SYMMETRIC SQUARE L-FUNCTIONS AND SHAFAREVICH-TATE GROUPS, II

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ABSTRACT. We re-examine some critical values of symmetric square L-functions for cusp forms of level one. We construct some more of the elements of large prime order in Shafarevich-Tate groups, demanded by the Bloch-Kato conjecture. For this we use the Galois interpretation of Kurokawa-style congruences between vector-valued Siegel modular forms of genus two (cusp forms and Klingen Eisenstein series), making further use of a construction due to Urban. We must assume that certain 4-dimensional Galois representations are symplectic. Our calculations with Fourier expansions use the Eholzer-Ibukiyama generalisation of the Rankin-Cohen brackets. We also construct some elements of global torsion which should, according to the Bloch-Kato conjecture, contribute a factor to the denominator of the rightmost critical value of the standard L-function of the Siegel cusp form. Then we prove, under certain conditions, that the factor does occur.

1. INTRODUCTION

Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a normalised, cuspidal Hecke eigenform of weight k' for $SL(2,\mathbb{Z})$. Associated to f is its L-function

$$L_f(s) = \sum_{n=1}^{\infty} a_n n^{-s} = \prod_p (1 - a_p p^{-s} + p^{k'-1-2s})^{-1} = \prod_p [(1-\alpha_p p^{-s})(1-\beta_p p^{-s})]^{-1}.$$

The series converges for $\Re(s)$ sufficiently large, but there is a holomorphic continuation to the whole of $\mathbb{C}$. Likewise there is a symmetric square L-function

$$D_f(s) = \prod_p [(1-\alpha_p^2 p^{-s})(1-\beta_p^2 p^{-s})(1-\alpha_p\beta_p p^{-s})]^{-1}.$$

In [Du] we looked at those weights ($k' = 12, 16, 18, 20, 22, 26$) for which the dimension of the space of cusp forms is one. Using a method of Zagier [Z], we calculated the critical values of $D_f(s)$. For example, here are the values of $D_f(r + k' - 1)/(f, f)\pi^{2r+k'-1}$ for odd r with $1 \leq r \leq k' - 1$, firstly when $k' = 22$. (The other critical values are related to these ones by the functional equation $\Lambda(s) =$

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$\Lambda(2k-1-s)$, where $\Lambda(s) = \Gamma_{\mathbb{R}}(s-k+2)\Gamma_{\mathbb{C}}(s)D_f(s)$.)

$k' = 22$

$r = 1$	$2^{25}/3^9.5^4.7^3.11.13.17.19$
3	$2^{28}/3^{10}.5^5.7^2.11^2.17.19^2.23$
5	$2^{26}.59/3^{11}.5^6.7^3.11^2.13.17.19^2.23$
7	$2^{28}.239/3^{15}.5^8.7^3.11^2.13^2.19^2.23$
9	$2^{28}.25537/3^{16}.5^8.7^4.11^2.13^2.17.19^2.23.29$
11	$2^{28}.\mathbf{131}.\mathbf{593}/3^{17}.5^9.7^5.11.13^2.17.19^2.23.29.31$
13	$2^{28}.2436904891/3^{20}.5^{10}.7^5.11^4.13^2.17^2.19^2.23.29.31$
15	$2^{32}.98513941/3^{22}.5^{10}.7^6.11^4.13^2.17^2.19^2.23.29.31$
17	$2^{34}.545715463/3^{21}.5^{12}.7^7.11^4.13^2.17^2.19^2.23.29.31.37$
19	$2^{37}.281.286397/3^{21}.5^{10}.7^7.11^4.13^4.17^3.19^2.23.29.31.37$
21	$2^{37}.\mathbf{61}.\mathbf{103}/3^{21}.5^8.7^4.11^4.13^2.17.19^2.23.29.31.37.41.131.593$

The Bloch-Kato conjecture [BK] provides a conjectural formula for these values. Large primes in the numerators ought to be the orders of elements in generalised Shafarevich-Tate groups. In [Du] we gave conditional constructions for some of these elements, in particular for the primes set in boldface in the above table. The Eisenstein primes appearing for $r = k'/2$ (when it is odd) were somehow related to Ramanujan congruences and the vanishing of $L_f(k'/2)$. The primes ℓ appearing for $r = k'-1$ are the moduli for congruences of a type discovered by Kurokawa [Ku] and generalised by Mizumoto [M], between (the Hecke eigenvalues of) a Klingen-Eisenstein series attached to f and a cuspidal Hecke eigenform F for the Siegel modular group of genus 2. The congruence forces a 4-dimensional $\pmod{\ell}$ Galois representation attached to F (by Laumon and Weissauer [L],[W], see also [Ta]) to be reducible. Exploiting this, a construction due to Urban [U] provides the required elements of order ℓ in Selmer groups attached to the pre-motivic structure for $D_f(k'-1)$. They come from extensions inside the $\pmod{\ell}$ representation, which can be made non-trivial, using the absolute irreducibility of the ℓ-adic representation. This is a higher dimensional analogue of Ribet's construction of elements in class groups of cyclotomic fields, which uses congruences between classical Eisenstein series and cusp forms [R]. In the situation dealt with by Urban, the coefficient ring is "large", parametrising cyclotomic twists of cusp forms in a p-adic family. We adapt his construction to the simpler case of ℓ-adic coefficients.

The goal of this paper is to use Urban's construction again, but this time to get away from the right-hand edge of the critical range by using vector-valued Siegel modular forms (of genus 2). For example, here are the values of $D_f(r+k'-1)/(f,f)\pi^{2r+k'-1}$ for odd r with $1 \leq r \leq k'-1$ when $k' = 16$.

$k' = 16$

$r = 1$	$2^{20}/3^6.5^3.7^2.11.13$
3	$2^{21}/3^8.5^3.7.11.13^2.17$
5	$2^{22}/3^8.5^3.7^2.11.13^2.17.19$
7	$2^{22}.23^2/3^{11}.5^5.7^3.11^2.13^2.17.19$
9	$2^{21}.\mathbf{2243}/3^{13}.5^5.7^3.11^2.13^2.17.19.23$
11	$2^{24}.\mathbf{839}/3^{12}.5^8.7^5.11^3.13^2.17.19.23$
13	$2^{28}.\mathbf{373}/3^{14}.5^8.7^5.11^3.13^2.17.19.23$
15	$2^{30}/3^{13}.5^6.7^2.11.13^2.17.19.23.29.31.3617$

We shall explain the factors $\ell = 373, 839, 2243$ using congruences between Klingen-Eisenstein series and cusp forms of type $\det^{14} \otimes \mathrm{Sym}^2(\mathbb{C}^2)$, $\det^{12} \otimes \mathrm{Sym}^4(\mathbb{C}^2)$ and $\det^{10} \otimes \mathrm{Sym}^6(\mathbb{C}^2)$ respectively. The congruence $\pmod{373}$ had already been proved by Satoh [Sa], following Kurokawa's method (as we shall too). To produce explicitly the vector-valued Siegel modular forms that we need, we shall use a generalisation, due to Eholzer and Ibukiyama [EI], of the Rankin-Cohen brackets. In these cases we can take this elementary approach because the space of vector-valued cusp forms is 1-dimensional. However, to illustrate a different method (modelled on the proofs of Propositions 3.4 and 3.5 of [R]), by which most of the remaining cases can be handled, we prove a suitable congruence modulo (a divisor of) $\ell = 9385577$. This is one of the large primes appearing in the table, below, of values of $D_f(r+k'-1)/(f,f)\pi^{2r+k'-1}$ for odd r with $1 \leq r \leq k'-1$ when $k' = 20$. This method is less elementary in that it depends on a formula of Böcherer, Satoh and Yamazaki for the Fourier coefficients of vector-valued Klingen-Eisenstein series [BSY]. For the most general statement about the congruences we may prove by this method, see Proposition 4.4. It covers all the large primes in numerators in the following table, and for $k' = 22$ above, only $\ell = 59$ and $\ell = 239$ are omitted.

In order to apply Urban's construction, we need the 4-dimensional λ-adic Galois representation, attached to the vector-valued, genus 2 cusp form F, to take values in $\mathrm{GSp}(4, K_\lambda)$, not just $\mathrm{GL}(4, K_\lambda)$. This is expected to be true in general, and is certainly true in the $k' = 16$ examples where $\dim S_{j,k} = 1$. (See following Proposition 6.4 for further comment.)

$k' = 20$	
$r = 1$	$2^{23}/3^8.5^3.7^2.11.13.17.19$
3	$2^{25}/3^{10}.5^4.7^3.11.17^2.19$
5	$2^{27}/3^{12}.5^4.7^2.11.13.17^2.19.23$
7	$2^{26}.\mathbf{2593}/3^{12}.5^7.7^3.11^2.13^2.17^2.19.23$
9	$2^{28}.\mathbf{8831}/3^{15}.5^7.7^5.11^2.13^2.17^2.19.23$
11	$2^{27}.\mathbf{304793977}/3^{19}.5^8.7^5.11^3.13^3.17^2.19.23.29$
13	$2^{25}.\mathbf{40706077}/3^{20}.5^9.7^5.11^2.13^2.17^2.19.23.29.31$
15	$2^{28}.\mathbf{9385577}/3^{19}.5^9.7^7.11^3.13^2.17^2.19.23.29.31$
17	$2^{32}.\mathbf{439367}/3^{19}.5^{11}.7^5.11^4.13^3.17^2.19.23.29.31$
19	$2^{32}.\mathbf{71^2}/3^{18}.5^9.7^3.11^2.13^2.17^2.19.23.29.31.37.283.617$

It is interesting to compare with the situation for critical values of $L_f(s)$. When $k'/2$ is odd, Brown [Br] has shown that, if $\lambda \mid \ell$ is a large prime dividing the "algebraic part" of $L_f((k'/2)+1)$, then one can construct a non-zero element of λ-torsion in an appropriate Selmer group, using the Galois representation attached to a (non-CAP) scalar-valued Siegel cusp form of genus two, with Hecke eigenvalues congruent $\pmod{\lambda}$ to those of the Saito-Kurokawa lift of f. To move away from this near-central critical value, one can try to use vector-valued Siegel cusp forms of genus two. Now there is no longer a Saito-Kurokawa lift, but Harder [Ha] has conjectured that cusp forms (of genus two) satisfying the necessary congruences exist anyway, and there is good computational evidence for his conjecture in special cases [FvdG], [vdG].

In §2 we introduce some definitions and notation. In §3 we describe (some of) Eholzer and Ibukiyama's operators, and write down the Fourier expansions of some Siegel Eisenstein series. In §4 we use these ingredients to produce Fourier expansions for some vector-valued Siegel modular forms, which are then used to prove the congruences modulo 839 and 2243 (for $k' = 16$), 9385577 (for $k' = 20$) and more generally (Proposition 4.4). In §5 we examine the predictions of the Bloch-Kato conjecture, applied to the special case of critical twists of the symmetric square of the pre-motivic structure attached to f. The Galois representations attached to Siegel cusp forms are introduced in §6, and the construction of Urban is described. In §7 it is proved that this element of Galois cohomology satisfies the necessary local conditions to lie in the Bloch-Kato Selmer group. The main results on congruences and Selmer groups are summarised in Propositions 4.4 and 7.3 respectively.

In §8 we construct a non-zero Galois-fixed element in a certain $\mathbb{F}_\lambda[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-module associated to the rightmost critical value of the standard L-function of the Siegel cusp form F. Then in §9 we show that, in favourable cases, λ does appear in certain ratios of critical values, as predicted by the Bloch-Kato conjecture.

