_f^+\Omega_f^-\Omega_g^+\Omega_g^-\Omega_h^+\Omega_h^-),$$

where $\delta_f := \Omega_f^+\eta_f^- - \Omega_f^-\eta_f^+$, etc. Now $\delta_f = \Omega_f^+\eta_f^- - \Omega_f^-\eta_f^+$ is the determinant of the isomorphism $M_{f,B} \otimes_K \mathbb{C} \simeq M_{f,\mathrm{dR}} \otimes_K \mathbb{C}$ (with respect to the chosen $O_K[1/S]$-bases). As on p.2 of [DFG2], $\wedge^2 M_f \simeq K(1-k)$ (the right-hand-side being a twist of the trivial pre-motivic structure), with $\wedge^2\mathfrak{M}_f$ mapping to $\eta O_K[1/S](1-k)$, for some integral ideal η. Since comparison maps are functorial, δ_f is the scalar (up to units in $O_K[1/S]$) giving the comparison map from $K(1-k)_B \otimes_K \mathbb{C}$ to $K(1-k)_{\mathrm{dR}} \otimes_K \mathbb{C}$ with respect to the natural integral bases. This is $(2\pi i)^{1-k}$ (c.f. 1.1.3 of [DFG1]). Hence $\Omega^+_{f,g,h} = -2(2\pi i)^{3-3k}\Omega_f^+\Omega_f^-\Omega_g^+\Omega_g^-\Omega_h^+\Omega_h^-$. For $\Omega^-_{f,g,h}$ we just exchange the superscripts $+$ and $-$ everywhere. This changes only the sign of the result. ∎

We shall need the elements $\mathfrak{M}_{f,\mathfrak{q}}$ of the S-integral premotivic structure, for each prime $\mathfrak{q}$ of O_K. These are as in 1.6.2 of [DFG1]. For each $\mathfrak{q}$, $\mathfrak{M}_{f,\mathfrak{q}}$ is a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable $O_{\mathfrak{q}}$-lattice in $M_{f,\mathfrak{q}}$. Taking tensor products, we get $\mathfrak{M}_{f,g,h,\mathfrak{q}}$, a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable $O_{\mathfrak{q}}$-lattice in $M_{f,g,h,\mathfrak{q}}$.

Let $A_{f,\mathfrak{q}} := M_{f,\mathfrak{q}}/\mathfrak{M}_{f,\mathfrak{q}}$, and $A_f[\mathfrak{q}] := A_{f,\mathfrak{q}}[\mathfrak{q}]$, the $\mathfrak{q}$-torsion subgroup. Similarly, let $A_{f,g,h,\mathfrak{q}} := M_{f,g,h,\mathfrak{q}}/\mathfrak{M}_{f,g,h,\mathfrak{q}}$, and $A_{f,g,h}[\mathfrak{q}] = A_{f,g,h,\mathfrak{q}}[\mathfrak{q}]$. Let $\check{A}_{f,g,h,\mathfrak{q}} := \check{M}_{f,g,h,\mathfrak{q}}/\check{\mathfrak{M}}_{f,g,h,\mathfrak{q}}$, where $\check{M}_{f,g,h,\mathfrak{q}}$ and $\check{\mathfrak{M}}_{f,g,h,\mathfrak{q}}$ are the $K_{\mathfrak{q}}$-vector space and $O_{\mathfrak{q}}$-lattice dual to $M_{f,g,h,\mathfrak{q}}$ and $\mathfrak{M}_{f,g,h,\mathfrak{q}}$ respectively, with the natural $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action. Let $A_{f,g,h} := \oplus_{\mathfrak{q}} A_{f,g,h,\mathfrak{q}}$, etc.

