# A SIMPLE TRACE FORMULA FOR ALGEBRAIC MODULAR FORMS

NEIL DUMMIGAN

Abstract. We derive an elementary formula for the trace of a Hecke operator acting on a space of algebraic modular forms, as a sum of character values. We describe explicit computations in the case of the unitary group $U(4)$, allowing the determination of the eigenvalues of a certain Hecke operator. This produces numerical evidence for a $U(2,2)$ analogue of Harder's conjecture, on congruences between Hecke eigenvalues modulo divisors of critical $L$-values.

## 1. Introduction–the formula and its proof

Let $G/\mathbb{Q}$ be a reductive group, with the property that for any open compact subgroup $K_f$ of $G(\mathbb{A}_f)$, $K_f \cap G(\mathbb{Q})$ is finite (where $\mathbb{A}$ is the adèle ring of $\mathbb{Q}$, and $\mathbb{A}_f$ is its "finite part", excluding the factor $\mathbb{R}$). Examples are the multiplicative group $G = \mathbb{G}_m$, and any $G$ for which $G(\mathbb{R})$ is compact. See [Gross 99, Proposition 1.4] for several equivalent conditions. Let $K = K_\infty \times K_f = K_\infty \times \prod K_q$ be an open subgroup of $G(\mathbb{A})$, with $K_\infty$ containing the connected component of the identity $G(\mathbb{R})^0$, and each $K_q$ compact, equal to $G(\mathbb{Z}_q)$ for all but finitely many $q$. ($G$ is understood to have a natural model over some open subset of $\mathrm{Spec}(\mathbb{Z})$.) Let $V/\overline{\mathbb{Q}}$ be (the space of) an algebraic representation of $G$.

Define

$$A(G,K,V) := \{\tilde{f} : G(\mathbb{Q})\backslash G(\mathbb{A}) \to V \mid$$
$$\tilde{f}(gk_f) = \tilde{f}(g),\ \tilde{f}(gk_\infty) = k_\infty^{-1}\tilde{f}(g),\ \forall\, g \in G(\mathbb{A}), k_f \in K_f,\ k_\infty \in K_\infty\}.$$

Via $f(g) = g_\infty \tilde{f}(g)$, we get $A(G,K,V)$

$$\simeq \{f : G(\mathbb{A}) \to V \mid f(gk) = f(g),\ f(\gamma g) = \gamma f(g),\ \forall\, g \in G(\mathbb{A}), k \in K,\ \gamma \in G(\mathbb{Q})\}.$$

This can be thought of as $H^0(G(\mathbb{Q})\backslash G(\mathbb{A})/K, \tilde{V})$, where $\tilde{V}$ is the sheaf on the space $G(\mathbb{Q})\backslash G(\mathbb{A})/K$ [Gross 99, Proposition 4.3] associated to the representation $V$ of $G$. So the isomorphism between the two descriptions of $A(G,K,V)$ (which is elementary) may be viewed as an analogue of the Shimura isomorphism. This $A(G,K,V)$ is what Gross would call a space of "algebraic modular forms". Here we shall not insist on $V$ being defined over $\mathbb{Q}$.

Let $h := \#(G(\mathbb{Q})\backslash G(\mathbb{A})/K)$, and let $z_1, \ldots, z_h \in G(\mathbb{A})$ be a set of representatives. Then any $f \in A(G,K,V)$ is determined by the vector $(f(z_m)) \in V^h$. There is a constraint on what elements of $V^h$ may occur. If $\gamma \in \Gamma_m := G(\mathbb{Q}) \cap z_m K z_m^{-1}$ (a finite group) then $\gamma f(z_m) = f(\gamma z_m) = f(z_m)$. In fact ([Gross 99, Proposition 4.5]),

$$A(G,K,V) \simeq \oplus_{m=1}^{h} V^{\Gamma_m}.$$

Given any $r \in G(\mathbb{A}_f)$, let $\chi_{K_f r K_f} : G(\mathbb{A}_f) \to \mathbb{Z}$ be the characteristic function of the double coset $K_f r K_f$. Any compactly supported, left and right $K_f$-invariant