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2. Klingen-Eisenstein series

Let $\mathfrak{H}_2$ be the Siegel upper half plane of 2 by 2 complex symmetric matrices with positive-definite imaginary part. For $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_2 := \mathrm{Sp}(4,\mathbb{Z})$ and $Z \in \mathfrak{H}_2$, let $M\langle Z\rangle := (AZ+B)(CZ+D)^{-1}$ and $J(M,Z) := CZ+D$. Let V be the space of the representation $\rho_{j,k} := \det^k \otimes \mathrm{Sym}^j(\mathbb{C}^2)$ of $\mathrm{GL}(2,\mathbb{C})$. Let $\{u_1^j, u_1^{j-1}u_2, \ldots, u_2^j\}$ be a standard basis for V. A holomorphic function $f : \mathfrak{H}_2 \to V$ is said to belong to the space $M_{j,k} = M_{j,k}(\Gamma_2)$ of Siegel modular forms of genus two and weight $\rho_{j,k}$ if

$$f(M\langle Z\rangle) = \rho_{j,k}(J(M,Z))f(Z)\ \forall M \in \Gamma_2, Z \in \mathfrak{H}_2.$$

(We consider only even j, to avoid necessitating $f = 0$.) Such an f has a Fourier expansion

$$f(Z) = \sum_{S\geq 0} a(S)\mathbf{e}(\mathrm{Tr}(SZ)) = \sum_{S\geq 0} a(S,f)\mathbf{e}(\mathrm{Tr}(SZ)),$$

where the sum is over all positive semi-definite matrices of the form $\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$ with $a, b, c \in \mathbb{Z}$, and $\mathbf{e}(z) := e^{2\pi i z}$.

Let $\Gamma_1 = \mathrm{SL}(2,\mathbb{Z})$ and $M_{k'}(\Gamma_1)$ the space of modular forms of weight k' for Γ_1. The Siegel operator Φ on $M_{j,k}(\Gamma_2)$ is defined by

$$\Phi f(z) = \lim_{t\to\infty} f\left(\begin{bmatrix} z & 0 \\ 0 & it \end{bmatrix}\right) \quad \text{for } z \in \mathfrak{H}_1, t \in \mathbb{R}.$$

By Lemma 1.1 of [A], for $f \in M_{j,k}(\Gamma_2)$, Φf is of the form gu_1^j for some modular form $g \in M_{j+k}(\Gamma_1)$, in fact if $j > 0$ then $g \in S_{j+k}(\Gamma_1)$. We also denote $g = \Phi f$, so we have a linear operator $\Phi : M_{j,k}(\Gamma_2) \to M_{j+k}(\Gamma_1)$. In terms of Fourier expansions,

$$\Phi f(z) = \sum_{n\geq 0} a\left(\begin{bmatrix} n & 0 \\ 0 & 0 \end{bmatrix}\right)\mathbf{e}(nz). \tag{1}$$

The kernel of Φ, denoted $S_{j,k}$, is the space of Siegel cusp forms of genus two and weight $\rho_{j,k}$. With respect to a natural inner product, there is an orthogonal decomposition $M_{j,k} = S_{j,k} \oplus N_{j,k}$, and, in the case $j > 0$, a linear isomorphism

$S_{j+k}(\Gamma_1) \simeq N_{j,k}$, where to $g \in S_{j+k}(\Gamma_1)$ we associate its vector-valued Klingen-Eisenstein series, denoted $[g]_j$. See §1 of [A] (which generalises [Kl]) for more details. In the case $j = 0$ there are still Klingen-Eisenstein series, but one needs also the Siegel Eisenstein series to complete a basis for $N_{j,k}$.

The Klingen-Eisenstein series $[g]_j$ satisfies $\Phi([g]_j) = g$. If g is a Hecke eigenform, with Hecke eigenvalues $\lambda_0(m)$ then $[g]_j$ is also a Hecke eigenform. If $\lambda(m)$ is the eigenvalue for $T(m)$ on $[g]_j$, then for prime p we have

$$\lambda(p) = (1 + p^{k-2})\lambda_0(p).$$

(See §2.1 of [A] or (2.12) of [Sa] for the definition of $T(m)$ in genus 2.) Consequently the Andrianov (spinor) L-function satisfies

$$L([g]_j, s) = L(g, s)L(g, s - (k - 2)).$$

See §§2 and 3 of [A] for more details.

3. Rankin-Cohen type operators

For even $k \geq 4$ let $E_k^{(2)} \in M_{0,k}$ be the (normalised) Siegel-Eisenstein series for Γ_2. It has the property that $\Phi E_k^{(2)} = E_k$, where $E_k = 1 - (2k/B_k) \sum_{n=1}^{\infty} \sigma_{k-1}(n)\mathbf{e}(nz)$ is the normalised Eisenstein series for Γ_1. It may be obtained from a modular form of weight $k - \frac{1}{2}$, via a Jacobi form of weight k and index 1, as described in §21 of [vdG]. This leads to a formula for its Fourier expansion:

$$E_k^{(2)} = 1 - \frac{2k}{B_k}\frac{2k-2}{B_{2k-2}} \sum_{S \geq 0} \left(\sum_{d|(a,b,c)} d^{k-1} H\left(k-1, \frac{4ac - b^2}{d^2}\right)\right) \mathbf{e}(\mathrm{Tr}(SZ)),$$

where $S = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$ and $H(r, N)$ is the function defined by Cohen in [C]. Note that if $D = (-1)^r N$ is the discriminant of a quadratic field then $H(r, N) = L(1 - r, \chi_D)$. Using the tables of $H(3, N)$ and $H(5, N)$ at the end of [C], we obtain (abbreviating $\mathbf{e}(\mathrm{Tr}(SZ))$ to S)

$$E_4^{(2)} = 1 + 240 \left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + 56 \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} + \right.$$

$$\left. 126 \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \right) + 2520 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \cdots \right]$$

and

$$E_6^{(2)} = 1 - 504 \left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - 88 \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \right.$$

$$\left. -330 \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \right) - 49368 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \cdots \right].$$

In later calculations we shall be especially concerned with the coefficients of $P := \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$ and $2P = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. The coefficients displayed above are chosen with this in mind, given that

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

It will be useful also to let $J := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

In the next section we shall use these Siegel-Eisenstein series, together with Eholzer and Ibukiyama's generalisation of the Rankin-Cohen bracket operators [EI], to produce elements of $M_{4,12}$ and $M_{6,10}$ that will be useful to us. Note that by Tsushima's formula [Ts], $S_{4,12}$ and $S_{6,10}$ are both 1-dimensional. (See the table in §25 of [vdG].)

For $Z = \begin{bmatrix} z_1 & z_3 \\ z_3 & z_2 \end{bmatrix}$, let $R_i = \frac{\partial}{\partial z_i}$. Similarly, for $Z' = \begin{bmatrix} z_1' & z_3' \\ z_3' & z_2' \end{bmatrix}$, let $R_i' = \frac{\partial}{\partial z_i'}$. For positive integers k_1, k_2 and ν, let $D_{k_1,k_2,\nu}$ be the result of substituting $S = R_1 u_1^2 + R_2 u_2^2 + 2R_3 u_1 u_2$ and $S' = R_1' u_1^2 + R_2' u_2^2 + 2R_3' u_1 u_2$ into the polynomial

$$\sum_{r+s=\nu} (-1)^r \binom{\nu + k_2 - 1}{r} \binom{\nu + k_1 - 1}{s} S^r S'^s. \tag{2}$$

Now for $F \in M_{0,k_1}(\Gamma_2), G \in M_{0,k_2}(\Gamma_2)$, let

$$[F, G]_{2\nu} = D_{k_1,k_2,\nu}(F(Z)G(Z'))|_{Z=Z'}.$$

Then by §5.3 and Proposition 6.1 of [EI], $[F, G]_{2\nu} \in M_{2\nu,k_1+k_2}(\Gamma_2)$. The bracket defined and used by Satoh in [Sa] is a scalar multiple of $[,]_2$. If we were to replace F and G by modular forms (for Γ_1), and S, S' by $\frac{\partial}{\partial z}, \frac{\partial}{\partial z'}$, we would obtain the Rankin-Cohen brackets of §7 of [C].

The following is part of Theorem 4.1 of [Sa]. Let η_{14} be a generator of the 1-dimensional space $S_{2,14}$, and for any Hecke eigenform $f \in M_{j,k}$ let $\nu(m, f)$ be the corresponding eigenvalue of $T(m)$.

Proposition 3.1 (Satoh). *For all* $m \geq 1$,

$$\nu(m, \eta_{14}) \equiv \nu(m, [\Delta_{16}]_2) \pmod{373}.$$

His proof follows the method employed by Kurokawa [Ku] in the scalar case. We use the same method in the next section to prove some more congruences of similar type.

4. Some more congruences

The results of §§4.1 and 4.2 are subsumed by Proposition 4.4, but we include them for their more elementary proofs.

4.1. $\ell = 839$.

Satoh used $[E_6^{(2)}, E_8^{(2)}]_2$ and $[E_4^{(2)}, E_{10}^{(2)}]_2$ to produce elements of $M_{2,14}$. Similarly we can use $[E_6^{(2)}, E_6^{(2)}]_4$ and $E_4^{(2)}[E_4^{(2)}, E_4^{(2)}]_4$ to produce elements of $M_{4,12}$. The coefficient of u_1^4 in $D_{4,4,2}$ is $10R_1^2 + 10R_1'^2 - 25R_1 R_1'$. The coefficient of u_1^4 in $D_{6,6,2}$ is $21R_1^2 + 21R_1'^2 - 49R_1 R_1'$. Looking just at the terms for $S = J, P$ and $2P$ (now S is a matrix again), we find that the u_1^4 part of $[E_6^{(2)}, E_6^{(2)}]_4$ is

$$-504\,(42\mathbf{e}(\mathrm{Tr}(JZ)) - 3696\mathbf{e}(\mathrm{Tr}(PZ)) + 42311808\mathbf{e}(\mathrm{Tr}(2PZ)) + \ldots).$$

We write $[E_6^{(2)}, E_6^{(2)}]_4 =: -504\phi_{12}$. It must be that $\Phi\phi_{12} = 42\Delta_{16}$, where $\Delta_{16} = q + 216q^2 + \ldots$ is the normalised cusp form of weight 12 for Γ_1. (The coefficient of $\mathbf{e}(JZ)$ turns into the coefficient of $q = \mathbf{e}(z)$, by (1).) Note that ϕ_{12} has integer Fourier coefficients.

We find also that the u_1^4 part of $E_4^{(2)}[E_4^{(2)}, E_4^{(2)}]_4$ is

$$240\,(20\mathbf{e}(\mathrm{Tr}(JZ)) + 1120\mathbf{e}(\mathrm{Tr}(PZ)) + 19872000\mathbf{e}(\mathrm{Tr}(2PZ)) + \dots)\,.$$

Hence $\Phi(42(1/240)E_4^{(2)}[E_4^{(2)}, E_4^{(2)}]_4 - 20\phi_{12}) = 0$. Dividing by $120960 = 2^7 3^3 5 \cdot 7$, we get $\eta_{12} \in S_{4,12}$, where η_{12} has rational Fourier coefficients, integral at least for any $\ell > 7$, and the u_1^4 part is

$$\mathbf{e}(\mathrm{Tr}(PZ)) - 96\mathbf{e}(\mathrm{Tr}(2PZ)) + \dots .$$

Since $S_{4,12}$ is 1-dimensional, η_{12} is necessarily a Hecke eigenform.

Proposition 4.1. *For all* $m \geq 1$,

$$\nu(m, \eta_{12}) \equiv \nu(m, [\Delta_{16}]_4) \pmod{839}.$$

Note that, since $\Delta_{16} \in S_{j+k}(\Gamma_1)$ with $j = 4, k = 12$, so that $k - 2 = 10$, we have, for any prime p, $\nu(p, [\Delta_{16}]_4) = (1 + p^{10})\nu(p, \Delta_{16})$.