Following [BK] (Section 3), for $\ell \neq q$ (including $\ell = \infty$) let

$$H^1_f(\mathbb{Q}_\ell, M_{\mathfrak{q}}(t)) := \ker\big(H^1(D_\ell, M_{\mathfrak{q}}(t)) \to H^1(I_\ell, M_{\mathfrak{q}}(t))\big).$$

Here D_ℓ is a decomposition subgroup at a prime above ℓ, I_ℓ is the inertia subgroup, and $M_{\mathfrak{q}}(t)$ is a Tate twist of $M_{\mathfrak{q}}$, etc. The cohomology is for continuous cocycles and coboundaries. For $\ell = q$ let

$$H^1_f(\mathbb{Q}_q, M_{\mathfrak{q}}(t)) := \ker\big(H^1(D_q, M_{\mathfrak{q}}(t)) \to H^1(D_q, M_{\mathfrak{q}}(t) \otimes_{\mathbb{Q}_q} B_{\mathrm{crys}})\big).$$

(See Section 1 of [BK], or §2 of [Fo], for the definition of Fontaine's ring B_{crys}.) Let $H^1_f(\mathbb{Q}, M_{\mathfrak{q}}(t))$ be the subspace of those elements of $H^1(\mathbb{Q}, M_{\mathfrak{q}}(t))$ which, for all primes ℓ, have local restriction lying in $H^1_f(\mathbb{Q}_\ell, M_{\mathfrak{q}}(t))$. There is a natural exact sequence

$$0 \longrightarrow \mathfrak{M}_{\mathfrak{q}}(t) \longrightarrow M_{\mathfrak{q}}(t) \xrightarrow{\ \pi\ } A_{\mathfrak{q}}(t) \longrightarrow 0.$$

Let $H^1_f(\mathbb{Q}_\ell, A_{\mathfrak{q}}(t)) = \pi_* H^1_f(\mathbb{Q}_\ell, M_{\mathfrak{q}}(t))$. Define the $\mathfrak{q}$-Selmer group $H^1_f(\mathbb{Q}, A_{\mathfrak{q}}(t))$ to be the subgroup of elements of $H^1(\mathbb{Q}, A_{\mathfrak{q}}(t))$ whose local restrictions lie in $H^1_f(\mathbb{Q}_\ell, A_{\mathfrak{q}}(t))$ for all primes ℓ. Note that the condition at $\ell = \infty$ is superfluous unless $q = 2$. Define the Shafarevich-Tate group

$$Ш(t) = \bigoplus_{\mathfrak{q}} \frac{H^1_f(\mathbb{Q}, A_{\mathfrak{q}}(t))}{\pi_* H^1_f(\mathbb{Q}, M_{\mathfrak{q}}(t))}.$$

Conjecture 5.2 (Case of Bloch-Kato)**.** *Suppose that $k \leq t \leq 2k-2$. Then we have the following equality of fractional ideals of $O_K[1/S]$:*

$$(5.1)\qquad \frac{L(f\otimes g\otimes h, t)}{(2\pi i)^{4t}\,\Omega^{(-1)^t}_{f,g,h}} = \frac{\prod_{\ell\leq\infty} c_\ell(t)\ \#Ш(t)}{\#H^0(\mathbb{Q}, A_{f,g,h}(t))\#H^0(\mathbb{Q}, \check{A}_{f,g,h}(1-t))}.$$

The Tamagawa factors $c_\ell(t)$ will be defined in the last section. It is more convenient to use $\parallel f \parallel^2$ than $\Omega^{\pm}_f$, so we consider the relation between them. Bearing in mind §6 of [Hi], using Lemma 5.1.6 of [De] and the latter part of 1.5.1 of [DFG1], one recovers the well-known fact that, up to S-units,

$$(5.2)\qquad \parallel f \parallel^2 = i^{k-1}\Omega^+_f\Omega^-_f c(f),$$

where $c(f)$, the "cohomology congruence ideal", is, as the cup-product of basis elements for $\mathfrak{M}_{f,B}$, an integral ideal. (It is certainly trivial in those cases for which $\dim(S_k) = 1$.) Recall that by Lemma 5.1 above,

$$\Omega^{\pm}_{f,g,h} = 2(2\pi i)^{3-3k}\Omega_f^+\Omega_f^-\Omega_g^+\Omega_g^-\Omega_h^+\Omega_h^-.$$

Via the duality $M_f \times M_f \to K(1-k)$, $\check{A}_{f,g,h,\mathfrak{q}} \simeq A_{f,g,h,\mathfrak{q}}(3k-3)$. (Recall that $K \supset \mathbb{Q}(\{a_n\})$, and here $K(1-k)$ is a twist of the trivial premotivic structure over $\mathbb{Q}$ with coefficients in K.) Therefore (5.1) becomes, for $k \leq t \leq 2k-2$, the conjecture that

