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TAMAGAWA FACTORS FOR CERTAIN SEMI-STABLE REPRESENTATIONS

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Abstract. An attempt is made to generalise Bloch and Kato's calculation of some of the local Tamagawa factors appearing in their conjecture, from the crystalline case to the semi-stable case. Success is attained only under severe conditions, but the result applies to the symmetric powers of an elliptic curve with multiplicative reduction.

1. Introduction

Let $E/\mathbb{Q}$ be an elliptic curve and $L_E(s) = \prod_p \frac{1}{P_p(p^{-s})}$ its L-function. Suppose for simplicity that $L_E(1) \neq 0$. The conjecture of Birch and Swinnerton-Dyer predicts that

$$\frac{L(E,1)}{\Omega} = \frac{\prod_p c_p \ \#\text{Ш}}{(\#E(\mathbb{Q})_{\text{tors.}})^2},$$

where Ω is (possibly twice) the real period of a Néron differential on the global minimal model, and Ш is the Shafarevich-Tate group. The factor c_p is trivial if p is a prime of good reduction. In general it is equal to $[E(\mathbb{Q}_p) : E^0(\mathbb{Q}_p)]$, where $E^0(\mathbb{Q}_p)$ is the subgroup of points with non-singular reduction.

$c_p P_p(p^{-1})$ may be interpreted as a local measure of the group $E(\mathbb{Q}_p)$, as Ω is a measure of $E(\mathbb{R})$, and the Birch-Swinnerton-Dyer conjecture may be interpreted as a kind of Tamagawa number formula. Using this viewpoint, and assuming various conjectures, Bloch and Kato [**1**] were able to formulate a conjecture on the values of the L-functions attached to motives of weight at most -1. (In other words, the integer at which the L-function is evaluated should not be to the left of the point of symmetry of the conjectured functional equation. This weight restriction was

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later removed by Fontaine and Perrin-Riou [**18**], without assuming the functional equation.)

A crucial point in the work of Bloch and Kato is the definition of the p-part of the local measure at p. They do this using their exponential map, which is defined using Fontaine's rings B_{dR} and B_{cris}. Suppose that p is a prime of good reduction. Then by the comparison theorems of Fontaine-Messing and Faltings [**17**], [**11**], the p-adic realisation of the motive is a crystalline representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. In this case it follows from Theorem 4.1(iii) of [**1**] that, with appropriate choices of lattices in the de Rham and p-adic realisations of the motive (and under certain conditions), the p-part of the local measure at p is just the p-part of the relevant value of the Euler factor at p. In other words, the p-part of the analogue of c_p is trivial.

Our goal in this paper is to generalise the calculation of Bloch and Kato's Theorem 4.1(iii) to the case that the p-adic realisation of the motive is a semi-stable representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, using the work of Fontaine [**14**] and Breuil [**2**]. We need to impose a strong condition on the availability of special types of lattices in the de Rham realisation. This helps to make the result we are able to prove in this paper (Theorem 6.1) inapplicable to higher-weight Hecke eigenforms of level N with $p \mid\mid N$, but it is applicable to the symmetric powers of an elliptic curve with multiplicative reduction at p. We recover the main theorem of [**10**] by this new method, and end with a numerical example for the symmetric sixth power.

2. Some big rings

For more details see [**13**]. There is a $\mathbb{Q}_p$-algebra B_{dR} with an action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. It is the fraction field of a complete local ring B_{dR}^+, which has a natural uniformiser t, on which $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts via the cyclotomic character. Powers of t define a filtration of B_{dR}, with $B_{\mathrm{dR}}^+ = \mathrm{Fil}^0 B_{\mathrm{dR}}$. The quotient $\mathrm{Fil}^m B_{\mathrm{dR}}/\mathrm{Fil}^{m+1} B_{\mathrm{dR}}$ is isomorphic to $\mathbb{C}_p(m)$, a Tate twist of the completion of the algebraic closure of $\mathbb{Q}_p$. For any finite extension $F/\mathbb{Q}_p$, $(B_{\mathrm{dR}}^+)^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)} \simeq B_{\mathrm{dR}}^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)} \simeq F$.

There are subrings $A_{\mathrm{cris}} \subset B^+_{\mathrm{cris}} \subset B_{\mathrm{cris}} \subset B_{\mathrm{dR}}$, with $B^+_{\mathrm{cris}} = A_{\mathrm{cris}}[1/p]$ and $B_{\mathrm{cris}} = B^+_{\mathrm{cris}}[1/t]$. They all have induced filtrations and actions of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, and $t \in A_{\mathrm{cris}}$. They also possess compatible Frobenius endomorphisms ϕ such that $\phi(t) = pt$. The ring A_{cris} is called B_∞ in §4 of [**1**]. B^+_{cris} is a subring of B^+_{dR}.