Proof. Since $\Phi[\Delta_{16}]_4 = \Delta_{16}$, $\Phi\phi_{12} = 42\Delta_{16}$ and $\Phi\eta_{12} = 0$, and since η_{12} and ϕ_{12} must span the 2-dimensional space $M_{4,12}$, there must be a relation of the form

$$[\Delta_{16}]_4 = \frac{1}{42}\phi_{12} + c\eta_{12}.$$

Now

$$a(2P, [\Delta_{16}]_4) = a(P, T(2)[\Delta_{16}]_4)$$

(using (2.13) of [Sa])

$$= \nu(2, [\Delta_{16}]_4)a(P, [\Delta_{16}]_4),$$

and $\nu(2, [\Delta_{16}]_4) = (1 + 2^{10})\nu(2, \Delta_{16}) = 1025 \cdot 216 = 221400$, hence

$$a(2P, [\Delta_{16}]_4) = 221400a(P, [\Delta_{16}]_4).$$

Substituting $[\Delta_{16}]_4 = \frac{1}{42}\phi_{12} + c\eta_{12}$ and looking at u_1^4 parts, we obtain

$$1007424 - 96c = 221400(-88 + c),$$

hence $221496c = 20490624$, so

$$c = \frac{77616}{839}.$$

Now applying $T(m) - \nu(m, [\Delta_{16}]_4)$ to both sides of $[\Delta_{16}]_4 = \frac{1}{42}\phi_{12} + c\eta_{12}$, we see that the coefficients of $\frac{77616}{839}(\nu(m, \eta_{12}) - \nu(m, [\Delta_{16}]_4))\eta_{12}$ are integral at $\ell = 839$, forcing the congruence to hold. (We use (2.13) of [Sa], and $k \geq 2$, to see that $T(m)$ preserves integrality of Fourier coefficients.) $\square$

Note that the coefficient 221496 of c, which we divided by, is $221400 - (-96) = \nu(2, [\Delta_{16}]_4) - \nu(2, \eta_{12})$. The proof worked because this happened to be a multiple of 839, in other words we deduced the general case from the case $m = 2$.

4.2. $\ell = 2243$.
We can use $[E_4^{(2)}, E_6^{(2)}]_6$ to produce an element of $M_{6,10}$ with integer polynomials as Fourier coefficients. The coefficient of u_1^6 in $D_{4,6,3}$ is $20R_1'^3 - 120R_1R_1'^2 + 168R_1^2R_1' - 56R_1^3$. Calculating as before, we find that $[E_4^{(2)}, E_6^{(2)}]_6 = 2^5 3 \cdot 5 \cdot 7\phi_{10}$, where ϕ_{10} has Fourier coefficients integral at any $\ell > 7$, and its u_1^6-part is

$$40\mathbf{e}(\mathrm{Tr}(PZ)) + 7342272\mathbf{e}(\mathrm{Tr}(2PZ)) + \dots.$$

Also $\Phi\phi_{10} = -7\Delta_{16}$.

This time we cannot produce another element of $M_{6,10}$, independent of $[E_4^{(2)}, E_6^{(2)}]_6$, using brackets and Siegel-Eisenstein series. However, we do know that the space $S_{6,10}$ is 1-dimensional. Proposition 5.1 of [BSY] shows that $[\Delta_{16}]_6$ has rational (polynomial) Fourier coefficients. We shall see below that it is not simply a multiple of ϕ_{10}. Therefore some linear combination of $[\Delta_{16}]_6$ and ϕ_{10} gives a non-zero cusp form $\eta_{10} \in S_{6,10}$ with rational polynomial Fourier coefficients. We choose η_{10} in such a way that (the coefficients of) its Fourier coefficients are integers with no non-trivial common divisor. Then for some integer a, (the u_1^6 part of)

$$\eta_{10} = a\mathbf{e}(\mathrm{Tr}(PZ)) + \nu(2,\eta_{10})a\mathbf{e}(\mathrm{Tr}(2PZ)) + \dots.$$

G. van der Geer has kindly provided me with the eigenvalue $\nu(2,\eta_{10}) = 1680$, calculated by the method described in [FvdG] and [vdG], which involves counting points on hyperelliptic curves over finite fields. [Strictly speaking, their method relies on an assumption (Conjecture 2 in §24 of [vdG]) about an "endoscopic contribution", which appears to be closely related to (7) in Hypothesis A of [W].] Now copying the proof of the previous proposition, we find this time $c = \frac{-1}{7a}\left(\frac{213408}{2243}\right)$. No matter what a is, we obtain then the following.

Proposition 4.2. *For all* $m \geq 1$,

$$\nu(m,\eta_{10}) \equiv \nu(m,[\Delta_{16}]_6) \pmod{2243}.$$

Since $\Delta_{16} \in S_{j+k}(\Gamma_1)$ with $j = 6, k = 10$, so that $k - 2 = 8$, this time we have, for any prime p, $\nu(p,[\Delta_{16}]_6) = (1 + p^8)\nu(p,\Delta_{16})$.

4.3. **Where the congruences come from.**
There is a slightly different approach to congruences such as the above, which illuminates their origin. In a relation such as

$$[\Delta_{16}]_4 = \frac{1}{42}\phi_{12} + c\eta_{12},$$

the important thing is to show that c has a factor of ℓ (in this case 839) in the denominator. If it didn't, this would imply that all the Fourier coefficients of $[f]_j$ are integral at ℓ. This is rendered extremely unlikely by Theorem 5.3 of [BSY], a formula for the Fourier coefficients (for non-singular S) of $[f]_j$, in which appears a multiple of $D_f(2k' - 2 - j)$ in the denominator. Hence the appearance of ℓ in the normalised $D_f(2k' - 2 - j)$ explains its appearance in the denominator of c, hence the existence of the congruence. This was noted at the end of §4 of [Sa], and was the motivation for [BSY]. Now we see what we can actually prove using this.

Let Δ_{20} be the normalised cusp form of weight $k' = 20$ for Γ_1. Let $\ell = 9385577$.

Proposition 4.3. *There exists a Hecke eigenform* $\eta_{16} \in S_{4,16}$ *such that, for all* $m \geq 1$,

$$\nu(m,\eta_{16}) \equiv \nu(m,[\Delta_{20}]_4) \pmod{\lambda},$$

where λ is a prime ideal dividing ℓ.

Note that, since $\Delta_{20} \in S_{j+k}(\Gamma_1)$ with $j = 4, k = 16$, so that $k-2 = 14$, we have, for any prime p, $\nu(p, [\Delta_{20}]_4) = (1+p^{14})\nu(p, \Delta_{20})$.

Proof. Since $\dim(S_{4,16}) = 3 > 1$ (see §25 of [vdG]), the method used in previous examples does not apply.

Let $c(S, \mathbf{u})$ (a polynomial in the entries of $\mathbf{u} = (u_1, u_2)$) be the coefficient of $\mathbf{e}(\mathrm{Tr}(SZ))$ in $[f]_j$, where $f = \Delta_{20} = \sum_{n=1}^{\infty} a(n)q^n$ and $j = 4$. In the case that $\det S$ is square-free, Theorem 5.3 of [BSY] tells us that

$$H^{(1)}_{k,j}(S)\, c(S, \mathbf{u}) = \frac{2\gamma(k, j, 2)(\det S)^{(3/2)-k}}{\alpha_{k,j} C_{k,j,1} D_f(2k'-2-j)} H(k'-1, \det S) \frac{\Gamma(k'-1)}{(4\pi)^{k'-1}} \sum_{\mathbf{x}\in\mathbb{Z}^2} \frac{a(S[\mathbf{x}])}{S[\mathbf{x}]^{k'-1}},$$

where $S[\mathbf{x}] := \mathbf{x}S^t\mathbf{x}$. In their theorem, our j is their l, $p = 2, q = 1, n = 3$. Their R is our S, and their outer sum has only one term, with $T = S$ and $\omega_1 = I_2$. We must check that the factor of ℓ contributed by $D_f(2k'-2-j)/\langle f, f\rangle\pi^{2r+k'-1}$ is not cancelled by anything else.

Up to powers of π, the constants $\alpha_{k,j}$ and $C_{k,j,1}$ (defined on pages 15 and 9 respectively of [BSY]) are ℓ-units (using $\ell > 2k'$), so may be disregarded. By Lemma 5.2 of [BSY], the generalised gamma factor $H^{(1)}_{k,j}(S)$ almost cancels with the constant $\gamma(k, j, 2)$. In the case we shall concentrate on below, namely $S = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$ (i.e. $S = P$), the factors of $\gamma(k, j, 2)/H^{(1)}_{k,j}(S)$ (a power of $\det(S) = 3$ and "$P(S, S, S)/(\mathbf{u}^t\mathbf{v})^j$", which may be evaluated using their (5.7)) are ℓ-units.

The inner sum $\sum_{\mathbf{x}\in\mathbb{Z}^2} \frac{a(S[\mathbf{x}])}{S[\mathbf{x}]^{k'-1}}$ is the value at $k'-1$ of the Rankin convolution of f and the weight-one theta series of S. Choosing $S = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$, it becomes $6L(f, k'-1)L(f, \chi_{-3}, k'-1)$. Define $L_{\mathrm{alg}}(f, k'-1) := L(f, k'-1)/((2\pi i)^{k'-1}\Omega^-)$ and $L_{\mathrm{alg}}(f, \chi_{-3}, k'-1) := L(f, \chi_{-3}, k'-1)/((2\pi i)^{k'-1}\Omega^+)$, where Ω^+ and Ω^- are as in the next section. Observing, as in (4) in the next section, that the ratio of $\langle f, f\rangle$ to $i\Omega^+\Omega^-$ is an integer $\pm c(f)$, and that the Petersson norm of f may be cancelled with the one in $D_f(2k'-2-j)$, we see that, to prove that $\mathrm{ord}_\ell(c(S, \mathbf{u})) < 0$, it suffices to check that

$$\ell \nmid H(k'-1, 3)L_{\mathrm{alg}}(f, k'-1)L_{\mathrm{alg}}(f, \chi_{-3}, k'-1).$$

Using Stein's table "Rational parts of the special values of the L-functions of level 1" [St], we find that $\ell \nmid L_{\mathrm{alg}}(\Delta_{20}, 19)$ (by which I mean $\mathrm{ord}_\ell(L_{\mathrm{alg}}(\Delta_{20}, 19)) \leq 0$), and using modular symbols as explained in §§8 and 2 of [MTT], we find that $\ell \nmid L_{\mathrm{alg}}(\Delta_{20}, \chi_{-3}, 19)$. Now $H(19, 3) = L(1-19, \chi_{-3}) = -B_{19,\chi_{-3}}/19 = \frac{2\cdot 7\cdot 19\cdot 7691\cdot 8609}{3}$. Certainly $\ell \nmid H(19, 3)$. We conclude that the Fourier coefficients of $[\Delta_{20}]_4$ are not all integral at ℓ.

The coefficient of $\mathbf{e}(\mathrm{Tr}(JZ))$ in the u_1^4-part of $E_4^{(2)}[E_6^{(2)}, E_6^{(2)}]_4$ is $c = -504\left(2\binom{7}{2}\right)$. Therefore the coefficient of q in $\Phi(E_4^{(2)}[E_6^{(2)}, E_6^{(2)}]_4)$ is c. Since $\dim S_{20}(\Gamma_1) = 1$, it must be that $\Phi(E_4^{(2)}[E_6^{(2)}, E_6^{(2)}]_4) = c\Delta_{20}$. Let $g = c^{-1}E_4^{(2)}[E_6^{(2)}, E_6^{(2)}]_4$. Then $\Phi(g) = \Delta_{20}$, and since $\ell \nmid c$, the Fourier coefficients of g are integral at ℓ.