(5.3)

$$\frac{L(f\otimes g\otimes h, t)}{(2\pi i)^u\, i^{3-3k} \parallel f \parallel^2 \parallel g \parallel^2 \parallel h \parallel^2} = \frac{\prod_{\ell\leq\infty} c_\ell(t)\ \#\text{Ш}(t)}{\#H^0(\mathbb{Q}, A(t))\#H^0(\mathbb{Q}, A(3k-2-t))\ c(f)c(g)c(h)},$$

where $u = 4t+3-3k$.

6. Global torsion

Let $p \equiv 3 \pmod 4$ be a prime, and let $k := (p+1)/2$. Suppose that $h(\sqrt{-p}) > 1$. According to Theorem 1.1,

$$\sum_{a,b,c=1}^{\dim S_k} \widehat{L}(f_a\otimes f_b\otimes f_c, 2k-2)^{\mathsf{alg}} \in p^{-1}\mathbb{Z}_{(p)}^{\times}.$$

Bearing in mind that the functional equation implies $\widehat{L}(f_a\otimes f_b\otimes f_c, 2k-2) = -\widehat{L}(f_a\otimes f_b\otimes f_c, k)$, there must exist normalised, cuspidal Hecke eigenforms f, g, h for $\mathrm{SL}_2(\mathbb{Z})$, of weight k, and a prime $\mathfrak{p} \mid p$ of $K = \mathbb{Q}(\{a_n(f), a_n(g), a_n(h)\})$ such that

$$\frac{L(f\otimes g\otimes h, k)}{i\pi^{k+3} \parallel f \parallel^2 \parallel g \parallel^2 \parallel h \parallel^2}$$

is not integral at $\mathfrak{p}$. Strictly speaking we do not know that the $\mathfrak{p}$-part of $\text{Ш}(k)$ is trivial, but it is at least integral. The Bloch-Kato conjecture then demands that, for some f, g and h, the product of the other factors on the right-hand-side of (5.3), for $t = k$, is not integral, at some $\mathfrak{p} \mid p$. One of the terms appearing in this denominator has $\mathfrak{p}$-part $H^0(\mathbb{Q}, A_{f,g,h,\mathfrak{p}}(2k-2))$. So the following proposition provides what is required, when combined with the result from the next section, that $\mathrm{ord}_{\mathfrak{p}}(\prod_{\ell\leq\infty} c_\ell(k)) \leq 0$.

Proposition 6.1. *Let $p \equiv 3 \pmod 4$ be a prime, and let $k := (p+1)/2$. Suppose that $h(\sqrt{-p}) > 1$. Then there exist normalised, cuspidal Hecke eigenforms f, g, h for $\mathrm{SL}_2(\mathbb{Z})$, of weight k, and a prime $\mathfrak{p} \mid p$ of $K = \mathbb{Q}(\{a_n(f), a_n(g), a_n(h)\})$ such that $H^0(\mathbb{Q}, A_{f,g,h,\mathfrak{p}}(2k-2))$ is non-trivial.*

Proof. It suffices to find a non-zero element of $H^0(\mathbb{Q}, A_{f,g,h}[\mathfrak{p}](2k-2))$ (for some f, g and h). Since $2k-2 = p-1$ and the $(p-1)^{\text{st}}$ power of the cyclotomic character is trivial $\pmod p$, this is the same as $H^0(\mathbb{Q}, A_{f,g,h}[\mathfrak{p}])$. Choose a non-trivial character $\tau : \mathrm{Gal}(H/F) \to \overline{\mathbb{F}}_p^{\times}$ and let $g = h = g_\tau$ as in Proposition 4.2. Let $f = g_{\tau^{-2}}$. There is a basis $\{x_\tau, y_\tau\}$ for $A_g[\mathfrak{p}]$ such that $\mathrm{Gal}(H/F)$ acts on x_τ and y_τ by the characters τ and τ^{-1} respectively, and $\mathrm{Gal}(F/\mathbb{Q})$ swaps the one-dimensional spaces spanned by x_τ and y_τ. Similarly we have a basis $\{x_{\tau^{-2}}, y_{\tau^{-2}}\}$ for $A_f[\mathfrak{p}]$. Now $A_{f,g,h}[\mathfrak{p}] = A_f[\mathfrak{p}] \otimes A_g[\mathfrak{p}] \otimes A_h[\mathfrak{p}]$, and the element we seek is $x_{\tau^{-2}} \otimes x_\tau \otimes x_\tau + y_{\tau^{-2}} \otimes y_\tau \otimes y_\tau$. □