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function on $G(\mathbb{A}_f)$ is a finite linear combination of such functions. Letting $S = A(G,K,V)$, we may define a Hecke operator $T = T_{K_f\,rK_f} : S \to S$ by, for $g \in G(\mathbb{A})$ and $f \in S$,

$$T(f)(g) = \int_{G(\mathbb{A}_f)} \chi_{K_f\,rK_f}(h) f(gh)\, dh.$$

The integration is with respect to some right-invariant Haar measure $\mu_f$ such that $\mu(K_f) = 1$. In fact, we take $\mu = \prod \mu_q$, with each $\mu_q(K_q) = 1$. Let $K_f\,rK_f = \bigsqcup_{i=1}^n r_i K_f$. Given a representative $z_m$ for a class in $G(\mathbb{Q})\backslash G(\mathbb{A})/K$, there will be a subset of the $1 \leq i \leq n$ such that $G(\mathbb{Q})z_m r_i K = G(\mathbb{Q})z_m K$. For each such $i$, choose $k_{m,i} \in K$ and $\gamma_{m,i} \in G(\mathbb{Q})$ such that $z_m r_i k_{m,i} = \gamma_{m,i} z_m$. The choice of $\gamma_{m,i}$ is unique up to right multiplication by $\Gamma_m$. Let $\mathrm{ch}_V$ be the character of the representation $V$.

**Theorem 1.1.** *The trace of $T$ acting on $S$ is*

$$\mathrm{Tr}(T) = \sum_{m=1}^{h} \frac{1}{|\Gamma_m|} \sum_{\gamma \in \Gamma_m, \gamma_{m,i}} \mathrm{ch}_V(\gamma_{m,i}\gamma).$$

Note that in the special case $r = \mathrm{id}.$ (so $T$ is the identity operator), where $n = 1$ and we may take $r_1 = \gamma_{m,1} = \mathrm{id}.$, we get $\sum_{m=1}^h \frac{1}{|\Gamma_m|} \sum_{\gamma \in \Gamma_m} \mathrm{ch}_V(\gamma)$, which is $\sum_{i=1}^h \dim V^{\Gamma_m} = \dim S$, as expected.

*Proof.* Since $f \in S$ is right $K_f$-invariant and $\mu(K_f) = 1$, $T(f)(g) = \sum_{i=1}^n f(gr_i)$. If $G(\mathbb{Q})\backslash G(\mathbb{A})/K = \{[z_m]\}$, $\mathrm{tr}(T)$ is a sum over $m$ of the contributions of $f$ supported on $[z_m]$. The subspace of such $f$ is identified with $V^{\Gamma_m}$ via $f \mapsto f(z_m)$. For such a $[z_m]$, only those $r_i$ such that $z_m r_i k_{m,i} = \gamma_{m,i} z_m$ (i.e. $[z_m] = [z_m r_i]$) make a non-zero contribution to the trace, and clearly this contribution is the trace of $\sum_i \gamma_{m,i}$ acting on $V^{\Gamma_m}$. Since $\frac{1}{|\Gamma_m|}\sum_{\gamma \in \Gamma_m} \gamma$ is the projector from $V$ to $V^{\Gamma_m}$, this is the same as the trace of $(\sum_i \gamma_{m,i}) \left(\frac{1}{|\Gamma_m|}\sum_{\gamma \in \Gamma_m} \gamma\right)$ acting on $V$. Now just sum over $m$ from 1 to $h$. □

*Remarks.*

(1) The formula two lines before (9) in 3.1 of [Shimizu 65] appears to be equivalent to this, in the case that $G$ is the multiplicative group of a definite quaternion algebra over $\mathbb{Q}$. In fact, Shimizu's formula is given even for definite quaternion algebras over totally real fields, where one has to mod out by the units of the ring of integers of the field.