Let s be such that the Fourier coefficients of $\ell^s[\Delta_{20}]_4$ are integral at ℓ and not all divisible by ℓ. Then $s > 0$. With g as above, let $h = \ell^s[\Delta_{20}]_4 - \ell^s g$. Let

$S_{4,16}(\mathbb{Z}_{(\ell)})$ denote the set of elements of $S_{4,16}$ with Fourier coefficients in $\mathbb{Z}_{(\ell)}$. Then $h \in S_{4,16}(\mathbb{Z}_{(\ell)})$, since $\Phi(g) = \Delta_{20}$. Furthermore, since the Fourier coefficients of g are integral at ℓ, the image of h in $S_{4,16}(\mathbb{Z}_{(\ell)})/\ell S_{4,16}(\mathbb{Z}_{(\ell)})$ is a Hecke eigenform (with the same eigenvalues as $[\Delta_{20}]_4$). (We use here that the Hecke operators preserve integrality of Fourier coefficients, which can be seen from (2.13) of [Sa].) The proof of the proposition is now completed by an application of the Deligne-Serre Lemma (Lemme 6.11 of [DeSe]). $\square$

Proposition 4.4. *Let* f *a normalised generator of* $S_{k'}$, *with* $k' = 16, 18, 20, 22$ *or* 26. *Suppose that* $\ell > 2k'$ *is a prime such that* $\ell \mid D_f(r+k'-1)/\langle f,f\rangle\pi^{2r+k'-1}$, *with* r *odd and* $1 \leq r \leq k'-1$. *Suppose further that* $r \geq 7$, *if* $k'/2$ *is even, and* $r \geq 9$ *(but* $r \neq 11$*) if* $k'/2$ *is odd. Let* $j = k'-1-r$ *and* $k = k'-j = r+1$. *Then there exists a Hecke eigenform* $F \in S_{j,k}$ *such that, for all* $m \geq 1$,

$$\nu(m, F) \equiv \nu(m, [f]_j) \pmod{\lambda},$$

where λ *is a prime ideal dividing* ℓ *in a sufficiently large field.*

Proof. This can be done using exactly the same method as in Proposition 4.3 for $k' = 20, r = 15$. I have checked in all cases that ℓ does not divide (the numerator of) $L_{\mathrm{alg}}(f, k'-1)$, $L_{\mathrm{alg}}(f, \chi_{-3}, k'-1)$ or $H(k'-1, 3)$. Further, using a Maple program, I have checked that in none of the cases does ℓ appear in the factorisation of $P(S,S,S)/(\mathbf{u}^t\mathbf{v})^j$ (with $S = P$ again). The condition on r is necessary in order to produce, using $E_4^{(2)}, E_6^{(2)}$ and brackets, a form with non-zero coefficient of $u_1^j\,\mathbf{e}(\mathrm{Tr}(JZ))$ (which we can then divide by to get our g).

In a bit more detail, using (2) with $\nu = j/2$, the coefficient c of $u_1^j\,\mathbf{e}(\mathrm{Tr}(JZ))$ in $[E_4, E_4]_\nu$ is $(1+(-1)^\nu)\binom{\nu+3}{\nu}$. In $[E_6, E_6]_\nu$ it is $(1+(-1)^\nu)\binom{\nu+5}{\nu}$. For ν even these are non-zero and coprime to ℓ, but for ν odd they are zero. When ν is odd, the coefficient d of $u_1^j\,\mathbf{e}(\mathrm{Tr}(JZ))$ in $[E_4, E_6]_\nu$ is $\frac{(\nu+3)(\nu+2)\dots 6}{\nu!}[(\nu+5)(\nu+4)-(5)(4)] = \frac{(\nu+3)(\nu+2)\dots 6}{\nu!}\nu(\nu+9)$, which is non-zero and coprime to ℓ. When ν is even, it is $\frac{(\nu+3)(\nu+2)\dots 6}{\nu!}[(\nu+5)(\nu+4)+(5)(4)] = \frac{(\nu+3)(\nu+2)\dots 6}{\nu!}(\nu^2+9\nu+40)$, so we require the condition $\ell \nmid (j^2+18j+160)$.

When $k'/2$ is even, for $k = 8$ we take $g = c^{-1}[E_4, E_4]_\nu$, with $\nu = (k'/2)-4$, which is even. For $k = 10$ we take $g = d^{-1}[E_4, E_6]_\nu$, with $\nu = (k'/2)-5$, which is odd. Each time k goes up by 4, ν goes down by 2, and we multiply by a factor of E_4.

When $k'/2$ is odd, for $k = 14$ we take $g = c^{-1}E_6[E_4, E_4]_\nu$, with $\nu = (k'/2)-7$, which is even. For $k = 16$ we take $g = d^{-1}E_6[E_4, E_6]_\nu$, with $\nu = (k'/2)-8$, which is odd. Again, each time k goes up by 4, ν goes down by 2, and we multiply by a factor of E_4. For $k = 10$ we take $g = d^{-1}[E_4, E_6]_\nu$, where $\nu = (k'/2)-5$, which is even, so we require the condition $\ell \nmid (j^2+18j+160)$, which certainly holds in the cases $k' = 18, 22, 26$. $\square$

See [Du] for the full table of values of $D_f(r+k'-1)/(f,f)\pi^{2r+k'-1}$.

In the case $\dim S_{k'}(\Gamma_1) > 1$ it would be desirable to be able to apply the Deligne-Serre lemma directly to the reduction of $\lambda^s\,[f]_j$, which is a $\pmod{\lambda}$ cusp form (i.e. a section of some sheaf, as in [Hi2]), without having to produce an explicit cusp form like h. For this we would need to know that $\pmod{\lambda}$ cusp forms (of genus 2) can be lifted to characteristic-zero cusp forms.

5. The Bloch-Kato conjecture

Let $f \in S_{k'}(\Gamma_1)$ (necessarily for some even $k' \geq 12$) be a normalised Hecke eigenform, K some number field containing all the Hecke eigenvalues of f. Attached to f is a "premotivic structure" M_f over $\mathbb{Q}$ with coefficients in K. Thus there are 2-dimensional K-vector spaces $M_{f,B}$ and $M_{f,\mathrm{dR}}$ (the Betti and de Rham realisations) and, for each finite prime λ of O_K, a 2-dimensional K_λ-vector space $M_{f,\lambda}$, the λ-adic realisation. These come with various structures and comparison isomorphisms, such as $M_{f,B} \otimes_K K_\lambda \simeq M_{f,\lambda}$. See 1.1.1 of [DFG1] for the precise definition of a premotivic structure, and 1.6.2 of [DFG1] for the construction of M_f. The λ-adic realisation $M_{f,\lambda}$ comes with a continuous linear action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. For each prime number p, the restriction to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ may be used to define a local L-factor, and the Euler product is precisely $L_f(s)$. Let $M'_f := \mathrm{Sym}^2 M_f$. Then similarly from M'_f one obtains $D_f(s)$.

On $M_{f,B}$ there is an action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, and the eigenspaces $M^{\pm}_{f,B}$ are 1-dimensional. On $M_{f,\mathrm{dR}}$ there is a decreasing filtration, with F^j a 1-dimensional space precisely for $1 \leq j \leq k'-1$. The de Rham isomorphism $M_{f,B} \otimes_K \mathbb{C} \simeq M_{f,\mathrm{dR}} \otimes_K \mathbb{C}$ induces isomorphisms between $M^{\pm}_{f,B} \otimes \mathbb{C}$ and $(M_{f,\mathrm{dR}}/F) \otimes \mathbb{C}$, where $F := F^1 = \ldots = F^{k'-1}$. Define $\Omega^{\pm}$ to be the determinants of these isomorphisms. These depend on the choice of K-bases for $M^{\pm}_{f,B}$ and $M_{f,\mathrm{dR}}/F$, so should be viewed as elements of $\mathbb{C}^\times/K^\times$. Note that if we consider the twist $M_f(j)$ (with $1 \leq j \leq k'-1$), then $(M_f(j))_B = (2\pi i)^j M_f$, so $(M_f(j))^+_B = (2\pi i)^j M^{(-1)^j}_{f,B}$ and the Deligne period of $M_f(j)$, as the determinant of the isomorphism from $(M_f(j))^+_B \otimes_K \mathbb{C}$ to $(M_f(j)_{\mathrm{dR}}/F^0 M_f(j)_{\mathrm{dR}}) \otimes_K \mathbb{C} = (M_{f,\mathrm{dR}}/F^j M_{f,\mathrm{dR}}) \otimes_K \mathbb{C}$, is $(2\pi i)^j \Omega^{(-1)^j}$.

The eigenspace $M'^{+}_{f,B}$ is 2-dimensional. On $M'_{f,\mathrm{dR}}$ there is a decreasing filtration, with F^t a 1-dimensional space precisely for $k' \leq t \leq 2k'-2$. The de Rham isomorphism $M'_{f,B} \otimes_K \mathbb{C} \simeq M'_{f,\mathrm{dR}} \otimes_K \mathbb{C}$ induces an isomorphism between $M'^{+}_{f,B} \otimes \mathbb{C}$ and $(M'_{f,\mathrm{dR}}/F') \otimes \mathbb{C}$, where $F' := F^{k'} = \ldots = F^{2k'-2}$. Define $\Omega \in \mathbb{C}^\times/K^\times$ to be the determinant of this isomorphism. Note that t as above is critical only when it is even, since this is when the dimension of $(M'_f(t))^+_B = (2\pi i)^t {M'_{f,B}}^{(-1)^t}$ matches that of $(M'_f(t))_{\mathrm{dR}}/F^0(M'_f(t))_{\mathrm{dR}} = M'_{f,\mathrm{dR}}/F^t M'_{f,\mathrm{dR}}$. In this case, the Deligne period of $M'_f(t)$ is $(2\pi i)^{2t}\Omega$.

We shall choose an O_K-submodule $\mathfrak{M}_{f,B}$, generating $M_{f,B}$ over K, but not necessarily free, and likewise an $O_K[1/S]$-submodule $\mathfrak{M}_{f,\mathrm{dR}}$, generating $M_{f,\mathrm{dR}}$ over K, where S is the set of primes less than or equal to k'. We take these as in 1.6.2 of [DFG1]. They are part of the "S-integral premotivic structure" $\mathfrak{M}_f$ associated to f. Actually, it will be convenient to enlarge S so that $O_K[1/S]$ is a principal ideal domain, then replace $\mathfrak{M}_{f,B}$ and $\mathfrak{M}_{f,\mathrm{dR}}$ by their tensor products with the new $O_K[1/S]$. These will now be free, as will be any submodules and quotients. Choosing bases, and using these to calculate the above determinants, we pin down the values of $\Omega^{\pm}$ (up to S-units). Setting $\mathfrak{M}'_{f,B} := \mathrm{Sym}^2\mathfrak{M}_{f,B}$ and $\mathfrak{M}'_{f,\mathrm{dR}} := \mathrm{Sym}^2\mathfrak{M}_{f,\mathrm{dR}}$, similarly we pin down Ω (up to S-units). We just have to imagine not including in S any prime we care about (like $373, 839$, 2243 or 9385577).

Lemma 5.1. $\Omega = 2(2\pi i)^{1-k'}\Omega^+\Omega^-$.