7. Tamagawa factors

The goal of this subsection is to show that if $p = 2k-1$ is prime, and $\mathfrak{p} \mid p$, then the factor $\prod_{\ell \leq \infty} c_\ell(k)$ contributes a non-positive power of $\mathfrak{p}$ to the $\mathfrak{p}$-part of the right-hand-side of (5.3) in the case $t = k$, with (eventually) $f = g_{\tau^{-2}}, g = h = g_\tau$ as in the previous section.

For a finite prime ℓ, let $H^1_f(\mathbb{Q}_\ell, \mathfrak{M}_\mathfrak{q}(k))$ be the inverse image of $H^1_f(\mathbb{Q}_\ell, M_\mathfrak{q}(k))$ under the natural map. (Recall that $M = M_{f,g,h}$.) Suppose now that $\ell \neq q$. Now $H^0(\mathbb{Q}_\ell, M_\mathfrak{q}(k))$ is trivial, since the eigenvalues of Frob_ℓ^{-1} acting on $M_\mathfrak{q}$ are algebraic integers with absolute value $\ell^{3(k-1)/2}$. Hence, by inflation-restriction, we find that $H^1_f(\mathbb{Q}_\ell, M_\mathfrak{q}(k)) \simeq (M_\mathfrak{q}(k)^{I_\ell})/(1-\mathrm{Frob}_\ell)(M_\mathfrak{q}(k)^{I_\ell})$ is trivial, so $H^1_f(\mathbb{Q}_\ell, \mathfrak{M}_\mathfrak{q}(k))$ is the torsion part of $H^1(\mathbb{Q}_\ell, \mathfrak{M}_\mathfrak{q}(k))$. Again using the triviality of $H^0(\mathbb{Q}_\ell, M_\mathfrak{q}(k))$, we identify $H^1_f(\mathbb{Q}_\ell, \mathfrak{M}_\mathfrak{q}(k))$ with $H^0(\mathbb{Q}_\ell, A_\mathfrak{q}(k))$. This has a subgroup that is given by $(M_\mathfrak{q}(k)^{I_\ell}/\mathfrak{M}_\mathfrak{q}(k)^{I_\ell})^{\mathrm{Frob}_\ell = \mathrm{id}}$, whose order is the $\mathfrak{q}$-part of $P_\ell(\ell^{-k})$, where $P_\ell(\ell^{-s}) = \det(1 - \mathrm{Frob}_\ell^{-1}\ell^{-s} | M_\mathfrak{q}{}^{I_\ell})$ is the Euler factor at ℓ in $L(f \otimes g \otimes h, s)$ (strictly speaking, its reciprocal). When ℓ is a prime of "good reduction", so that $M_\mathfrak{q}(k)^{I_\ell} = M_\mathfrak{q}(k)$ maps surjectively to $A_\mathfrak{q}(k)$, the subgroup is the whole of $H^0(\mathbb{Q}_\ell, A_\mathfrak{q}(k))$, but in general we define the $\mathfrak{q}$-part of the Tamagawa factor $c_\ell(k)$ to be the index of the subgroup. For us, every ℓ is a prime of good reduction (i.e. $M_\mathfrak{q}$ is unramified at ℓ), because f, g and h have level one, so we get the following straight from the definition.