(2) If one takes the Grothendieck-Lefschetz-Verdier trace formula [Milne 92, Theorem C.1] and applies it to the finite space $X = G(\mathbb{Q})\backslash G(\mathbb{A})/K$, with the local system $\tilde{V}$ associated to $V$ and the Hecke correspondence associated to $KrK$, one obtains the same formula. In slightly more detail, if $K' = K \cap rKr^{-1}$ and $Y = G(\mathbb{Q})\backslash G(\mathbb{A})/K'$ then we have two maps $\alpha, \beta : Y \to X$ given by $\alpha(G(\mathbb{Q})zK') = G(\mathbb{Q})zK$ and $\beta(G(\mathbb{Q})zK') = G(\mathbb{Q})zrK$. The fixed points of the correspondence, those with the same image under $\alpha$ and $\beta$, are the points $G(\mathbb{Q})z_m r_i r^{-1} K'$ of $Y$, just for those $r_i$ such that $G(\mathbb{Q})z_m r_i K = G(\mathbb{Q})z_m K$, as above. The stalk of $\tilde{V}$ at the image of such a fixed point is $V^{\Gamma_m}$. The trace of the action of the Hecke correspondence on $H^0(X, \tilde{V})$ is $\mathrm{tr}(T)$, while the contribution of the fixed point $G(\mathbb{Q})z_m r_i r^{-1} K'$, to the

right hand side of the Grothendieck-Lefschetz-Verdier trace formula, is the trace of $\gamma_{m,i}$ acting on the stalk $V^{\Gamma_m}$.

(3) In the case that $V$ is the trivial representation, all the character values are 1, and the contribution of the class $G(\mathbb{Q})z_mK$ to $\mathrm{tr}(T)$ is $\#\{\gamma_{m,i}\}$, which can be viewed as a diagonal entry in an $h \times h$ generalised Brandt matrix (the original case being when $G$ is a definite quaternion algebra over $\mathbb{Q}$).

Explicit computations of spaces of algebraic modular forms, for various groups $G$, with $V$ of small dimension, have been carried out by various authors. See, for example, [Consani and Scholten 01, Dembélé 07, Ibukiyama 84, Lansky and Pollack 02, Loeffler 08, Pizer 80, Socrates and Whitehouse 05]. The example with $G = U(4)$ examined in this paper is the "smallest" we can use to test a $U(2,2)$ analogue of Harder's conjecture [Harder 08] (see Section 6 below), and throws up a representation of dimension 5220, necessitating the use of Theorem 1.1 rather than explicit calculation of matrices representing Hecke operators. The remainder of the paper is devoted entirely to this example.

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## 2. The case $G = U(4)$

Let $F$ be an imaginary quadratic extension of $\mathbb{Q}$. Let $G$ be the group-scheme over $\mathbb{Z}$ such that for any commutative ring $R$, $G(R) = \{A \in \mathrm{GL}_n(R \otimes_{\mathbb{Z}} O_F) | \, A\overline{A}^t = I\}$. We shall also denote by $G$ the reductive group over $\mathbb{Q}$ that is $G \otimes_{\mathbb{Z}} \mathbb{Q}$. Since $G(\mathbb{R}) = U(n)$, the compact unitary group, we also write $G = U(n)$, though we must remember that the rational structure depends on the choice of $F$. Choose $K$ such that $K_\infty = G(\mathbb{R})$ and, for all prime numbers $q$, $K_q = G(\mathbb{Z}_q)$. Then $G(\mathbb{Q}) \cap K = G(\mathbb{Z}) = \{A \in \mathrm{GL}_n(O_F) | \, A\overline{A}^t = I\}$. If $A = (a_{ij}) \in G(\mathbb{Z})$ then, for each $i$, $\sum_j a_{ij}\overline{a_{ij}} = 1$, but each term is a non-negative integer, so the only possibility is that (fixing $i$) one $a_{ij}$ is a unit and the rest are 0. Since distinct rows are orthogonal, the unit entry must be in a different column for each row. We see that $G(\mathbb{Z})$ is a semi-direct product of $(O_F^\times)^n$ (down the leading diagonal) and $S_n$ (realised as permutation matrices). From now on, we specialise to the case $n = 4$ and $F = \mathbb{Q}(\sqrt{-3})$. Then $O_F^\times$ is the group of $6^{\text{th}}$ roots of unity, and $|G(\mathbb{Z})| = 6^4 \cdot 4! = 31104$.