Proof. We follow Proposition 7.7 of [De]. Let e^+ and e^- be generators of $\mathfrak{M}^+_{f,B}$ and $\mathfrak{M}^-_{f,B}$ respectively. Let $\{x, y\}$ be an $O_K[1/S]$-basis for $\mathfrak{M}_{f,\mathrm{dR}}$, with y generating the submodule F. Under the isomorphism $M_{f,B}\otimes_K \mathbb{C}\simeq M_{f,dR}\otimes_K\mathbb{C}$ we have

$$e^+\mapsto \Omega^+x+\eta^+y,\ e^-\mapsto \Omega^-x+\eta^-y,$$

for some η^+,η^-, so under the isomorphism between $M'^{+}_{f,B}\otimes\mathbb{C}$ and $(M'_{f,dR}/F')\otimes\mathbb{C}$ we have

$$(e^+)^2\mapsto (\Omega^+)^2x^2+2\Omega^+\eta^+xy,$$
$$(e^-)^2\mapsto (\Omega^-)^2x^2+2\Omega^-\eta^-xy.$$

Hence $\Omega=2\Omega^+\Omega^-\delta$, where $\delta=\Omega^+\eta^- - \Omega^-\eta^+$ is the determinant of the isomorphism $M_{f,B}\otimes_K\mathbb{C}\simeq M_{f,dR}\otimes_K\mathbb{C}$ (with respect to the chosen $O_K[1/S]$-bases). As on p.2 of [DFG2], $\wedge^2 M_f\simeq K(1-k')$ (the right-hand-side being a twist of the trivial pre-motivic structure), with $\wedge^2\mathfrak{M}_f$ mapping to $\eta O_K[1/S](1-k')$, for some integral ideal η. Since comparison maps are functorial, δ is the scalar (up to units in $O_K[1/S]$) giving the comparison map from $K(1-k')_B\otimes_K\mathbb{C}$ to $K(1-k')_{dR}\otimes_K\mathbb{C}$ with respect to the natural integral bases. This is $(2\pi i)^{1-k'}$ (c.f. 1.1.3 of [DFG1]). □

We shall need the elements $\mathfrak{M}_{f,\lambda}$ of the S-integral premotivic structure, for each prime λ of O_K. These are as in 1.6.2 of [DFG1]. For each λ, $\mathfrak{M}_{f,\lambda}$ is a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice in $M_{f,\lambda}$. Taking symmetric squares, we get $\mathfrak{M}'_{f,\lambda}$, a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice in $M'_{f,\lambda}$.

Let $A_\lambda:=M_{f,\lambda}/\mathfrak{M}_{f,\lambda}$, and $A[\lambda]:=A_\lambda[\lambda]$ the λ-torsion subgroup. Similarly, let $A'_\lambda:=M'_{f,\lambda}/\mathfrak{M}'_{f,\lambda}$, and $A'[\lambda]=A'_\lambda[\lambda]$. Let $\check{A}'_\lambda:=\check{M}'_{f,\lambda}/\check{\mathfrak{M}}'_{f,\lambda}$, where $\check{M}'_{f,\lambda}$ and $\check{\mathfrak{M}}'_{f,\lambda}$ are the vector space and O_λ-lattice dual to $M'_{f,\lambda}$ and $\mathfrak{M}'_{f,\lambda}$ respectively, with the natural $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action. Let $A':=\oplus_\lambda A'_\lambda$, etc.

Following [BK] (Section 3), for $p\neq\ell$ (including $p=\infty$) let

$$H^1_f(\mathbb{Q}_p,M'_{f,\lambda}(t))=\ker\big(H^1(D_p,M'_{f,\lambda}(t))\to H^1(I_p,M'_{f,\lambda}(t))\big).$$

Here D_p is a decomposition subgroup at a prime above p, I_p is the inertia subgroup, and $M'_{f,\lambda}(t)$ is a Tate twist of $M'_{f,\lambda}$, etc. The cohomology is for continuous cocycles and coboundaries. For $p=\ell$ let

$$H^1_f(\mathbb{Q}_\ell,M'_{f,\lambda}(t))=\ker\big(H^1(D_\ell,M'_{f,\lambda}(t))\to H^1(D_\ell,M'_{f,\lambda}(t)\otimes_{\mathbb{Q}_\ell}B_{\mathrm{crys}})\big).$$

(See Section 1 of [BK], or §2 of [Fo], for the definition of Fontaine's ring B_{crys}.) Let $H^1_f(\mathbb{Q},M'_{f,\lambda}(t))$ be the subspace of those elements of $H^1(\mathbb{Q},M'_{f,\lambda}(t))$ that, for all primes p, have local restriction lying in $H^1_f(\mathbb{Q}_p,M'_{f,\lambda}(t))$. There is a natural exact sequence

$$0\longrightarrow \mathfrak{M}'_{f,\lambda}(t)\longrightarrow M'_{f,\lambda}(t)\xrightarrow{\pi} A'_\lambda(t)\longrightarrow 0.$$

Let $H^1_f(\mathbb{Q}_p,A'_\lambda(t))=\pi_*H^1_f(\mathbb{Q}_p,M'_{f,\lambda}(t))$. Define the λ-Selmer group $H^1_f(\mathbb{Q},A'_\lambda(t))$ to be the subgroup of elements of $H^1(\mathbb{Q},A'_\lambda(t))$ whose local restrictions lie in $H^1_f(\mathbb{Q}_p,A'_\lambda(t))$ for all primes p. Note that the condition at $p=\infty$ is superfluous unless $\ell=2$. Define the Shafarevich-Tate group

$$Ш(t)=\bigoplus_\lambda\frac{H^1_f(\mathbb{Q},A'_\lambda(t))}{\pi_*H^1_f(\mathbb{Q},M'_{f,\lambda}(t))}.$$

Conjecture 5.2 (Case of Bloch-Kato)**.** *Suppose that* $1\le r\le k'-1$ *is odd, and* $t=r+k'-1$. *Then we have the following equality of fractional ideals of* $O_K[1/S]$*:*

$$\frac{D_f(t)}{(2\pi i)^{2t}\Omega}=\frac{\prod_{p\le\infty}c_p(t)\ \#Ш(t)}{\#H^0(\mathbb{Q},A'(t))\#H^0(\mathbb{Q},\check{A}'(1-t))}. \tag{3}$$

We omit the definitions of the factors $c_p(t)$, but note that, since M_f has "everywhere good reduction", $c_p(t)$ is at worst a power of p (or of 2 if $p = \infty$). Furthermore, a direct application of Theorem 4.1(iii) of [BK] shows that $c_p(t)$ is trivial for all finite $p > 2k'-1$ (the length of the Hodge filtration of M'_f). Therefore if we are looking at the λ part of the conjecture, with $\lambda \mid \ell > 2k'$, then these factors need not concern us. Note that $A'[\lambda](k'-1) \simeq \mathrm{End}^0(A[\lambda])$ (endomorphisms of trace 0), and that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariants correspond to endomorphisms commuting with the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action. Unless possibly if $\ell \leq k'+1$ or $\ell \mid B_{k'}$, $A[\lambda]$ is absolutely irreducible (by Lemma 8 of [SD]), and it follows from Schur's lemma, in most cases, that the factors in the denominator have trivial λ part. The exception is if $A[\lambda]$ is isomorphic to a non-trivial twist, but this can only happen if $\ell = 2k'-1$ and $r = k'-1$. (See the last section of [DH] for more details.)

It is more convenient to use $\langle f, f\rangle$ than Ω, so we consider the relation between the two. Bearing in mind §6 of [Hi1], using Lemma 5.1.6 of [De] and the latter part of 1.5.1 of [DFG1], one recovers the well-known fact that, up to S-units,

$$\langle f, f\rangle = i^{k'-1}\Omega^+\Omega^- c(f), \tag{4}$$

where $c(f)$, the "cohomology congruence ideal", is, as the cup-product of basis elements for $\mathfrak{M}_{f,B}$, an integral ideal. Recall that by Lemma 5.1 above,

$$\Omega = 2(2\pi i)^{1-k'}\Omega^+\Omega^-.$$

Therefore (3) becomes, for $t = r + k' - 1$ with $1 \leq r \leq k'-1$ odd,

$$\frac{D_f(r+k'-1)}{\pi^{2r+k'-1}\langle f, f\rangle} = \frac{\prod_{p\leq\infty} c_p(t)\ \#\text{Ш}(t)}{\#H^0(\mathbb{Q}, A'(t))\#H^0(\mathbb{Q}, \check{A}'(1-t))c(f)}. \tag{5}$$

Hence any large prime (e.g. $\ell = 373, 839, 2243$ or 9385577) dividing the left-hand side should be the order of an element in $\text{Ш}(r+k'-1)$. We shall construct such elements using the Galois interpretation of the congruences in §4, for $r = k'-1-j$.

6. Galois representations

The following is part of Theorem I of [W]

Proposition 6.1 (Weissauer). *Suppose that Π is a unitary, irreducible, automorphic representation of* $\mathrm{GSp}(4, \mathbb{A}_\mathbb{Q})$ *for which Π_∞ belongs to the discrete series of weight (k_1, k_2). Let S denote the set of ramified places of the representation Π. Put $w = k_1 + k_2 - 3$. Then there exists a number field E such that*

(1) *for any prime $p \notin S$, if $L_p(p^{-s}) = L_p(\Pi_p, s - w/2)$ is the (shifted) local factor in the spinor L-function, then $L_p(X)^{-1} \in E[X]$;*
(2) *for any prime λ of O_E, there exists a finite extension K of E (and K_λ of E_λ), and a 4-dimensional semisimple Galois representation*

$$\rho_{\Pi,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(4, K_\lambda),$$

unramified outside $S \cup \{\ell\}$ (where $\lambda \mid \ell$), such that for each prime $p \notin S \cup \{\ell\}$,

$$L_p(\Pi_p, s - w/2) = \det(I - \rho_{\Pi,\lambda}(\mathrm{Frob}_p^{-1})p^{-s})^{-1}.$$

The main theorems in [W] depend on hypotheses (A and B), whose proofs are currently only in preprint form. Section 3 of [U] gives an alternative proof of the above, in the case that $\ell \notin S$, $k_1 > k_2 > 3$ and the Newton polygon of $L_\ell(X)^{-1}$ has four distinct ℓ-slopes.

Take a Hecke eigenform $F \in S_{j,k}$ (with $k \geq 3$). To F may be associated a cuspidal automorphic form $\Phi_F \in L^2_0(Z(\mathbb{A}_\mathbb{Q})\mathrm{GSp}(4,\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{A}_\mathbb{Q}))$. This "lifting procedure" is described in detail in §3 of [AS] (§3.1 for the scalar-valued case, §3.5 for the vector-valued case). In the scalar-valued case one easily recovers F from Φ_F. I am grateful to R. Schmidt for explaining to me how this can be done also in the vector-valued case, even though F is vector-valued and Φ_F is scalar-valued. The translates of Φ_F under a maximal compact subgroup $K_\infty \simeq U(2)$ of $\mathrm{GSp}(4,\mathbb{R})$ generate a space of functions realising a representation of $U(2)$ isomorphic to the restriction of the contragredient of $\rho_{j,k}$. (This is the "lowest K_∞-type" in a discrete series representation of $\mathrm{GSp}(4,\mathbb{R})$.) One may assume that Φ_F itself corresponds to a highest-weight vector (by choosing the linear form L in §3.5 of [AS] to be a highest weight vector). Let V be the space of $\rho_{j,k}$ and V^* the dual space. Given Φ_F, one obtains a V-valued function $\tilde{\Phi}$ (from which the V-valued F may be recovered) as follows. It suffices to specify the value (given $g \in \mathrm{GSp}(4,\mathbb{A}_\mathbb{Q})$) of $\tilde{\Phi}(g)$ on any $f \in V^*$. But f corresponds to some K_∞-translate Φ_f of Φ_F, and we may take $(\tilde{\Phi}(g))(f) = \Phi_f(g)$.

Let Π_F be any irreducible constituent of the unitary representation of $\mathrm{GSp}(4,\mathbb{A}_\mathbb{Q})$ generated by right translates of Φ_F, as in 3.4 of [AS]. They are all isomorphic, in fact this unitary representation is expected to be irreducible already. In any case, we may choose F in such a way that Φ_F generates an irreducible representation, and then Φ_F may be recovered from Π_F as in Lemma 3.4.2 of [AS]. (In the vector-valued case, as explained to me by R. Schmidt, one must add, to the conditions in Lemma 3.4.2 of [AS], one about being killed by compact positive roots.) A consequence of the preceding discussion is the following, which will be used after Proposition 6.4.