Lemma 7.1. *If ℓ is a finite prime, and $\mathfrak{q}$ divides $q \neq \ell$, then the $\mathfrak{q}$-part of $c_\ell(k)$ is trivial.*

Note that the triviality of $H^0(\mathbb{Q}_\ell, M_\mathfrak{q}(k))$ is equivalent to $P_\ell(\ell^{-k}) \neq 0$. The Tamagawa factor $c_\infty(k)$ is, by definition, the order of the group

$$\frac{(M_B(k)/\mathfrak{M}_B(k))^+}{M_B(k)^+/\mathfrak{M}_B(k)^+}.$$

This is at worst a power of 2, so need not concern us.

It remains to consider the $\mathfrak{q}$-part of $c_\ell(k)$ in the case that $q = \ell$. It is known that $M_{f,\mathfrak{q}}, M_{g,\mathfrak{q}}$ and $M_{h,\mathfrak{q}}$ are crystalline representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$, as long as $q > k$.

(Recall that the level $N = 1$ for us.) For a careful discussion, referring to [Fa], see 1.2.5 of [DFG1]. It follows that $M_{\mathfrak{q}} = M_{f,g,h,\mathfrak{q}}$ is a crystalline representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$. Furthermore, $\mathbb{V}(\mathfrak{M}_{f,\mathrm{dR}} \otimes O_{\mathfrak{q}}) = \mathfrak{M}_{f,\mathfrak{q}}$, and likewise for g and h. (Note that $\mathfrak{M}_{f,\mathrm{dR}} \otimes O_{\mathfrak{q}}$ is really the crystalline realisation $\mathfrak{M}_{f,\mathfrak{q}\text{-crys}}$, or $\mathfrak{M}_{f,\mathrm{crys}}$ for short.) For the definitions of the modified Fontaine-Lafaille functor $\mathbb{V}$ and the categories $O_{\mathfrak{q}}$-$\mathfrak{MF}^a$ of filtered Dieudonné modules, see 1.1.2 of [DFG1].
For $\ell = q$, $P_q(T)$ may be realised as $\det(1 - \phi T | D(M_{\mathfrak{q}}))$, where, for a q-adic representation V of $\mathrm{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$, $D(V)$ is the filtered ϕ-module $(V \otimes_{\mathbb{Q}_q} B_{\mathrm{crys}})^{\mathrm{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)}$. Recall that $P_q(q^{-k}) \neq 0$. It now follows from Theorem 4.1(ii) of [BK] that the Bloch-Kato exponential map gives an isomorphism

$$(M_{\mathrm{dR}} \otimes K_{\mathfrak{q}})/\mathrm{Fil}^k(M_{\mathrm{dR}} \otimes K_{\mathfrak{q}}) \simeq H^1_f(\mathbb{Q}_q, M_{\mathfrak{q}}(k)).$$

The norm of the $\mathfrak{q}$-part of the Tamagawa factor $c_q(k)$ is

$$\mu(H^1_f(\mathbb{Q}_q, \mathfrak{M}_{\mathfrak{q}}(k)))/|P_q(q^{-k})|_q^{-1},$$

where μ is the Haar measure of $H^1_f(\mathbb{Q}_q, M_{\mathfrak{q}}(k)))$ induced via the exponential map from that measure on $(M_{\mathrm{dR}} \otimes K_{\mathfrak{q}})/\mathrm{Fil}^k(M_{\mathrm{dR}} \otimes K_{\mathfrak{q}})$ giving $(\mathfrak{M}_{\mathrm{dR}} \otimes O_{\mathfrak{q}})/\mathrm{Fil}^k(\mathfrak{M}_{\mathrm{dR}} \otimes O_{\mathfrak{q}})$ volume 1. By $\mu(H^1_f(\mathbb{Q}_q, \mathfrak{M}_{\mathfrak{q}}(k)))$ we really mean μ of its image in $H^1_f(\mathbb{Q}_q, M_{\mathfrak{q}}(k)))$, multiplied by the order of its torsion subgroup. The following is a direct consequence of Theorem 4.1(iii) of [BK].