**Proposition 2.1.** *With $F = \mathbb{Q}(\sqrt{-3})$, $G = U(4)$ and $K$ as above, $\#(G(\mathbb{Q})\backslash G(\mathbb{A})/K) = 1$.*

D. Loeffler kindly checked this, thus enabling me to proceed. He applied the mass formula of Gan, Hanke and Yu [Gan, Hanke, Yu 01], but due to the fact that their open compact subgroup $K_3$ is different from ours, the application was far from trivial, in fact I was unable to reconstruct it. It was only much later, after completing this work, that I discovered that the fact was already known to Schiemann, by a different method, and may be read off from Table 1 of [Schiemann 98], which counts classes of unimodular hermitian lattices. In the notation of earlier sections, we have then $h = 1$, may take $z_1 = \mathrm{id}.$, and there is a unique $\Gamma := \Gamma_1 = G(\mathbb{Q}) \cap K = G(\mathbb{Z})$.

Let $V$ be (the space of) an irreducible algebraic representation of $G$, defined over $\mathbb{C}$. Such $V$ are in one-to-one correspondence with irreducible algebraic representations of $\mathrm{GL}_4$. (The latter may be restricted from $\mathrm{GL}_4(\mathbb{C})$ to $G(\mathbb{R})$.) Each is determined by a choice of highest weight $\mathrm{diag}(t_1, t_2, t_3, t_4) \mapsto t_1^{a_1} t_2^{a_2} t_3^{a_3} t_4^{a_4}$, with integers $a_1 \geq a_2 \geq a_3 \geq a_4$. We denote such a highest weight by $\Lambda = [a_1, a_2, a_3, a_4] = \sum_i a_i e_i$. When restricted to the diagonal $(\mathbb{C}^\times)^4$, the representation decomposes as a sum of characters (one of which is $\Lambda$). This sum of characters may be expressed formally using the Weyl character formula, as

$$\frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\Lambda+\rho)-\rho}}{\prod_{\alpha>0}(1 - e^{-\alpha})}.$$

In our case, the Weyl group $W \simeq S_4$, acting in the natural way on the characters $e_i$. The sign $(-1)^{\ell(w)}$ is $\pm 1$ depending on whether $w$ is an even or an odd permutation. The $\alpha > 0$ are the positive roots $\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3$, where $\alpha_i := e_i - e_{i+1}$, and $\rho = \frac{1}{2}(3\alpha_1 + 4\alpha_2 + 3\alpha_3)$ is half their sum. For a character $\lambda$, $e^\lambda$ is just multiplicative notation for the same character.

**Proposition 2.2.** *Let $V$ be the representation of $G = U(4)$ (for $F = \mathbb{Q}(\sqrt{-3})$) with highest weight $\Lambda = [5, 1, -1, -5] = 5\alpha_1 + 6\alpha_2 + 5\alpha_3$.*

1. $\dim V = 5220$.
2. $\dim S = 2$, *where $S = A(G, K, V)$, with $K = G(\mathbb{R}) \times \prod G(\mathbb{Z}_q)$ as above.*

*Proof.* (1) This is a direct application of the Weyl dimension formula [Carter, Segal and Macdonald 95, I.3.4].