Now let $F/\mathbb{Q}_p$ be a finite extension, with maximal unramified subextension $F_0/\mathbb{Q}_p$, and choose a uniformiser π for F. Depending on these choices, one may define a sub-B^+_{cris}-algebra B^+_{st} of B^+_{dR}, with induced filtration and $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-action, and a subring $B_{\mathrm{st}} := B^+_{\mathrm{st}}[1/t]$ of B_{dR}. Let $F' \subset \overline{\mathbb{Q}}_p$ be any finite extension of F, with maximal unramified (over $\mathbb{Q}_p$) subfield F'_0. Then $(B^+_{\mathrm{st}})^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F')} \simeq B_{\mathrm{st}}^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F')} \simeq F'_0$. The Frobenius endomorphism ϕ extends to B^+_{st} and B_{st}, which also have a B_{cris}-derivation N such that $N\phi = p\phi N$. The kernel of N on B_{st} is B_{cris}. The action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ commutes with ϕ and N.

For simplicity we assume from this point onwards that $F = F_0$, i.e. that $F/\mathbb{Q}_p$ is a finite, unramified extension. It is only with this assumption that we prove our main theorem (and with which Bloch and Kato proved their Theorem 4.1(iii) in the crystalline case).

There is an A_{cris}-algebra $\widehat{A_{\mathrm{st}}}$ with a filtration, ϕ, N and $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-action. The part of $\widehat{A_{\mathrm{st}}}[1/p]$ on which N is nilpotent may be identified with B^+_{st}, though the induced filtration on B^+_{st} is coarser than the true filtration coming from B_{dR}. The part of $\widehat{A_{\mathrm{st}}}$ fixed by $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ is naturally identified with the ring $S := \left\{\sum_{n=0}^{\infty} w_n \frac{u^n}{n!}, |w_n \in O_F,\, w_n \to 0\right\}$. S inherits ϕ and N, and a filtration given by $\mathrm{Fil}^i S = S \cap (u-p)^i S[1/p]$. In fact, $\widehat{A_{\mathrm{st}}}$ is naturally an S-algebra.

3. Crystalline and semi-stable representations

Let V be a finite-dimensional $\mathbb{Q}_p$-vector space with a continuous, linear action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$.

Definition 3.1. (1) $D_{\mathrm{cris}}(V) := (V \otimes B_{\mathrm{cris}})^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)}$.

(2) $D_{\mathrm{st}}(V) := (V \otimes B_{\mathrm{st}})^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)}$.

(3) $D_{\mathrm{dR}}(V) := (V \otimes B_{\mathrm{dR}})^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)}$.

These are all finite dimensional F-vector spaces. They inherit various structures from the big rings. Thus they all have decreasing filtrations, $D_{\mathrm{cris}}(V)$ has an injective semilinear Frobenius endomorphism ϕ, and $D_{\mathrm{st}}(V)$ has Frobenius and monodromy operators ϕ and N respectively, satisfying $N\phi = p\phi N$.

Definition 3.2. (1) *V is de Rham if* $\dim_F(D_{\mathrm{dR}}(V)) = \dim_{\mathbb{Q}_p}(V)$.
(2) *V is semi-stable if* $\dim_F(D_{\mathrm{st}}(V)) = \dim_{\mathbb{Q}_p}(V)$.
(3) *V is crystalline if* $\dim_F(D_{\mathrm{cris}}(V)) = \dim_{\mathbb{Q}_p}(V)$.

These properties are preserved by taking duals and tensor products. They are listed in ascending order of strength. In all cases, the dimension on the left is less than or equal to the dimension on the right.

A (finite-dimensional) filtered ϕ-module over F that is $D_{\mathrm{cris}}(V)$ for some crystalline representation V of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ is said to be admissible. Likewise, a filtered (ϕ, N)-module over F that is $D_{\mathrm{st}}(V)$ for some semi-stable representation V of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ is said to be admissible. Fontaine defined an intrinsic property of a filtered ϕ-module (or (ϕ, N)-module) over F, called weak admissibility (for the definition, see 4.4 of [**14**] or 1.2 of [**4**]), and proved that admissibility implies weak admissibility (see Proposition 5.4.2 of [**14**]). He conjectured the converse, and this was eventually proved by him and Colmez [**6**], without the assumption $F = F_0$. The crystalline case, with $F = F_0$ and filtration length at most p, was proved in [**16**].

The functor D_{cris} gives an equivalence of categories from crystalline representations to weakly admissible filtered ϕ-modules. The inverse functor is such that

$$V_{\mathrm{cris}}(D) = \{z \in \mathrm{Fil}^0(D \otimes_F B_{\mathrm{cris}}) | \phi(z) = z\}.$$

Here the filtration on $D \otimes_F B_{\mathrm{cris}}$ is the "tensor product" of those on the two factors. The functor D_{st} gives an equivalence of categories from semi-stable representations to weakly admissible filtered (ϕ, N)-modules. The inverse functor is such that

$$V_{\mathrm{st}}(D) = \{z \in \mathrm{Fil}^0(D \otimes_F B_{\mathrm{st}}) | \phi(z) = z,\, N(z) = 0\}.$$

These equivalences respect tensor products and duals. For a crystalline representation V, $D_{\mathrm{st}}(V) \simeq D_{\mathrm{cris}}(V)$ with the same filtration and ϕ, and $N = 0$.