Lemma 6.2. *If the Hecke eigenspace of $F \in S_{j,k}$ is 1-dimensional, then the multiplicity of Π_F in $L^2_0(Z(\mathbb{A}_\mathbb{Q})\mathrm{GSp}(4,\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{A}_\mathbb{Q}))$ is 1.*

Proposition 6.3. *Let $f = \sum_{n=1}^\infty a_n q^n \in S_{k'}$ be a normalised Hecke eigenform. Let $\lambda \mid \ell > 2k'$ in a field K large enough to contain the Hecke eigenvalues of f. Let $F \in S_{j,k}$ (with $j+k = k'$, $k \geq 4$ and $j \geq 0$ even) be a Hecke eigenform, and suppose that,*

(1) *for all primes p,*

$$\nu(p,F) \equiv \nu(p,[f]_j) = a_p(1+p^{k-2}) \pmod{\lambda};$$

(2) *$\ell \nmid B_{k'}$ (a Bernoulli number).*

Then the following hold.

(1) *If K is sufficiently large then there exists a 4-dimensional semisimple Galois representation*

$$\rho_{F,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(4, K_\lambda),$$

unramified outside $\{\ell\}$, such that for each prime $p \neq \ell$, $\det(I-\rho_{F,\lambda}(\mathrm{Frob}_p^{-1})p^{-s})^{-1}$ is the local factor in the (shifted) spinor L function of F.

(2) *Choose a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant O_λ-lattice S in W (the space of $\rho_{F,\lambda}$) and consider the representation $\overline{\rho}_{F,\lambda}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $S/\lambda S$. Then the composition factors of $\overline{\rho}_{F,\lambda}$ are $\overline{\rho}_{f,\lambda}$ and $\overline{\rho}_{f,\lambda}(2-k)$.*

(3) *ρ_F is absolutely irreducible.*

Proof. (1) This is a direct consequence of Proposition 6.1, applied to Π_F. Note that here $k_1 = j+k, k_2 = k$, and the condition $k \geq 3$ (which, of course, implies $k \geq 4$) is necessary to ensure that Π_∞ is discrete series. Also Π_F is unramified at all primes p, since F is for the full modular group Γ_2.

(2) The congruence, with conclusion (1), implies that $\mathrm{tr}(\overline{\rho}_{F,\lambda}(\mathrm{Frob}_p^{-1})) = (1 + p^{k-2})\mathrm{tr}(\overline{\rho}_{f,\lambda}(\mathrm{Frob}_p^{-1}))$. It remains to observe that $\overline{\rho}_{f,\lambda}$ is (absolutely) irreducible, a consequence of $\ell > k' + 1$ and $\ell \nmid B_{k'}$ (by Lemma 8 of [SD]).

(3) If $\rho_{F,\lambda}$ were (absolutely) reducible, then its semisimplification would be on the list in the proof of Theorem 3.2.1 of [SU]. The fact that the composition factors of $S/\lambda S$ are $\overline{\rho}_{f,\lambda}$ and $\overline{\rho}_{f,\lambda}(2-k)$ is incompatible with Cas A, and the condition $\ell > k - 1$ excludes Cas B (v), leaving only Cas B (iv), and hence implying that π_F is cuspidal associated to a Klingen parabolic. By Corollary 4.5 of [PS], this is impossible.

□

Proposition 6.4. *In the situation of Proposition 6.3, suppose further that there is a symplectic form on W with respect to which the image of $\rho_{F,\lambda}$ is contained in* $\mathrm{GSp}(4, K_\lambda)$.

(1) *There is a choice of Galois invariant O_λ-lattice S in W such that $B[\lambda] := S/\lambda S$ has a submodule, but not a quotient, realising the representation $\overline{\rho}_{f,\lambda}(2-k)$.*

(2) *Let*

$$0 \longrightarrow A[\lambda](2-k) \xrightarrow{\ i\ } B[\lambda] \xrightarrow{\ \pi\ } A[\lambda] \longrightarrow 0.$$

be the exact sequence of $\mathbb{F}_\lambda[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-modules implied by (1). Let s be any section for π as a linear map of $\mathbb{F}_\lambda$-vector spaces. Then there is an $\mathbb{F}_\lambda$-valued skew-symmetric form $(\,,)$ on $B[\lambda]$ such that

(a) *each of $i(A[\lambda](2-k))$ and $s(A[\lambda])$ is an isotropic subspace;*

(b) *if $\{v_1, v_2\}$ is a basis for $A[\lambda](2-k)$ and $\{w_1, w_2\}$ is the dual basis of $A[\lambda]$ (relative to the natural duality $A[\lambda] \times A[\lambda] \to \mathbb{F}_\lambda(1-j-k)$) then*

$$(s(w_i), i(v_j)) = \delta_{ij}.$$

Recall that the composition factors of $B[\lambda]$ are isomorphic to $A[\lambda]$ and $A[\lambda](2-k)$. The proof of Proposition 6.4 is now a direct application of Theorem 1.1 and Proposition 1.1 of [U]. In our situation, condition (iv) of his Proposition 1.1 amounts to $\chi_\ell^{3-j-2k} \neq \chi_\ell^{1-j-k}$ (where χ_ℓ is the $\pmod{\ell}$ cyclotomic character), which holds because $\ell - 1 > k - 2$. The statements about isotropy are a consequence of this condition. The fact that $\overline{\rho}_{f,\lambda}(2-k)$ cannot be both a quotient and a submodule of $B[\lambda]$, i.e. that s cannot be a splitting of $\mathbb{F}_\lambda[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-modules, is a consequence of the absolute irreducibility of $\rho_{F,\lambda}$.

The condition that the image of $\rho_{F,\lambda}$ should be contained in $\mathrm{GSp}(4, K_\lambda)$, is expected to hold in general. See the remark following Theorem IV of [W] for how it might be possible to prove it. According to Theorem IV of [W], the condition does hold if the multiplicity of Π_F in $L_0^2(Z(\mathbb{A}_\mathbb{Q})\mathrm{GSp}(4,\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{A}_\mathbb{Q}))$ is 1, or even if Π_F is "weakly equivalent" to a representation for which the multiplicity is 1. Alternatively, Proposition 3.5 of [U] provides a proof (given the existence of $\rho_{F,\lambda}$) that if the multiplicity of Π_F is 1 then the condition (that the image of $\rho_{F,\lambda}$ is contained in $\mathrm{GSp}(4, K_\lambda)$) holds. Thanks to Lemma 6.2, we know that the condition will hold whenever the Hecke eigenspace of F is 1-dimensional. This includes, of course, those cases where $\dim S_{j,k} = 1$, i.e. $k' = 16$, $\ell = 373, 839$ or 2243. It also includes the scalar-valued cases $(k', \ell) = (20, 71), (22, 61), (22, 103)$ (where F

spans a complement to the Maass space of Saito-Kurokawa lifts, see §3 of [M]) and $(26,163),(26,187273)$ (by the calculations in [Sk]). If the assumption (about the endoscopic contribution) underlying the calculations of [FvdG] is justified, then the condition also holds in the cases $(j,k,\ell)=(2,16,541),(2,16,2879),(4,14,19501)$. In these examples $\dim S_{j,k}=2$, and the eigenvalues of $T(2)$ are distinct quadratic conjugates of each other, as described in §27 of [vdG].

Following Urban, in the situation of Proposition 6.4, we define a cocycle $C:\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to\mathrm{Hom}(A[\lambda],A[\lambda](2-k))$ by

$$C(g)(w)=i^{-1}(g.s(g^{-1}.w)-s(w)).$$

This cocycle is not a coboundary, since s is not a splitting of $\mathbb{F}_\lambda[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-modules. A different choice of s would only change $C(g)$ by a coboundary, so we have a well-defined, non-zero cohomology class $[C]\in H^1(\mathbb{Q},\mathrm{End}(A[\lambda])(2-k))$.

Lemma 6.5. $[C]\in H^1(\mathbb{Q},\mathrm{End}^0(A[\lambda])(2-k))=H^1(\mathbb{Q},A'[\lambda](j+1))$.

Proof. Let $C(g)(w_1)=av_1+cv_2$ and $C(g)(w_2)=bv_1+dv_2$. Since $s(A[\lambda])$ is isotropic and g preserves the symplectic form (up to scalar multiples), we must have

$$\langle s(w_1)+av_1+cv_2, s(w_2)+bv_1+dv_2\rangle=0,$$

hence $b-c=0$. Under the identification of $A[\lambda]$ with $A[\lambda](2-k)$ obtained by just forgetting the twist, $\{w_1,w_2\}$ corresponds to $\{v_2,-v_1\}$ (if we scale the v_i appropriately), so the condition $b-c=0$ is that the trace of the endomorphism $C(g)$ of $A[\lambda]$ is 0. Alternatively, using the duality to identify $\mathrm{Hom}(A[\lambda],A[\lambda])(2-k)$ with $A[\lambda]\otimes A[\lambda](j+k-1+(2-k))=A[\lambda]\otimes A[\lambda](j+1)$,

$$C(g)=av_1\otimes v_1+cv_1\otimes v_2+bv_2\otimes v_1+dv_2\otimes v_2,$$

so the condition $b=c$ tells us that C takes values in $\mathrm{Sym}^2(A[\lambda])(j+1)=A'[\lambda](j+1)$. □

7. Checking local conditions

With assumptions as in Proposition 6.4, we have, following Urban, constructed a non-zero element $[C]\in H^1(\mathbb{Q},A'[\lambda](j+1))$. As explained in §5, $H^0(\mathbb{Q},A'[\lambda](j+1))$ (and hence $H^0(\mathbb{Q},A'_\lambda(j+1))$) is trivial, given that $\ell>2k'$ and $\ell\nmid B_{k'}$, so $H^1(\mathbb{Q},A'[\lambda](j+1))$ injects into $H^1(\mathbb{Q},A'_\lambda(j+1))$, giving us a non-zero element c. We would like to show that $c\in H^1_f(\mathbb{Q},A'_\lambda(j+1))$, by showing that, for each prime p, the restriction to a decomposition subgroup satisfies $\mathrm{res}_p(c)\in H^1_f(\mathbb{Q}_p,A'_\lambda(j+1))$.

Lemma 7.1. $\mathrm{res}_p(c)\in H^1_f(\mathbb{Q}_p,A'_\lambda(j+1))$ *if* $p\neq\ell$.

Proof. When restricted to an inertia subgroup I_p, the exact sequence

$$0\longrightarrow A[\lambda](2-k)\xrightarrow{i}B[\lambda]\xrightarrow{\pi}A[\lambda]\longrightarrow 0.$$

splits, since each term is a trivial I_p-module. Hence the image of $[C]$ in $H^1(I_p,A'[\lambda](j+1))$ (and therefore of c in $H^1(I_p,A'_\lambda(j+1))$) is trivial. It follows easily, from this and the fact that $A'_\lambda(j+1)$ is unramified at p, that $\mathrm{res}_p(c)\in H^1_f(\mathbb{Q}_p,A'_\lambda(j+1))$. (See, for instance, Lemma 7.4 of [Br].) □

Contrary to the suggestion at the end of [Du], it is straightforward to prove the local condition at $p=\ell$. Note that Urban uses a different local condition at ℓ, and imposes an ordinarity condition. Recall that we are already assuming that

$\ell > 2(j+k)$ and $\ell \nmid B_{j+k}$. To prove the local condition at $p = \ell$, we need a somewhat stronger condition on ℓ.

Lemma 7.2. *If* $\ell > 2j + 3k - 2$ *then* $\mathrm{res}_\ell(c) \in H^1_f(\mathbb{Q}_\ell, A'_\lambda(j+1))$.