(2) $S = V^\Gamma$, and the dimension of this is $\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} \mathrm{ch}_V(\gamma)$. Each $\gamma$ is a unitary matrix, so its adjoint with respect to the standard inner product is $\overline{\gamma}^t = \gamma^{-1}$. In particular, $\gamma$ commutes with its adjoint so is diagonalisable, i.e. is conjugate in $\mathrm{GL}_4(\mathbb{C})$ to a diagonal matrix, with eigenvalues $t_1, t_2, t_3, t_4$ on the diagonal. Since $V$ is the restriction of a representation of $\mathrm{GL}_4(\mathbb{C})$, $\mathrm{ch}_V(\gamma) = \mathrm{ch}_V(\mathrm{diag}(t_1, t_2, t_3, t_4))$. Working through the 24 elements of $W$, the numerator of Weyl's character formula may be evaluated, then (with the help of the computer package Maple), the quotient by the denominator can be simplified, and the result is $\frac{1}{x^5y^6z^5}$ times a polynomial of degree 32 in $x := e^{\alpha_1}, y := e^{\alpha_2}$ and $z := e^{\alpha_3}$, filling up two screens of Maple. Into this we must substitute $x = t_1/t_2$, $y = t_2/t_3$ and $z = t_3/t_4$, to find $\mathrm{ch}_V(\gamma)$. For speed, we must use floating point approximations. For $\gamma \in \Gamma$ it is not difficult to find precisely the eigenvalues $t_i$, which, $\Gamma$ being a finite group, are roots of unity, but later, for other elements $\gamma_i\gamma$ of $G(\mathbb{Q})$, we resort to using the Eigenvalues command in Maple. Anyway, using loops to work through the elements of $\Gamma$, Maple computed (in a few minutes) an approximation to $\dim S$, necessarily an integer, and to 9 decimal places obtained the answer 2.

□

## 3. The quasi-split unitary group

Let $U(2, 2)$ be defined as for $U(4)$, but with the equation $A\overline{A}^t = I$ replaced by $AJ\overline{A}^t = J$, where $J := \begin{pmatrix} 0_2 & -I_2 \\ I_2 & 0_2 \end{pmatrix}$. Then $U(2, 2)(\mathbb{R})$ is not compact.

**Proposition 3.1.** *Consider the case $F = \mathbb{Q}(\sqrt{-3})$. For each prime number $q$, there is an element $B \in \mathrm{GL}_4(F \otimes \mathbb{Q}_q) \cap \mathrm{M}_4(O_F \otimes \mathbb{Z}_q)$ such that $A \mapsto B^{-1}AB$ gives an isomorphism $\theta_q : U(2, 2)(\mathbb{Q}_q) \simeq U(4)(\mathbb{Q}_q)$, restricting (for $q \neq 3$) to $U(2, 2)(\mathbb{Z}_q) \simeq U(4)(\mathbb{Z}_q)$.*

*Proof.* First we show that it suffices to produce a $B \in \mathrm{GL}_4(F \otimes \mathbb{Q}_q) \cap \mathrm{M}_4(O_F \otimes \mathbb{Z}_q)$ such that $B\overline{B}^t = \mu J$, with $\mu$ a unit in $O_F \otimes \mathbb{Z}_q$ when $q \neq 3$. Note that when $\mu$ is a unit, $B$ is invertible in $\mathrm{GL}_4(O_F \otimes \mathbb{Z}_q)$. Now if $AJ\overline{A}^t = J$ then $(B^{-1}AB)\overline{(B^{-1}AB)}^t = B^{-1}A(B\overline{B}^t)\overline{A}^t(\overline{B}^t)^{-1} = B^{-1}A(\mu J)\overline{A}^t(\overline{B}^t)^{-1} = B^{-1}\mu J(\overline{B}^t)^{-1} = \mathrm{id}$. Conversely, if $(B^{-1}AB)\overline{(B^{-1}AB)}^t = \mathrm{id}$. then $B^{-1}A(\mu J)\overline{A}^t(\overline{B}^t)^{-1} = \mathrm{id}$., so $A(\mu J)\overline{A}^t = B\overline{B}^t = \mu J$, so $AJ\overline{A}^t = J$. Thus we have a bijection between $U(2,2)(\mathbb{Q}_q)$ and $U(4)(\mathbb{Q}_q)$. When $B$ is invertible in $\mathrm{GL}_4(O_F \otimes \mathbb{Z}_q)$, this restricts to a bijection between $U(2,2)(\mathbb{Z}_q)$ and $U(4)(\mathbb{Z}_q)$, because these are the intersections with $\mathrm{GL}_4(O_F \otimes \mathbb{Z}_q)$ of $U(2,2)(\mathbb{Q}_q)$ and $U(4)(\mathbb{Q}_q)$, respectively. To produce $B$, we consider three cases.