Lemma 7.2. *If $q > 3k - 2$ and $\mathfrak{q} \mid q$ then the $\mathfrak{q}$-part of $c_q(k)$ is trivial.*

(This $3k-2$ is the length of the Hodge filtration of $M_{f,g,h,\mathrm{dR}}$.) Since we are especially interested in the choice $q = p := 2k - 1$ (when it is prime), this is not good enough for our purposes, so we shall have to try harder, after a few preliminaries.
We assume that f is ordinary at $\mathfrak{q}$ (as it is in our application with $q = 2k - 1$ and $f = g_{\tau^{-2}}$). Then by a theorem of Mazur and Wiles (a special case of Theorem 2 of [Wile]), there is a filtration

$$0 \longrightarrow \mathfrak{M}^1_{f,\mathfrak{q}} \longrightarrow \mathfrak{M}_{f,\mathfrak{q}} \longrightarrow \mathfrak{M}^2_{f,\mathfrak{q}} \longrightarrow 0 \qquad (7.1)$$

of $O_{\mathfrak{q}}[\mathrm{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)]$-modules. Furthermore, we can identify the composition factors. For $a \in O_{\mathfrak{q}}$, let $O_{\mathfrak{q}}(a)$ denote the rank-one $O_{\mathfrak{q}}$-module on which the action of $\mathrm{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$ is unramified, with Frob_q acting as multiplication by a. For $t \in \mathbb{Z}$ let $O_{\mathfrak{q}}(a;t)$ be its t^{th} Tate twist (i.e. multiply by the t^{th} power of the q-adic cyclotomic character). Then $\mathfrak{M}^1_{f,\mathfrak{q}} \simeq O_{\mathfrak{q}}(a_q^{-1})$ and $\mathfrak{M}^2_{f,\mathfrak{q}} \simeq O_{\mathfrak{q}}(a_q; 1-k)$. (Note that our $\rho_{f,\mathfrak{q}}$ is the dual of the one in [Wile].) There is then a filtration

$$0 \longrightarrow \mathfrak{M}^1_{f,\mathrm{crys}} \longrightarrow \mathfrak{M}_{f,\mathrm{crys}} \longrightarrow \mathfrak{M}^2_{f,\mathrm{crys}} \longrightarrow 0$$

of filtered $O_{\mathfrak{q}}$-Dieudonné modules, which transforms under $\mathbb{V}$ to (7.1). We have $\mathfrak{M}^1_{f,\mathrm{crys}} \simeq O_{\mathfrak{q}}[a_q^{-1}]$ and $\mathfrak{M}^2_{f,\mathrm{crys}} \simeq O_{\mathfrak{q}}[a_q; 1-k]$, where $O_{\mathfrak{q}}[a;t]$ is a free rank-one $O_{\mathfrak{q}}$-module, concentrated in degree $-t$, on which the Frobenius map ϕ acts as multiplication by $p^{-t}a^{-1}$. It is such that $\mathbb{V}(O_{\mathfrak{q}}[a;t]) = O_{\mathfrak{q}}(a;t)$.
Tensoring with $\mathfrak{M}_g \otimes \mathfrak{M}_h$, we get filtrations

$$0 \longrightarrow \mathfrak{M}^1_{\mathrm{crys}} \longrightarrow \mathfrak{M}_{\mathrm{crys}} \longrightarrow \mathfrak{M}^2_{\mathrm{crys}} \longrightarrow 0 \tag{7.2}$$

and

$$0 \longrightarrow \mathfrak{M}^1_{\mathfrak{q}} \longrightarrow \mathfrak{M}_{\mathfrak{q}} \longrightarrow \mathfrak{M}^2_{\mathfrak{q}} \longrightarrow 0,$$

with $\mathbb{V}$ taking the first to the second. Similarly we have filtrations

$$0 \longrightarrow M^1_{\mathrm{dR}} \longrightarrow M_{\mathrm{dR}} \longrightarrow M^2_{\mathrm{dR}} \longrightarrow 0$$

and

$$0 \longrightarrow M^1_{\mathfrak{q}} \longrightarrow M_{\mathfrak{q}} \longrightarrow M^2_{\mathfrak{q}} \longrightarrow 0$$

of non-integral structures.