Proof. Both $\rho_{f,\lambda}$ and $\rho_{F,\lambda}$, restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$, are crystalline. (For a careful discussion of $\rho_{f,\lambda}$, referring to [Fa], see 1.2.5 of [DFG1]. For $\rho_{F,\lambda}$, see Theorem 3.2(ii) of [U], which refers to [Fa] and [CF].) It follows that there exist filtered Dieudonné O_λ-modules $\mathcal{D}$ and $\mathcal{E}$ such that $\mathbb{V}(\mathcal{D}) = \mathfrak{M}_{f,\lambda}$ and $\mathbb{V}(\mathcal{E}) = S$ (with S as in the proof of Proposition 6.3). For this, and for the definitions of the modified Fontaine-Lafaille functor $\mathbb{V}$ and the categories O_λ-$\mathfrak{M}\mathcal{F}^a$ of filtered Dieudonné O_λ-modules, see 1.1.2 of [DFG1]. We simply note that if $\mathcal{M}$ is a filtered Dieudonné O_λ-module (i.e. $\mathcal{M} \in O_\lambda$-$\mathfrak{M}\mathcal{F}$), and if $\mathrm{Fil}^a\mathcal{M} = \mathcal{M}$ and $\mathrm{Fil}^{a+\ell-1}\mathcal{M} = \{0\}$, then $\mathcal{M} \in O_\lambda$-$\mathfrak{M}\mathcal{F}^a$. Let 1_{FD} be such that $\mathbb{V}(1_{\mathrm{FD}}) = O_\lambda$. Below, we will need to work in a category containing both the dual of $\mathcal{D}$ (with graded pieces of degrees $1-j-k$ and 0) and $\mathcal{D}\{2-k\}$ (with graded pieces of degrees $k-2$ and $(k-2)+(j+k-1)$), as well as 1_{FD}. The condition on ℓ ensures that O_λ-$\mathfrak{M}\mathcal{F}^{1-j-k}$ is such a category. Let $\overline{\mathcal{D}}$ denote $\mathcal{D}/\lambda\mathcal{D}$, etc. Note that in §4 of [BK], $\mathbb{V}$ is "T".

The exact sequence

$$0 \longrightarrow A[\lambda](2-k) \xrightarrow{\ i\ } B[\lambda] \xrightarrow{\ \pi\ } A[\lambda] \longrightarrow 0.$$

is the image by $\mathbb{V}$ of an exact sequence

$$0 \longrightarrow \overline{\mathcal{D}}\{2-k\} \xrightarrow{\ i\ } \overline{\mathcal{E}} \xrightarrow{\ \pi\ } \overline{\mathcal{D}} \longrightarrow 0.$$

Apply $\mathrm{Hom}_{O_\lambda}(\overline{\mathcal{D}}, -)$ to this sequence, then pull back by the inclusion of 1_{FD} in $\mathrm{Hom}_{O_\lambda}(\overline{\mathcal{D}}, \overline{\mathcal{D}})$ to get an element of $\mathrm{Ext}^1(1_{\mathrm{FD}}, \mathrm{Hom}_{O_\lambda}(\overline{\mathcal{D}}, \overline{\mathcal{D}}\{2-k\})$. It is easy to check that this is sent by $\mathbb{V}$ to the extension of $\mathbb{F}_\lambda[\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)]$-modules corresponding to the class $\mathrm{res}_\ell([C])$.

Lemma 4.4 of [BK] gives a description of $\mathrm{Ext}^1(1_{\mathrm{FD}}, \mathcal{M})$ as a quotient of $\mathcal{M}$, namely $\mathcal{M}/(1-\phi^0)(\mathrm{Fil}^0\mathcal{M})$. From this it is obvious that the natural map from $\mathrm{Ext}^1(1_{\mathrm{FD}}, \mathrm{Hom}_{O_\lambda}(\mathcal{D}, \mathcal{D}\{2-k\}))$ to $\mathrm{Ext}^1(1_{\mathrm{FD}}, \mathrm{Hom}_{O_\lambda}(\overline{\mathcal{D}}, \overline{\mathcal{D}}\{2-k\}))$ is surjective. Application of $\mathbb{V}$ to any element of $\mathrm{Ext}^1(1_{\mathrm{FD}}, \mathrm{Hom}_{O_\lambda}(\mathcal{D}, \mathcal{D}\{2-k\}))$ produces an extension of O_λ by $\mathrm{Hom}(\mathfrak{M}_{f,\lambda}, \mathfrak{M}_{f,\lambda}(2-k))$, an invariant O_λ-lattice in a crystalline extension of K_λ by $\mathrm{Hom}(M_{f,\lambda}, M_{f,\lambda}(2-k))$. It is associated to a class in $H^1_f(\mathbb{Q}_\ell, \mathrm{Hom}(\mathfrak{M}_{f,\lambda}, \mathfrak{M}_{f,\lambda}(2-k)))$ (the inverse image of $H^1_f(\mathbb{Q}_\ell, \mathrm{Hom}(M_{f,\lambda}, M_{f,\lambda}(2-k))))$. [The subscript "f" is justified by the fact that the extension is crystalline.] Hence $\mathrm{res}_\ell([C]) \in H^1(\mathbb{Q}_\ell, \mathrm{Hom}(A[\lambda], A[\lambda](2-k)))$ is in the image of $H^1_f(\mathbb{Q}_\ell, \mathrm{Hom}(\mathfrak{M}_{f,\lambda}, \mathfrak{M}_{f,\lambda}(2-k)))$. It follows that $\mathrm{res}_\ell(c) \in H^1_f(\mathbb{Q}_\ell, A'_\lambda(j+1))$. □

What we have actually proved may be summarised as follows.

Proposition 7.3. *Let* $f = \sum_{n=1}^\infty a_n q^n \in S_{k'}$ *be a normalised Hecke eigenform. Let* $F \in S_{j,k}$ *(with* $j+k = k'$, $j \geq 0$ *even,* $k \geq 4$*) be a Hecke eigenform, and suppose that, for all primes* p,

$$\nu(p, F) \equiv \nu(p, [f]_j) = a_p(1 + p^{k-2}) \pmod{\lambda}.$$

Here, $\lambda \mid \ell > 2j + 3k - 2$ *in a field* K *large enough to contain the Hecke eigenvalues of* f. *Suppose also that* $\ell \nmid B_{k'}$. *Take* K *large enough for the representation*

$$\rho_{F,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(4, K_\lambda)$$

in Proposition 6.3 to exist. Suppose that the image of $\rho_{F,\lambda}$ *is contained in* $\mathrm{GSp}(4, K_\lambda)$. *(See the discussion following Proposition 6.4.)*

Then there exists a non-zero element in the Bloch-Kato Selmer group $H^1_f(\mathbb{Q}, A'_\lambda(j+1))$.

In the situation of Proposition 7.3, we have obtained a non-zero element $c \in H^1_f(\mathbb{Q}, A'_\lambda(j+1))$. Recall that j is even. The point $s = j+1$ is critical for $D_f(s)$ (i.e. the pre-motivic structure $M'_f(j+1)$ is critical), and non-central, so conjecturally $H^1_f(\mathbb{Q}, M'_{f,\lambda}(j+1))$ is trivial. Admitting this, we have an element of λ-torsion in $Ш(j+1)$. The functional equation pairs $s = j+1$ with $s = 2k' - 1 - (j+1) = 2k' - 2 - j$. Then by [Fl] we get a non-zero element of λ-torsion in $Ш(2k'-2-j)$, as required by the Bloch-Kato conjecture in the examples covered by Proposition 4.4. (Alternatively, just use the functional equation and look at $D_f(j+1)$ instead.)

8. Global torsion

We retain the notation of the previous three sections. There should exist a premotivic structure M_F over $\mathbb{Q}$ with coefficients in K (if K is chosen sufficiently large) such that W is the λ-adic realisation $M_{F,\lambda}$. The L-function attached to $M'_F := \wedge^2 M_F$ (equivalently to $\wedge^2 W$) would be $L(M'_F, s) = \zeta(s-(2k+j-3))L(F, s-(2k+j-3), \mathrm{St})$, where $L(F, s, \mathrm{St})$ is the "standard L-function". The right half of the set of critical points for $L(M'_F, s)$ should be $s = m + (2k+j-3)$ with m even and $0 < m \leq k-2$. The Bloch-Kato conjecture reads (with $t := m + (2k+j-3)$)

$$\frac{L(M'_F, m+(2k+j-3))}{(2\pi i)^{3m+6k+3j-9}\Omega'} = \frac{\prod_p c'_p(t)\ \#Ш'(t)}{\#H^0(\mathbb{Q}, B'(t))\ \#H^0(\mathbb{Q}, \check{B}'(1-t))}. \tag{6}$$

where the various terms are defined analogously to those in §5. It is really the λ part of this formula that concerns us, so it suffices to say that we use the lattice $\wedge^2 S$ in $\wedge^2 W$, so that $B'[\lambda] = \wedge^2 B[\lambda]$, with S and $B[\lambda]$ as in §6.

Putting ourselves in the situation of §6, recall the exact sequence

$$0 \longrightarrow A[\lambda](2-k) \xrightarrow{i} B[\lambda] \xrightarrow{\pi} A[\lambda] \longrightarrow 0.$$

of $\mathbb{F}_\lambda[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-modules. We see then that $B'[\lambda] = \wedge^2 B[\lambda]$ has a 1-dimensional submodule isomorphic to $(\wedge^2 A[\lambda])(4-2k) \simeq \mathbb{F}_\lambda(1-j-k)(4-2k) \simeq \mathbb{F}_\lambda(3-j-2k)(2-k)$. Hence $H^0(\mathbb{Q}, B'(t))$, with $t = (k-2) + (2k+j-3)$, has a non-trivial λ-part, which should contribute to the first term in the denominator of the right hand side of (6). Since $A[\lambda]$ is irreducible (as in the proof of Proposition 6.3), the only 1-dimensional composition factors of $B'[\lambda]$ are $\mathbb{F}_\lambda(5-j-3k)$ and $\mathbb{F}_\lambda(1-j-k)$. It is easy to check that they contribute nothing to any of the other denominator terms for any critical value in the right half of the range. The λ-parts of all the $c'_p(t)$ should be trivial (if $\ell > j+3k-3$), so short of the unlikely event that λ appears in one of the $\#Ш'(t)$ terms, the Bloch-Kato conjecture leads us to expect that for any even m with $0 < m < k-2$,

$$\mathrm{ord}_\lambda\left(\frac{\pi^{3(m-(k-2))}L(F, k-2, \mathrm{St})}{L(F, m, \mathrm{St})}\right) < 0.$$

(We suppose also that $\ell \nmid B_{k-2}$, so we may disregard the $\zeta(k-2)$ factor.)

9. The rightmost critical value

Proposition 9.1. *Suppose that $S_{j+k}(\Gamma_1)$ is 1-dimensional, and let f be a normalised generator, with $k \geq 3$. Let $k' = j + k$. Suppose that*

$$\mathrm{ord}_\ell \left(D_f(2k' - 2 - j)/(f, f)\pi^{3k'-3-2j} \right) = s > 0,$$

and that $\ell > 2k + j$. Suppose also that the Fourier coefficients of the Klingen-Eisenstein series $[f]_j \in M_{j,k}(\Gamma_2)$ become integral at ℓ when multiplied by ℓ^s, but not when multiplied by any smaller power of ℓ. Take an even m with $0 < m \leq k - 2$.