(1) $\mathbf{q = 2}$. The prime 2 is inert in $F$, and $O_F \otimes \mathbb{Z}_2 = \mathbb{Z}_2[(1+\sqrt{-3})/2]$. If we let $x = 1$, then $x \equiv 1 \pmod 4$ and $x^2 + 7 \equiv 0 \pmod 8$. Hensel's Lemma produces $x \in \mathbb{Z}_2$ such that $x \equiv 1 \pmod 4$ and $x^2 + 7 = 0$. Let $\alpha = \frac{x+\sqrt{-3}}{2}, \beta = \frac{1-\sqrt{-3}}{2} \in O_F \otimes \mathbb{Z}_2$. Then $\alpha\overline{\alpha} + \beta\overline{\beta} = (x^2+7)/4 = 0$, and $\alpha\overline{\beta} - \beta\overline{\alpha} = -\mu$, where $\mu = -\sqrt{-3}\left(\frac{1+x}{2}\right)$, which is a unit in $O_F \otimes \mathbb{Z}_2$. Now

$$B\overline{B}^t = \mu J \text{ if we let } B = \begin{pmatrix} \alpha & \beta & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ \beta & -\alpha & 0 & 0 \\ 0 & 0 & \beta & -\alpha \end{pmatrix}.$$

(2) $\mathbf{q \neq 2, 3}$. The element $\sqrt{-3}$ belongs either to $\mathbb{Z}_q$ (if $q$ splits) or to a quadratic extension. In $\mathbb{F}_q$ there are $(q+1)/2$ squares and $(q-1)/2$ non-squares, so among the elements $-y^2 - 6$, for $y \in \mathbb{F}_q$, there must be at least one square. Hence $x^2 + y^2 + 6 = 0$ has a solution in $\mathbb{F}_q^2$, which may be lifted, by Hensel's Lemma, to a solution in $\mathbb{Z}_q^2$. By flipping a sign if necessary, we may ensure that $x \not\equiv -y \pmod q$. (For $q = 3$ this is a problem, because $x = y = 0$.) Now let $\alpha = x + \sqrt{-3}, \beta = y + \sqrt{-3}$, and $B$ as above. We find that $\alpha\overline{\alpha} + \beta\overline{\beta} = x^2 + y^2 + 6 = 0$, and $\alpha\overline{\beta} - \beta\overline{\alpha} = 2(x+y)\sqrt{-3}$, which is a unit.

(3) $\mathbf{q = 3}$. Let $\alpha = x + \sqrt{-3}$, $\beta = y - \sqrt{-3}$, with $x, y \in \mathbb{Z}_3$ such that $x^2 + 2y^2 + 9 = 0$ and $x \equiv y \equiv 1 \pmod 3$. Then $\alpha\overline{\alpha} + 2\beta\overline{\beta} = 0$ and $\alpha^2 + 2\beta^2 = \sqrt{-3}\delta^{-1}$, where $\delta^{-1} := (2x - 4y) + 6\sqrt{-3}$, which is a unit since $x \equiv y \equiv 1 \pmod 3$.