If V is a semi-stable representation and $D_{\mathrm{st}}(V) = D$ then for the Tate twist by $r \in \mathbb{Z}$, $D_{\mathrm{st}}(V(r)) = D(r)$, where $D(r)$ is the same as D as an F-vector space and has the same N, but ϕ has been divided by p^r, and $\mathrm{Fil}^i D(r) = \mathrm{Fil}^{i+r} D$. For the trivial representation $\mathbb{Q}_p$ we have $D_{\mathrm{st}}(\mathbb{Q}_p) = F$, where $\mathrm{Fil}^i(F) = \begin{cases} F & i \leq 0 \\ \{0\} & i > 0 \end{cases}$, $\phi(1) = 1$ and $N = 0$. Recall that ϕ is Frob_p-semilinear.

4. The integral theory

We need only a watered-down version of this. Let V be a crystalline representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ and let $D = D_{\mathrm{cris}}(V)$ be the associated filtered module. If M is an O_F-lattice in D then M has a filtration $\mathrm{Fil}^i M = M \cap \mathrm{Fil}^i D$. M is said to be strongly divisible if $M = \sum_i p^{-i}\phi(\mathrm{Fil}^i M)$.

Suppose that there exist $j, k \in \mathbb{Z}$ such that $\mathrm{Fil}^j D = D, \mathrm{Fil}^k D = \{0\}$, and $k - j < p$. By Tate-twisting if necessary, we may suppose that $j = 2 - p$ and $k = 1$. Given a strongly divisible lattice M in D, let

$$T_{\mathrm{cris}}(M) := \{z \in \mathrm{Fil}^0(M \otimes_{O_F} A_{\mathrm{cris}}) | \phi(z) = z\}.$$

Then $T_{\mathrm{cris}}(M)$ is a $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$-stable $\mathbb{Z}_p$-lattice in V, and it follows from [**16**] (c.f. §1.1.2 of [**8**]) that all $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$-stable $\mathbb{Z}_p$-lattices in V arise in this manner.

It is possible to define a category of strongly divisible O_F-modules without reference to ambient spaces D, and then T_{cris} is a functor.

Now let V be a semi-stable representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ and let $D = D_{\mathrm{st}}(V)$ be the associated filtered (ϕ, N)-module. Let $\mathcal{D} = D \otimes_{O_F} S$, a free $S[1/p]$-module of finite rank. Via "$\phi = \phi \otimes \phi$" and "$N = N \otimes \mathrm{id} + \mathrm{id} \otimes N$" we get Frobenius and monodromy operators on $\mathcal{D}$, and there is a way of producing a filtration of $\mathcal{D}$ such that $N(\mathrm{Fil}^{i+1}\mathcal{D}) \subset \mathrm{Fil}^i\mathcal{D}$ for all $i \in \mathbb{Z}$. This is not the tensor product filtration, unless it happens to be the case that $N(\mathrm{Fil}^{i+1}D) \subset \mathrm{Fil}^i D$ for all $i \in \mathbb{Z}$. See, for example, §4.1 of [**4**].

Suppose that there exist $j,k\in\mathbb{Z}$ such that $\mathrm{Fil}^jD=D,\mathrm{Fil}^kD=\{0\}$, and $k-j<p$. By Tate-twisting if necessary, we may suppose that $j=0$ and $k=p-1$. Let $\mathcal{M}\subset\mathcal{D}$ be a free S-module of finite rank such that $\mathcal{M}[1/p]=\mathcal{D}$. $\mathcal{M}$ is said to be a strongly divisible lattice if $\frac{\phi}{p^{p-2}}(\mathrm{Fil}^{p-2}\mathcal{M})$ spans $\mathcal{M}$ over S. Given a strongly divisible lattice $\mathcal{M}$ in $\mathcal{D}$, let

$$T^*_{\mathrm{st}}(\mathcal{M}):=\mathrm{Hom}_{S,\phi,N,\mathrm{Fil}^{p-2}}(\mathcal{M},\widehat{A_{\mathrm{st}}}).$$

Then $T^*_{\mathrm{st}}(\mathcal{M})$ is a $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$-stable $\mathbb{Z}_p$-lattice in V^*, and all $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$-stable $\mathbb{Z}_p$-lattices in V^* arise in this manner, see Theorem 2.2.7(i) of [**4**].

It is possible to define a category of strongly divisible S-modules without reference to ambient spaces $\mathcal{D}$, and then T^*_{st} is a (contravariant) functor.

5. The exponential map and the local factor at p

Let V be a de Rham representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$. Let $D=D_{\mathrm{dR}}(V)$. Define $P(V,u):=\det_F(1-\phi^fu|D_{\mathrm{cris}}(V))$, where $f=[F:\mathbb{Q}_p]$. Bloch and Kato (Chapter 3 of [**1**]) define subspaces

$$H^1_e(F,V)\subset H^1_f(F,V)\subset H^1(F,V)$$

as follows:

$$H^1_e(F,V)=\mathrm{Ker}(H^1(F,V)\to H^1(F,B_{\mathrm{cris}}^{\phi=1}\otimes V));$$
$$H^1_f(F,V)=\mathrm{Ker}(H^1(F,V)\to H^1(F,B_{\mathrm{cris}}\otimes V)).$$

Given that V is crystalline, $H^1_f(F,V)$ corresponds to classes of those extensions of $\mathbb{Q}_p$ by V which are crystalline.