(1) *If $m = k-2$ then there exists a Hecke eigenform $F \in S_{j,k}$, and a prime ideal $\lambda \mid \ell$ in a suitable field, such that $\mathrm{ord}_\lambda \left(L(F, m, \mathrm{St})/\pi^{3m+2k+j-3}(F, F) \right) \leq -s$, with F normalised in such a way that its Fourier coefficients are integral at λ but not all divisible by λ.*

(2) *If $m < k-2$ and the Hecke eigenvalues of F are not all congruent $\pmod{\lambda}$ to those of some other Hecke eigenform in $S_{j,k}$, then*

$$\mathrm{ord}_\lambda \left(L(F, m, \mathrm{St})/\pi^{3m+2k+j-3}(F, F) \right) \geq 0.$$

Proof. (1) In our situation, the case $p = q = 2$ of Proposition 4.4 of [BSY] states that

$$L^{(j)} E_k^{(4)}(Z, W) = \alpha_{k,j} \left[C_{k,j,1} \Lambda(f)\, [f]_j(Z) [f]_j(W) + C_{k,j,2} \sum_F \Lambda(F)\, F(Z) F(W) \right],$$

where the sum is over a basis of Hecke eigenforms in $S_{j,k}$, and the various terms will be discussed below. First, $\Lambda(f) = (\zeta(k)\zeta(2k-2)\zeta(2k-4))^{-1} D_f(2k' - 2 - j)/(f, f)$ and $\Lambda(F) = (\zeta(k)\zeta(2k-2)\zeta(2k-4))^{-1} L(k-2, F, \mathrm{St})/(F, F)$. The constants $\alpha_{k,j}$ (defined in Remark 4.3 of [BSY]) and $C_{k,j,1}, C_{k,j,2}$ (defined in Proposition 3.1 of [BSY]) are powers of π times explicit rational numbers involving factorials and products of small numbers. Given our assumption that ℓ is sufficiently large, the prime factorisations of these rational numbers do not involve ℓ. As noted immediately before Theorem 2.2 of [Ka] (this is where we use $k \geq 3$), the Fourier coefficients of $(\zeta(k)\zeta(2k-2)\zeta(2k-4))E_k^{(4)}$ are integral at any prime greater than $2k-1$. (The ζ factors are incorporated into his definition of the Eisenstein series, not the normalised one we use here.) The differential operator $L^{(j)}$ maps $M_{0,k}(\Gamma_4)$ to $M_{j,k}(\Gamma_2) \otimes M_{j,k}(\Gamma_2)$, and the formula (2.7) of [BSY] shows that it is integral at any prime greater than $k + j - 1$. Hence the modified left hand side $(\zeta(k)\zeta(2k-2)\zeta(2k-4))L^{(j)} E_k^{(4)}(Z, W)$ has Fourier coefficients integral at ℓ. Since $[f]_j$ contributes twice to the first term on the right hand-side, but $\Lambda(f)$ can only cancel one factor of ℓ^s, some Fourier coefficient c in the first term on the right hand side is such that $\mathrm{ord}_\ell(c) = -s$ (ignoring powers of π). Considering the remaining sum on the right hand side, this is impossible unless F as stated exists.

(2) For $m \neq k-2$ there is (see (4.1) of [Koz]) a similar formula without the term involving $[f]_j$, with $\Lambda(F)$ involving $L(m, F, \mathrm{St})$, and $L^{(j)} \mathcal{D}_{m+2}^{k-2-m} E_{m+2}^{(4)}(Z, W)$ on the left. The differential operator $\mathcal{D}_{m+2}^{k-2-m}$ is as in [Bö], and using the definitions and formulas in that paper it is easy to see that it has the necessary integrality properties to show that again the (modified) left hand side will be integral at ℓ, this time forcing $\mathrm{ord}_\lambda(L(m, F, \mathrm{St})/\pi^{3m+2k+j-3}(F, F)) \geq$

0. Here F is the particular F appearing in (1), and the integrality of this one term is easily deduced from that of the sum, using the no-congruence condition. The absence of a term involving $[f]_j$ is due to the fact that $\mathcal{D}_{m+2}^{k-2-m} E_{m+2}^{(4)}(Z, W)$ is a cusp form, by Satz 7 of [Bö]. □

In those cases with $j + k = 16$ and $\ell = 373, 839$ or 2243, and also in the scalar valued cases $(k', \ell) = (20, 71), (22, 61), (22, 103), (26, 163)$ and $(26, 187273)$, the conditions of the proposition are certainly satisfied. In all these cases, the proposition confirms the prediction about the ratio of critical standard L-values, derived from the Bloch-Kato conjecture in the previous section.

Note that, as observed in the penultimate paragraph of §8 of [Du], there is an analogous phenomenon in the genus one case. Eisenstein primes in the denominators of rightmost normalised critical values of $D_f(s)$ are accounted for by global torsion.

References

[A] T. Arakawa, Vector valued Siegel's modular forms of degree two and the associated Andrianov L-functions, *Manuscripta Math.* **44** (1983), 155–185.

[AS] M. Asgari, R. Schmidt, Siegel modular forms and representations, *Manuscripta Math.* **104** (2001), 173–200.

[BK] S. Bloch, K. Kato, L-functions and Tamagawa numbers of motives, The Grothendieck Festschrift Volume I, 333–400, Progress in Mathematics, 86, Birkhäuser, Boston, 1990.

[Bö] S. Böcherer, Über die Fourier-Jacobi-Entwicklung Siegelscher Eisensteinreihen II, *Math. Z.* **189** (1985), 81–110.

[BSY] S. Böcherer, T. Satoh, T. Yamazaki, On the pullback of a differential operator and its application to vector valued Eisenstein series, *Comment. Math. Univ. St. Paul.* **42** (1992), 1–22.

[Br] J. Brown, Saito-Kurokawa lifts and applications to the Bloch-Kato conjecture, *Compositio Math.* **143** (2007), 290–322.

[CF] C.-L. Chai, G. Faltings, *Degeneration of Abelian Varieties,* Ergeb. Math. Grenzgeb. (3), Vol. 22, Springer, Berlin, 1990.

[C] H. Cohen, Sums involving the values at negative integers of L-functions of quadratic characters, *Math. Ann.* **217** (1975), 271–285.

[De] P. Deligne, Valeurs de Fonctions L et Périodes d'Intégrales, *AMS Proc. Symp. Pure Math.,* Vol. 33 (1979), part 2, 313–346.

[DeSe] P. Deligne, J.-P. Serre, Formes modulaires de poids 1, *Ann. Sci. E.N.S.* **7** (1974), 507–530.

[DFG1] F. Diamond, M. Flach, L. Guo, The Tamagawa number conjecture of adjoint motives of modular forms, *Ann. Sci. École Norm. Sup. (4)* **37** (2004), 663–727.

[DFG2] F. Diamond, M. Flach, L. Guo, The Bloch-Kato conjecture for adjoint motives of modular forms, *Math. Res. Lett.* **8** (2001), 437–442.

[Du] N. Dummigan, Symmetric square L-functions and Shafarevich-Tate groups, *Experiment. Math.* **10** (2001), 383–400.

[DH] N. Dummigan, B. Heim, Symmetric square L-values and dihedral congruences for cusp forms, submitted. http://neil-dummigan.staff.shef.ac.uk/papers.html

[EI] W. Eholzer, T. Ibukiyama, Rankin-Cohen type differential operators for Siegel modular forms, *Internat. J. Math.* **9** (1998), 443–463.

[FvdG] C. Faber, G. van der Geer, Sur la cohomologie des systèmes locaux sur les espaces de modules des courbes de genre 2 et des surfaces abéliennes, I, II, *C. R. Math. Acad. Sci. Paris* **338** (2004), 381–384 and 467–470.

[Fa] G. Faltings, Crystalline cohomology and p-adic Galois representations, *in* Algebraic analysis, geometry and number theory (J. Igusa, ed.), 25–80, Johns Hopkins University Press, Baltimore, 1989.

[Fl] M. Flach, A generalisation of the Cassels-Tate pairing, *J. reine angew. Math.* **412** (1990), 113–127.

[Fo] J.-M. Fontaine, Le corps de périodes $\mathfrak{p}$-adiques, *Astérisque* **223** (1994), 59–111.
[FL] J.-M. Fontaine, G. Lafaille, Construction de représentations $\mathfrak{p}$-adiques, *Ann. Sci. E.N.S.* **15** (1982), 547–608.
[Ha] G. Harder, A congruence between a Siegel and an elliptic modular form, manuscript, 2003, *reproduced in* The 1-2-3 of Modular Forms (J. H. Bruinier et. al.), 247–262, Springer-Verlag, Berlin Heidelberg, 2008.
[Hi1] H. Hida, Congruences of cusp forms and special values of their zeta functions, *Invent. Math.* **63** (1981), 225–261.
[Hi2] H. Hida, Control theorems of coherent sheaves on Shimura varieties of PEL type, *J. Inst. Math. Jussieu* **1** (2002), 1–76.
[Ka] H. Katsurada, Congruence of Siegel modular forms and special values of their standard zeta functions, *Math. Z.* **259** (2008), 97–111.
[Kl] H. Klingen, Zum Darstellungssatz fur Siegelsche Modulformen, *Math. Z.* **102** (1967), 30–43.
[Koz] N. Kozima, On special values of standard L-functions attached to vector valued Siegel modular forms, *Kodai Math. J.* **23** (2000), 255–265.
[Ku] N. Kurokawa, Congruences between Siegel modular forms of degree 2, *Proc. Japan Acad.* **55A** (1979), 417–422.
[L] G. Laumon, Fonctions zêtas des variétés de Siegel de dimension trois, *Astérisque* **302** (2005), 1–66.
[MTT] B. Mazur, J. Tate, J. Teitelbaum, On $\mathfrak{p}$-adic analogues of the conjectures of Birch and Swinnerton-Dyer, *Invent. Math.* **84** (1986), 1–48.
[M] S. Mizumoto, Congruences for eigenvalues of Hecke operators on Siegel modular forms of degree two, *Math. Ann.* **275** (1986), 149–161.
[PS] A. Pitale, R. Schmidt, Ramanujan-type results for Siegel cusp forms of degree 2, preprint, 2006. http://www.math.ou.edu/~rschmidt/papers/GSp4_satake.pdf. To appear in J. Ramanujan Math. Soc.
[R] K. Ribet, A modular construction of unramified $\mathfrak{p}$-extensions of $\mathbb{Q}(\mu_{\mathfrak{p}})$, *Invent. Math.* **34** (1976), 151–162.
[Sa] T. Satoh, On certain vector valued Siegel modular forms of degree two, *Math. Ann.* **274** (1986), 335–352.
[SU] C. Skinner, E. Urban, Sur les déformations $\mathfrak{p}$-adiques de certaines représentations automorphes, *J. Inst. Math. Jussieu* **5** (2006), 629–698.
[Sk] N.-P. Skoruppa, Computations of Siegel modular forms of genus two, *Math. Comp.* **58** (1992), 381–398.
[St] W. Stein, The Modular Forms Database: Tables, http://modular.ucsd.edu/Tables/tables.html
[SD] H. P. F. Swinnerton-Dyer, On $\mathfrak{l}$-adic representations and congruences for coefficients of modular forms, Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, 1972), pp. 1–55. Lecture Notes in Math., Vol. 350, Springer, Berlin, 1973.
[Ta] R. Taylor, On the ℓ-adic cohomology of Siegel threefolds, *Invent. Math.* **114** (1993), 289–310.
[Ts] R. Tsushima, An explicit dimension formula for the spaces of generalized automorphic forms with respect to $\mathrm{Sp}(2,\mathbb{Z})$, *Proc. Japan Acad.* **59A** (1983), 139-142.
[U] E. Urban, Selmer groups and the Eisenstein-Klingen ideal, *Duke Math. J.* **106** (2001), 485–525.
[vdG] G. van der Geer, Siegel Modular Forms and Their Applications, *in* The 1-2-3 of Modular Forms (J. H. Bruinier et. al.), 181–245, Springer-Verlag, Berlin Heidelberg, 2008.
[W] R. Weissauer, Four dimensional Galois representations, *Astérisque* **302** (2005), 67–150.
[Z] D. Zagier, Modular forms whose coefficients involve zeta-functions of quadratic fields,*Modular functions of one variable* VI, Lect. Notes Math. **627,** 105–169, Springer, 1977.

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