$$\text{We let } B = \begin{pmatrix} \delta\alpha & \delta\beta & \delta\beta & 0 \\ 0 & \delta\beta & -\delta\beta & \delta\alpha \\ \overline{\alpha} & \overline{\beta} & \overline{\beta} & 0 \\ 0 & \overline{\beta} & -\overline{\beta} & \overline{\alpha} \end{pmatrix}. \text{ Then } B\overline{B}^t = -\sqrt{-3}J.$$

□

## 4. A Hecke operator

Let $T = T_{K_f r K_f} : S \to S$, with $r \in G(\mathbb{A}_f)$, and $K_f r K_f = \bigsqcup_{i=1}^n r_i K_f$. In our special case with $F = \mathbb{Q}(\sqrt{-3})$ and $K = G(\mathbb{R}) \times \prod G(\mathbb{Z}_q)$, we have $\#(G(\mathbb{Q})\backslash G(\mathbb{A})/K) = 1$, and we are taking $z_1 = \mathrm{id}$. to represent the unique class. For each $r_i$ there are $k_i \in K$ and $\gamma_i \in G(\mathbb{Q})$ such that $r_i k_i = \gamma_i$. This $\gamma_i$ is unique only up to right multiplication by elements of $\Gamma = K \cap G(\mathbb{Q})$ (but we imagine a choice has been made, for each $i$). Notice that the $r_i$ may in fact be replaced by the $\gamma_i$: $K_f r K_f = \bigsqcup_{i=1}^n \gamma_i K_f$. Any $r$ may be factored as a product $r = \prod r_q$, where $r_q \neq \mathrm{id}$. at most in the $q$-component (so is identified with an element of $G(\mathbb{Q}_q)$), and $r_q \in K_q$ for all but finitely many $q$, so $T_{K_f\,rK_f}$ is a product of finitely many $T_{K_q\,r_qK_q}$. Hence, from now on we look only at local Hecke operators $T = T_{K_f\,rK_f} = T_{K_q\,rK_q}$ with $r \in G(\mathbb{Q}_q)$.

The formula $T(f)(g) = \int_{G(\mathbb{A}_f)} \chi_{K_f\,rK_f}(h)f(gh)\,dh$ becomes, with $g = \mathrm{id}\,.$ and $f$ identified with $v = f(\mathrm{id}\,.) \in V^\Gamma$, a finite sum $T(v) = \sum_i \gamma_i v$. To employ the method of D. Loeffler [Loeffler 08] to find a basis of Hecke eigenforms for $S = A(G, K, V)$, we would find the $\gamma_i$, and the 5220 by 5220 matrices by which they act on $V$, then apply these matrices to a basis for $S = V^\Gamma$, to make explicit the action of $T$ on $S$. Loeffler applied this method successfully with $G = U(3)$, $F = \mathbb{Q}(\sqrt{-7})$ and various $V$ of small dimension. However, in our case (with $V$ of highest weight $[5, 1, -1, -5]$) it is not feasible, because of the difficulty of working with semi-standard Young tableaux etc. for such a large dimension, to find the matrices and a basis for $V^\Gamma$. On the other hand, finding the $\gamma_i$ still turns out to be useful. I am grateful to Loeffler for his advice on what parts of his $U(3)$ work may or may not carry over to our example.