By Proposition 1.17 of [**1**] there is an exact sequence

(1) $$0\longrightarrow\mathbb{Q}_p\xrightarrow{\ \alpha\ }B_{\mathrm{cris}}\xrightarrow{\ \beta\ }B_{\mathrm{cris}}\oplus(B_{\mathrm{dR}}/B^+_{\mathrm{dR}})\longrightarrow 0,$$

where $\alpha(x)=x$ and $\beta(x)=(x-\phi(x),x)$. (This is also the sequence S_f in 3.1.1 of [**18**]). Tensoring (1) with V gives

(2)

$$0\longrightarrow V\xrightarrow{\ \alpha\ }V\otimes B_{\mathrm{cris}}\xrightarrow{\ \beta\ }V\otimes B_{\mathrm{cris}}\oplus V\otimes(B_{\mathrm{dR}}/B^+_{\mathrm{dR}})\longrightarrow 0,$$

Taking $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$-cohomology leads to a surjective map

$$\exp : D_{\mathrm{dR}}(V)/D_{\mathrm{dR}}(V)^0 \to H^1_e(\mathbb{Q}_p, V)$$

(see Corollary 3.8.4 of [**1**] or sequence S_e of [**18**]). Here, $D^0 := \mathrm{Fil}^0 D$.

Suppose that $P(V,1) \neq 0$. Then $H^0(F,V) = \{0\}$ and $D_{\mathrm{cris}}(V)/(1-\phi)D_{\mathrm{cris}}(V) = \{0\}$. Also, by Corollary 3.8.4 of [**1**], $H^1_e(F,V) = H^1_f(F,V)$ and exp is an isomorphism (Theorem 4.1(ii) of [**1**]).

Let T be some $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$-stable $\mathbb{Z}_p$-lattice in V and let M be some O_F-lattice in D. Let $M^0 = M \cap D^0$. Let μ_M be the invariant measure on $H^1_f(F,V)$ such that $\mu_M(\exp(M/M^0)) = 1$. Of course, it really only depends on the highest exterior power of the $\mathbb{Z}_p$-lattice M/M^0. Define

$$\mathrm{Tam}_M(T) := \mu_M(H^1_f(F,T))$$

and

$$\mathrm{Tam}^0_M(T) := \mathrm{Tam}_M(T)|P(V,1)|_p.$$

This is rather an ad hoc definition of $\mathrm{Tam}^0_M(T)$. See 11.3 and 11.5 of [**15**] for more. Here, by $\mu_M(H^1_f(F,T))$ I really mean μ_M of the image of $H^1_f(F,T)$ in $H^1_f(F,V)$, multiplied by the order of $H^1_f(F,T)_{\mathrm{tors}}$.

The following is Theorem 4.1(iii) of [**1**].

Theorem 5.1. *Suppose, as above, that V is a* crystalline *representation of* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ *(with $F = F_0$), and that $P(V,1) \neq 0$. Let $D = D_{\mathrm{cris}}(V)$, and suppose that there exist $j \leq 0, k > 0$ with $k - j < p$ and $\mathrm{Fil}^j D = D, \mathrm{Fil}^k D = \{0\}$. Let T be a $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$-stable $\mathbb{Z}_p$-lattice in V and let M be the strongly divisible lattice in D such that $T_{\mathrm{cris}}(M) = T$. Then* $\mathrm{Tam}_M(T) = |P(V,1)|_p^{-1}$*, i.e.* $\mathrm{Tam}^0_M(T) = 1$.

6. The main theorem

Slightly rearranging the sequence S_g in 3.1.1 of [**18**], we get

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B_{\mathrm{st}} \xrightarrow{(1-\phi,N,\mathrm{id})} R \oplus (B_{\mathrm{dR}}/B^+_{\mathrm{dR}}) \longrightarrow 0. \tag{3}$$

Here,

$$R := \{(x,y) \in B_{\mathrm{st}} \oplus B_{\mathrm{st}}(-1) | N(x) = (1-\phi)(y)\}.$$

The twist on B_{st} is as a filtered ϕ-module. Tensoring with V and taking $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$-cohomology gives

$$(4)\quad 0 \longrightarrow H^0(F,V) \longrightarrow D \xrightarrow{(1-\phi,N,\mathrm{id})} D'' \oplus (D/D^0) \longrightarrow H^1_{\mathrm{st}}(F,V)\,,$$

where

$$D'' := \{(x,y) \in D \oplus D(-1) | N(x) = (1-\phi)(y)\}$$

and $H^1_{\mathrm{st}}(F,V) := \ker(H^1(F,V) \to H^1(F, V \otimes B_{\mathrm{st}})$. Note that $H^1_f(F,V) \subset H^1_{\mathrm{st}}(F,V)$, and this recovers the Bloch-Kato exponential map. Given that V is semi-stable, $H^1_{\mathrm{st}}(F,V)$ corresponds to classes of those extensions of $\mathbb{Q}_p$ by V which are semi-stable.