Lemma 7.3. *There is an exact sequence*

$$0 \longrightarrow H^1_f(\mathbb{Q}_q, \mathfrak{M}^1_{\mathfrak{q}}(k)) \longrightarrow H^1_f(\mathbb{Q}_q, \mathfrak{M}_{\mathfrak{q}}(k)) \longrightarrow H^1_f(\mathbb{Q}_q, \mathfrak{M}^2_{\mathfrak{q}}(k)).$$

Proof. There is an exact sequence

$$0 \longrightarrow (M^1_{\mathrm{dR}} \otimes K_{\mathfrak{q}})/\mathrm{Fil}^k \longrightarrow (M_{\mathrm{dR}} \otimes K_{\mathfrak{q}})/\mathrm{Fil}^k \longrightarrow (M^2_{\mathrm{dR}} \otimes K_{\mathfrak{q}})/\mathrm{Fil}^k \longrightarrow 0,$$

the dimensions of the three non-zero terms being $3, 4, 1$. Applying the Bloch-Kato exponential map and Theorem 4.1(ii) of [BK], we have an exact sequence

$$0 \longrightarrow H^1_f(\mathbb{Q}_q, M^1_{\mathfrak{q}}(k)) \longrightarrow H^1_f(\mathbb{Q}_q, M_{\mathfrak{q}}(k)) \longrightarrow H^1_f(\mathbb{Q}_q, M^2_{\mathfrak{q}}(k)) \longrightarrow 0. \tag{7.3}$$

Also, since $P_{\mathfrak{q}}(q^{-k}) \neq 0$, $H^0(\mathbb{Q}_{\mathfrak{q}}, \mathfrak{M}^2_{\mathfrak{q}}(k))$ is trivial. Therefore, we have an exact sequence

$$0 \longrightarrow H^1(\mathbb{Q}_q, \mathfrak{M}^1_{\mathfrak{q}}(k)) \xrightarrow{\alpha} H^1(\mathbb{Q}_q, \mathfrak{M}_{\mathfrak{q}}(k)) \xrightarrow{\beta} H^1(\mathbb{Q}_q, \mathfrak{M}^2_{\mathfrak{q}}(k)), \tag{7.4}$$

which naturally maps (via "vertical" maps we shall call "θ") to a similarly exact sequence

$$0 \longrightarrow H^1(\mathbb{Q}_q, M^1_{\mathfrak{q}}(k)) \xrightarrow{\alpha} H^1(\mathbb{Q}_q, M_{\mathfrak{q}}(k)) \xrightarrow{\beta} H^1(\mathbb{Q}_q, M^2_{\mathfrak{q}}(k)). \tag{7.5}$$

Recall that $H^1_f(\mathbb{Q}_{\mathfrak{q}}, \mathfrak{M}_{\mathfrak{q}}(k))$ is the inverse image of $H^1_f(\mathbb{Q}_{\mathfrak{q}}, M_{\mathfrak{q}}(k))$, etc. We certainly have a sequence

$$0 \longrightarrow H^1_f(\mathbb{Q}_q, \mathfrak{M}^1_{\mathfrak{q}}(k)) \xrightarrow{\alpha} H^1_f(\mathbb{Q}_q, \mathfrak{M}_{\mathfrak{q}}(k)) \xrightarrow{\beta} H^1_f(\mathbb{Q}_q, \mathfrak{M}^2_{\mathfrak{q}}(k)).$$

We just have to show that it is exact. Clearly α is injective and $\mathrm{Im}(\alpha) \subset \ker(\beta)$. It remains to show that $\ker(\beta) \subset \mathrm{Im}(\alpha)$. Suppose that $x \in H^1_f(\mathbb{Q}_q, \mathfrak{M}_{\mathfrak{q}}(k))$ and

$\beta(x) = 0$. By (7.4), $x = \alpha(w)$, for some $w \in H^1(\mathbb{Q}_q, \mathfrak{M}^1_{\mathfrak{q}}(k))$. We need to show that $w \in H^1_f(\mathbb{Q}_q, \mathfrak{M}^1_{\mathfrak{q}}(k))$. Now $\alpha(\theta(w)) = \theta(\alpha(w)) \in H^1_f(\mathbb{Q}_q, M_{\mathfrak{q}}(k))$, but $\beta\alpha(\theta(w)) = 0$, so $\theta(w) \in H^1_f(\mathbb{Q}_q, M^1_{\mathfrak{q}}(k))$, by exactness of (7.3) and injectivity of α in (7.5). Hence $w \in H^1_f(\mathbb{Q}_q, \mathfrak{M}^1_{\mathfrak{q}}(k))$, as required. □