Theorem 6.1. *Suppose that V is a* semi-stable *representation of* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ *(with $F = F_0$), and that $P(V,1) \neq 0$. Let $D = D_{\mathrm{st}}(V)$, and suppose that there exist $j \leq 0, k > 0$ with $k - j < p$ and $\mathrm{Fil}^j D = D, \mathrm{Fil}^k D = \{0\}$. Suppose also that $N(\mathrm{Fil}^{i+1} D) \subset \mathrm{Fil}^i D$ for all $i \in \mathbb{Z}$. Let T be a $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$-stable $\mathbb{Z}_p$-lattice in V and let $\mathcal{M}$ be the strongly divisible lattice in $\mathcal{D} := D \otimes_{O_F} S$ such that $T^*_{\mathrm{st}}(\mathcal{M}) = T^* := \{f \in V^* | f(T) \subset \mathbb{Z}_p\}$. Suppose that $\mathcal{M}$ is of the form $M \otimes_{O_F} S$, where M is a strongly divisible, N-stable O_F-lattice in D. Then $\mathrm{Tam}_M(T) = [M'' : (1-\phi, N)M]$, where*

$$M'' := \{(x,y) \in M \oplus M(-1) | N(x) = (1-\phi)(y)\}.$$

(This index is well-defined, even though in general the submodule $(1-\phi,N)M$ of D is not contained in M''.) Note that if we start from a strongly divisible, N-stable O_F-lattice M in D, then $\mathcal{M} = M \otimes S$ is necessarily a strongly divisible S-module in $\mathcal{D}$, given that $N(\mathrm{Fil}^{i+1} D) \subset \mathrm{Fil}^i D$ for all $i \in \mathbb{Z}$.

Proof. Let $M' := \{m \in M | (1-\phi)m \in M\} = \{m \in M | \phi(m) \in M\}$. Note that, by strong divisibility, $M^0 \subset M'$.

Lemma 6.2. *There is a natural exact sequence*

$$0 \longrightarrow M' \xrightarrow{(1-\phi,\, N,\, \mathrm{id})} M'' \oplus (M'/M^0) \longrightarrow \mathrm{Ext}^1(O_F, M) \longrightarrow 0\,, \tag{5}$$

where the extensions are in the category of strongly divisible, N-stable O_F-lattices in (necessarily weakly admissible) filtered (ϕ, N)-modules E over F such that $\mathrm{Fil}^j E = E$ *and* $\mathrm{Fil}^k E = \{0\}$*, for some suitable fixed $j \leq 0$ and $k > 0$ with $k - j < p$.*

Proof. Let E be an extension of O_F by M. As an O_F-module, $E \simeq M \oplus O_F$. Since $\phi(1) = 1$ for $1 \in O_F$, we must have $\phi((0,1)) = (x, 1)$ for some $x \in M$. Since $N(1) = 0$ for $1 \in O_F$, we must have $N((0,1)) = (y, 0)$ for some $y \in M$. Since $N\phi = p\phi N$ on E, necessarily $(x, y(-1)) \in M''$. Since E is strongly divisible, we must have $(z, 1) \in \mathrm{Fil}^0 E$ for some $z \in M$, which could be replaced by addition of anything in M^0, so we also consider z as an element of M/M^0. By strong divisibility, $\phi((z,1))$ must belong to E, so in fact z must be chosen from M'/M^0.

To any extension we have associated $(x, y(-1), z) \in M'' \oplus (M'/M^0)$, so we have a surjective map as in the exact sequence above, which imbues the set of extensions with a group structure. It is easy to see that those $(x, y(-1), z)$ giving rise to isomorphic extensions are those differing by $((1-\phi)w, N(w)(-1), w)$ for $w \in M'$. (The isomorphism sends $(0,1)$ to $(w,1)$.) ☐

Lemma 6.3. *There is a natural isomorphism*

$$\mathrm{Ext}^1(O_F, M) \simeq \mathrm{Ext}^1(S, \mathcal{M}),$$

where the latter is extensions in a suitable category of strongly divisible S-modules.

Proof. There are two ways to recover M from $\mathcal{M} = M \otimes_{O_F} S$. One is $M = \mathcal{M}^{N-\mathrm{nil}}$, the part on which N is nilpotent. The other is $M \simeq \mathcal{M}/(u-\pi)\mathcal{M}$, i.e. apply something like the functor f_π of [**5**]. The quotient map $\mathcal{M} \to M$ respects the filtrations but not the actions of ϕ and N. Similarly $O_F \simeq S^{N-\mathrm{nil}} \simeq S/(u-\pi)S$.

Let $\mathcal{E}$ be an extension of S by $\mathcal{M}$. Taking N-nilpotent parts is only left-exact, while modding out by $(u-\pi)$ is only right-exact, so we get a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{M}^{N-\mathrm{nil}} & \longrightarrow & \mathcal{E}^{N-\mathrm{nil}} & \longrightarrow & S^{N-\mathrm{nil}} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{E} & \longrightarrow & S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathcal{M}/(u-\pi) & \longrightarrow & \mathcal{E}/(u-\pi) & \longrightarrow & S/(u-\pi) & \longrightarrow & 0
\end{array}$$

The middle composite vertical map is an injection with finite cokernel, which we can use to endow $\mathcal{E}/(u-\pi)$ with a ϕ and an N. We have now an extension

$$0 \longrightarrow \mathcal{M}^{N-\mathrm{nil}} \longrightarrow \mathcal{E}/(u-\pi) \longrightarrow S/(u-\pi) \longrightarrow 0,$$

which is of the form

$$0 \longrightarrow M \longrightarrow E \longrightarrow O_F \longrightarrow 0.$$

Conversely, starting from such an extension of strongly divisible, N-stable O_F-modules, tensoring with S gives an extension of strongly divisible S-modules. This uses the hypothesis $N(\mathrm{Fil}^{i+1}D) \subset \mathrm{Fil}^i D$ for all $i \in \mathbb{Z}$. □

Applying T^*_{st} identifies $\mathrm{Ext}^1(S, \mathcal{M})$ with the subset of $\mathrm{Ext}^1(T^*, \mathbb{Z}_p)$ corresponding to lattices in semi-stable representations. Taking duals, this is in turn identified with the subset of $\mathrm{Ext}^1(\mathbb{Z}_p, T)$ corresponding to lattices in semi-stable representations. We have used here the equivalence of categories in Theorem 2.2.7(i) of [**4**]. Thus we get

$$(6) \qquad 0 \longrightarrow M' \xrightarrow{(1-\phi,\, N,\, \mathrm{id})} M'' \oplus (M'/M^0) \longrightarrow H^1_{\mathrm{st}}(F,T) \longrightarrow 0\,,$$

where $H^1_{\mathrm{st}}(F,T)$ is the inverse image of $H^1_{\mathrm{st}}(F,V)$. Tensoring (6) with $\mathbb{Q}_p$ gives

$$(7) \qquad 0 \longrightarrow D \xrightarrow{(1-\phi,\, N,\, \mathrm{id})} D'' \oplus (D/D^0) \longrightarrow H^1_{\mathrm{st}}(F,V) \longrightarrow 0\,.$$

One can check this coincides with (4), giving us again the Bloch-Kato exponential map. Dimension counting forces $H^1_{\mathrm{st}}(F,V) = H^1_f(F,V)$, so $H^1_{\mathrm{st}}(F,T) = H^1_f(F,T)$, and the theorem now follows from (6). □

In the crystalline case, one may likewise prove the simpler Theorem 5.1, using an analogue of Lemma 6.2. One finds that $\mu_M(H^1_f(F,T)) = [M:(1-\phi)M]$. This in turn is equal to

$$\det_{\mathbb{Q}_p}((1-\phi)|D) = \det_F((1-\phi^f)|D) = |P(V,1)|_p^{-1}.$$

This is essentially the proof of [**1**], recast somewhat as in I.4.2.3 of [**18**].

7. Elliptic curves with multiplicative reduction, and their symmetric powers

Let $E/\mathbb{Q}_p$ be an elliptic curve with multiplicative reduction. Thanks to the Tate parametrisation there is an isomorphism of groups

$$E(\overline{\mathbb{Q}}_p) \simeq \overline{\mathbb{Q}}_p^*/q^{\mathbb{Z}}.$$

The natural actions of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ on the two sides are the same in the case of split multiplicative reduction, and differ only by an unramified quadratic character in the case of non-split multiplicative reduction. Here, $q \in \mathbb{Q}_p$ is such that $j(E) = \frac{1}{q} + 744 + 196884q + \dots$

Let $T = H^1_{et}(E \otimes \overline{\mathbb{Q}}_p, \mathbb{Z}_p) = T_p(E)(-1)$, where $T_p(E) = \varprojlim E[p^n]$ is the p-adic Tate module of E. Then $T^* \simeq T(1)$. Let $V = T \otimes \mathbb{Q}_p$.

Proposition 7.1. *Let $E/\mathbb{Q}_p$ be an elliptic curve with multiplicative reduction at $p \geq 3$, and Tate parameter q. Let $\alpha := \mathrm{ord}_p(q)$ and $\lambda := \log_p(q)$. Then the 2-dimensional p-adic representation V of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is semi-stable and corresponds (via D_{st}) to a (ϕ, N)-filtered module D with $\mathbb{Q}_p$-basis $\{e_0, e_1\}$, defined by*

$$\phi e_1 = \pm p e_1,\ \phi e_0 = \pm e_0,$$

$$N e_1 = \alpha e_0\ ,\ N e_0 = 0,$$

$$D^0 = D,\ D^1 = \langle e_1 - \lambda e_0 \rangle\ ,\ D^2 = \{0\}.$$

The upper or lower sign is taken when the reduction is split or non-split, respectively.

This follows from the proposition in 3.5 of [**20**]. Note that the condition $N(\mathrm{Fil}^{i+1}D) \subset \mathrm{Fil}^i D$ for all $i \in \mathbb{Z}$ is satisfied. Let M be the $\mathbb{Z}_p$-submodule of D generated by e_0 and e_1. Then M is N-stable and, since $p \mid \lambda$, strongly divisible. Let $\mathcal{M} = M \otimes_{\mathbb{Z}_p} S$.

Proposition 7.2. $T^*_{\mathrm{st}}(\mathcal{M}) = T^*$.

This is Proposition 6.8 of [**10**].