Substituting $\mathfrak{M}^i$ for $\mathfrak{M}$ and M^i for M (including in the Euler factor $\det(1-\phi T|D(M_{\mathfrak{q}}))$), we may define Tamagawa factors $c^i_q(k)$, for $i = 1, 2$. Since $M_{\mathfrak{q}}$ is crystalline,

$$\det(1 - \phi T|D(M_{\mathfrak{q}})) = \det(1 - \phi T|D(M^1_{\mathfrak{q}}))\det(1 - \phi T|D(M^2_{\mathfrak{q}})).$$

With (7.2) and Lemma 7.3, this implies the following.

Lemma 7.4. $\mathrm{ord}_{\mathfrak{q}}(c_q(k)) \leq \mathrm{ord}_{\mathfrak{q}}(c^1_q(k)) + \mathrm{ord}_{\mathfrak{q}}(c^2_q(k))$.

If we could show that the restriction of β to $H^1_f(\mathbb{Q}_q, \mathfrak{M}_{\mathfrak{q}}(k))$ surjects onto $H^1_f(\mathbb{Q}_q, \mathfrak{M}^2_{\mathfrak{q}}(k))$ then we would have equality. The bound on q in the following proposition is just good enough for our application to $q = p = 2k - 1$.

Proposition 7.5. *Suppose that f is ordinary at $\mathfrak{q}$, and that $q > 2k - 2$. Then* $\mathrm{ord}_{\mathfrak{q}}(c_q(k)) \leq 0$.

Lemma 7.4 above reduces this to the following.

Lemma 7.6. (1) *If $q > k + 1$ then* $\mathrm{ord}_{\mathfrak{q}}(c^1_q(k)) = 0$.
(2) *If $q > 2k - 2$ then* $\mathrm{ord}_{\mathfrak{q}}(c^2_q(k)) = 0$.

Proof. (1) $\mathfrak{M}^1_{\mathfrak{q}}(k) = O_{\mathfrak{q}}(a_q^{-1}; k) \otimes \mathfrak{M}_g \otimes \mathfrak{M}_h \simeq \mathrm{Hom}(\mathfrak{M}_g, \mathfrak{M}_h(a_q^{-1}; 1))$, since the dual of $\mathfrak{M}_g$ is $\mathfrak{M}_g(k - 1)$. As in the proof of Lemma 4.4 of [DH], we make a direct application of the proof of Proposition 2.16 of [DFG1] (that part before the statement of Lemma 2.17). This time we make the choices (in their notation) $\mathcal{D}_1 = \mathfrak{M}_{g,\mathrm{crys}}, \mathcal{D}_2 = \mathfrak{M}_{h,\mathrm{crys}}[a_q^{-1}; 1]$.

(2) $\mathfrak{M}^2_{\mathfrak{q}}(k) = O_{\mathfrak{q}}(a_q; 1) \otimes \mathfrak{M}_g \otimes \mathfrak{M}_h \simeq \mathrm{Hom}(\mathfrak{M}_g, \mathfrak{M}_h(a_q; 2 - k))$. Again we apply the proof of Proposition 2.16 of [DFG1], this time making the choices $\mathcal{D}_1 = \mathfrak{M}_{g,\mathrm{crys}}, \mathcal{D}_2 = \mathfrak{M}_{h,\mathrm{crys}}[a_q; 2-k]$. Note that for the bound on q, k is the length of the Hodge filtration of M_g or M_h, and we add to this the difference in twists. Thus, in case (2) for example, both $\mathfrak{M}_{g,\mathrm{crys}}$ (with graded pieces of degrees 0 and $k - 1$) and $\mathfrak{M}_{h,\mathrm{crys}}[a_q; 2 - k]$ (with graded pieces of degrees $k - 2$ and $2k - 3$) satisfy $\mathrm{Fil}^a\mathfrak{M} = \mathfrak{M}, \mathrm{Fil}^{a+q-1}\mathfrak{M} = \{0\}$, with $a = 0$.

□

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