For $V(1)$ we have $D(1)$ with

$$\phi e_1 = \pm e_1,\ \phi e_0 = \pm\frac{1}{p} e_0,$$

$$N e_1 = \alpha e_0\ ,\ N e_0 = 0,$$

$$D^{-1} = D,\ D^0 = \langle e_1 - \lambda e_0 \rangle\ ,\ D^1 = \{0\}.$$

Proposition 7.3. $\mathrm{ord}_p(\mathrm{Tam}^0_{M(1)}(T(1))) = \mathrm{ord}_p(\alpha)$.

Proof. We simply apply Theorem 6.1. $M(1)$ is spanned by e_0 and e_1. $(1-\phi, N)(M(1))$ is then spanned by $((1-p^{-1})e_0, 0)$ and $(0, \alpha e_0)$. But $M(1)'' := \{(x,y) \in M(1) \oplus M | N(x) = (1-\phi)(y)\}$ is spanned by $(e_0, 0)$ and $(0, e_0)$. (When calculating $(1-\phi)(y)$ we must remember the twist.) Since $(1-p^{-1}) = P(V(1), 1)$, the proposition follows. □

Using various comparison theorems [**3**], [**19**], [**22**], Lemma 6.9 of [**10**] identifies D with $H^1_{\mathrm{dR}}(E/\mathbb{Q}_p)$, and M with the $\mathbb{Z}_p$-lattice spanned by a Néron differential ω and an element η whose image in D/D^1 is Serre-dual to ω. Given this, Proposition 7.3 simply recovers the p-part of the familiar Birch-Swinnerton-Dyer fudge factor at p, since $\mathrm{ord}_p(q) = [E(\mathbb{Q}_p) : E^0(\mathbb{Q}_p)]$, where $E^0(\mathbb{Q}_p)$ is the non-singular reduction subgroup.

We can also recover the results of Theorem 5.1 and Proposition 7.1 of [**10**], about Tamagawa factors for $\mathrm{Sym}^2(E)$ at $s = 1$ and $s = 2$, without having to use anything about compatibility of the (Fontaine-Perrin-Riou-) Bloch-Kato conjecture with the functional equation for $L(\mathrm{Sym}^2(E), s)$.

Proposition 7.4. *Suppose that* $p \geq 5$.

(1) $\mathrm{ord}_p(\mathrm{Tam}^0_{\mathrm{Sym}^2(M)(1)}(\mathrm{Sym}^2(T)(1)) = \mathrm{ord}_p(\alpha)$;

(2) $\mathrm{ord}_p(\mathrm{Tam}^0_{\mathrm{Sym}^2(M)(2)}(\mathrm{Sym}^2(T)(2)) = \mathrm{ord}_p(\alpha) - 1$.

Proof. (1) $\mathrm{Sym}^2(M)(1)$ is spanned by e_0^2, e_0e_1 and e_1^2, so $(1-\phi, N)(\mathrm{Sym}^2(M)(1))$ is spanned by

$$((1-p^{-1})e_0^2, 0), (0, \alpha e_0^2) \text{ and } ((1-p)e_1^2, 2\alpha e_0e_1).$$

Meanwhile, $\mathrm{Sym}^2(M)(1)''$ is spanned by

$$(e_0^2, 0), (0, e_0^2), \text{ and } ((1-p)e_1^2, 2\alpha e_0e_1),$$

(noting that $(1-p)$ is a unit in $\mathbb{Z}_p$), and $P(\mathrm{Sym}^2(V)(1), 1) = (1-p^{-1})$.

(2) $(1-\phi, N)(\mathrm{Sym}^2(M)(2))$ is spanned by

$$((1-p^{-2})e_0^2, 0), ((1-p^{-1})e_0e_1, \alpha e_0^2) \text{ and } (0, 2\alpha e_0e_1).$$

Meanwhile, $\mathrm{Sym}^2(M)(1)''$ is spanned by

$$(e_0^2, 0), (e_0e_1, \alpha(1-p^{-1})^{-1}e_0^2), \text{ and } (0, e_0e_1),$$

and $P(\mathrm{Sym}^2(V)(2), 1) = (1-p^{-2})$.

□

For any $n \geq 1$, $\mathrm{Fil}^0(\mathrm{Sym}^n D) = \mathrm{Sym}^n D$ and $\mathrm{Fil}^{n+1}(\mathrm{Sym}^n D) = \{0\}$. Also, $N(\mathrm{Fil}^{i+1}\mathrm{Sym}^n D) \subset \mathrm{Fil}^i\mathrm{Sym}^n D$ for all $i \in \mathbb{Z}$, and $\mathrm{Sym}^n M$ is strongly divisible and N-stable. We can apply Theorem 6.1 to $\mathrm{Sym}^n(M)(j)$ as long as $p > n+1$ and $n+1-p < j < p-1$ (and also $j \neq 0$ so that $P(\mathrm{Sym}^n V(j), 1) \neq 0$. (Using induction on n, the method of [**10**] can also be made to work for integer $j \in [\frac{n}{2}, \frac{n}{2}+1]$.) We content ourselves with one more example.

Proposition 7.5. *Suppose that* $p \geq 11$.

$$\mathrm{ord}_p(\mathrm{Tam}^0_{\mathrm{Sym}^6(M)(4)}(\mathrm{Sym}^6(T)(4))) = \mathrm{ord}_p(\alpha) - 6.$$

Proof. $(1-\phi, N)(\mathrm{Sym}^6(M)(4))$ is spanned by

$$((1-p^{-4})e_0^6, 0),\ \ ((1-p^{-3})e_0^5e_1, \alpha e_0^6),\ \ ((1-p^{-2})e_0^4e_1^2, 2\alpha e_0^5 e_1),$$

$$((1-p^{-1})e_0^3e_1^3, 3\alpha e_0^4e_1^2),\ (0, 4\alpha e_0^3e_1^3), \dots,$$

while $\mathrm{Sym}^6(M)(4)''$ is spanned by

$$(e_0^6, 0),\ \ (e_0^5e_1, \alpha(1-p^{-3})^{-1}e_0^6),\ \ (e_0^4e_1^2, 2\alpha(1-p^{-2})^{-1}e_0^5e_1),$$

$$(e_0^3e_1^3, 3\alpha(1-p^{-1})^{-1}e_0^4e_1^2),\ (0, e_0^3e_1^3), \dots$$

(after the dots, the two bases coincide). $P(\mathrm{Sym}^6(V)(4), 1) = (1-p^{-4})$ and $3+2+1 = 6$. □

A numerical experiment We can illustrate this using the example of $E : y^2 + y = x^3 - x - 10x - 20$, the optimal curve $X_0(11)$ in the unique isogeny class of elliptic curves over $\mathbb{Q}$ of conductor $N = 11$. Following the standard recipe $\det(1 - \mathrm{Frob}_p^{-1} p^{-s} | V_\ell^{I_p})^{-1}$ for the Euler factor at p, the symmetric 6^{th}-power L-function attached to E is

$$L(\mathrm{Sym}^6 E, s) = (1 - 11^{-s})^{-1} \prod_{p \neq 11} \prod_{i=0}^{6} (1 - \alpha^i \beta^{6-i} p^{-s})^{-1},$$

where $1 - a_pT + pT^2 = (1-\alpha T)(1-\beta T)$. Note that the definition of the Euler factor at p is, as expected in general, independent of the chosen $\ell \neq p$, and, what is more relevant to us, that the Euler factor at $p = 11$ is the same as $P(\mathrm{Sym}^n V_p, p^{-s})^{-1}$. Recall that $P(\mathrm{Sym}^n(V_p), u) := \det(1 - \phi u | D_{\mathrm{cris}}(\mathrm{Sym}^n(V_p)))$, and note that $D_{\mathrm{cris}}(\mathrm{Sym}^n(V_p)))$ is generated by e_0^n.

The critical values for this L-function are $L(\mathrm{Sym}^6 E, 3)$ and $L(\mathrm{Sym}^6 E, 4)$. Note that if n is a multiple of 4 then $L(\mathrm{Sym}^n E, s)$ does not have any critical values. According to Deligne's conjecture,

$$\frac{L(\mathrm{Sym}^6 E, 4)}{(\Omega^+\Omega^-)^6 (2\pi i)^{-2}}$$

is a rational number, where $\Omega^+ \approx 1.269$ and $\Omega^- \approx 2.918$ are, respectively, the real and imaginary periods of a Néron differential. See 7.7 and 7.8 of [**7**] for why this is what the conjecture says in this case.

The Bloch-Kato conjecture [**1**] gives an interpretation of this rational number:

$$\frac{L(\mathrm{Sym}^6 E, 4)}{(\Omega^+\Omega^-)^6(2\pi i)^{-2}} = \frac{c_{11}\#\text{Ш}}{\#H^0(\mathbb{Q}, (\mathrm{Sym}^6 T/\mathrm{Sym}^6 V)(4))\#H^0(\mathbb{Q}, (\mathrm{Sym}^6 T/\mathrm{Sym}^6 V)(3))},$$

where Ш is a generalised Shafarevich-Tate group. We shall examine the 11-part of the right hand side.

According to Corollaire 2 to Proposition 21 of [**21**], the natural homomorphism

$$\phi_{11} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(E[11])$$

is surjective. Since $6 < 11$, the 6^{th} symmetric power of the standard representation of $\mathrm{GL}_2(\mathbb{F}_{11})$ is irreducible. It follows that the 11-part of the denominator of the above expression is trivial. There is no particular reason to expect the 11-part of Ш to be non-trivial. But the 11-part of c_{11} is $\mathrm{Tam}^0_{\mathrm{Sym}^6(M)(4)}(\mathrm{Sym}^6(T)(4)) = 11^{-6}$ by Proposition 7.5, since $\alpha = \mathrm{ord}_{11}(q) = \mathrm{ord}_{11}(\Delta) = 5$. Therefore we expect the power of 11 in $\frac{L(\mathrm{Sym}^6 E,4)}{(\Omega^+\Omega^-)^6(2\pi i)^{-2}}$ to be 11^{-6}.

I wrote an easy program for the computer package GP-Pari, calling on Dokchitser's program ComputeL [**9**], which computes approximations to values of L-functions, assuming that they satisfy the expected functional equations. Using the first $480,000$ coefficients of the Dirichlet series, and 33-digit accuracy, I obtained a number whose continued fraction expansion is indistinguishable from that of $5^7/(2^2 11^6)$.

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