G. Harder suggested finding $\dim S$ before worrying further about what to do next, since it could have been 0. As we shall see later, I had a reason to believe it should be non-zero, so when it turned out to be only 2, it was as if it was trying hard to be non-zero for this reason, but had no reason to be much bigger. Whether lucky or not, it was certainly useful that it was quite so small. As Harder pointed out, to calculate the eigenvalues of $T$ it now sufficed to calculate the traces of $T$ and $T^2$. One might expect these traces to be given by some version of Goresky and MacPherson's topological trace formula, i.e. a sum over conjugacy classes of elements $\gamma \in G(\mathbb{Q})$, of terms, each of which is a product of factors including $\mathrm{ch}_V(\gamma)$ and an orbital integral over the $G(\mathbb{A}_f)$-conjugacy class of $\gamma$ [Harder 93]. It seems very difficult to apply such a formula directly. The fact that the trace of the identity (i.e. $\dim V$) is given by a simple formula suggested that there might also be a simple formula for the trace of any Hecke operator, and this led to Theorem 1.1. In fact, the correct way to understand the relation between Theorem 1.1 and the topological trace formula is to note that the latter is deduced from the Grothendieck-Lefschetz-Verdier fixed point formula, whose relation to Theorem 1.1 was explained in Section 1.

We concentrate now on our special case, seeking the traces of $T$ and $T^2$, where $T$ is a certain Hecke operator concentrated locally at the inert prime $p = 2$. In $U(2,2)(\mathbb{Q}_p)$ is an element $\tilde{r} = \mathrm{diag}(p, 1, 1/p, 1)$. The double coset $U(2,2)(\mathbb{Z}_p)\,\tilde{r}\,U(2,2)(\mathbb{Z}_p)$ has a decomposition $\bigsqcup \tilde{r}_i\,U(2,2)(\mathbb{Z}_p)$, very much like (4.4) of [Klosin 11] (which is for right cosets rather than left cosets). The number of $\tilde{r}_i$ is $p^6 + p^4 + p^2 + 1 + 2(p-1) + (p-1)(p^2-1)$, which for $p = 2$ is equal to 90. Applying the isomorphism $\theta_2 : U(2,2)(\mathbb{Q}_2) \to U(4)(\mathbb{Q}_2)$, letting $r := \theta_2(\tilde{r})$ and each $r_i = \theta(\tilde{r}_i)$, we have $K_2\,rK_2 = \bigsqcup r_iK_2$, and let $T = T_{K_2\,rK_2}$.

We need to find the 90 elements $\gamma_i$. Since $2\tilde{r} \in \mathrm{GL}_4(\mathbb{Z}_2)$, as an element of $G(\mathbb{A}_f)$, $2r$ belongs to $\prod_q G(\mathbb{Z}_q)$, so each $2\gamma_i \in \mathrm{GL}_4(O_F)$. If $2\gamma_i = (a_{ij})$ then for each $j$, $\sum_{i=1}^4 a_{ij}\overline{a_{ij}} = 4$, while distinct columns are orthogonal, and the invariant factors of $2\gamma_i$ (using left and right multiplication by $\mathrm{GL}_4(\mathbb{Z}_2)$) are $2^2, 2, 2, 1$. We search for such $2\gamma_i$, up to right multiplication by $\Gamma$ (which amounts to permutations of columns and multiplying columns by $6^{\text{th}}$ roots of unity).

**Lemma 4.1.** *The $\gamma_i$ can be taken to be* 54 *left* $\Gamma$*-translates of* $(1/2)A$ *and* 36 *left* $\Gamma$*-translates of* $(1/2)B$*, where* $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & \sqrt{-3} & 0 & 0 \\ \sqrt{-3} & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

*Proof.* $A$ is the Kronecker square of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, which may be converted, by left and right multiplication by $\mathrm{GL}_2(\mathbb{Z})$, to $\mathrm{diag}(1,2)$, from which it follows that the invariant factors of $A$ are $1, 2, 2, 2^2$. In $B$, if we subtract $\sqrt{-3} \times$ row 1 from row 2 then we get $\mathrm{diag}(1,4,2,2)$, so again the invariant factors are right.

The number of left $\Gamma$-translates of $A$ is 31104, but many of them are right $\Gamma$-equivalent. We need to show that the number of right $\Gamma$-equivalence classes of left $\Gamma$-translates of $A$ is 54. The left action of $S_4$ by permutations of rows is generated by transpositions of rows, but it is easy to check that these are all equivalent to column operations, or sequences of such. For example, swapping rows 2 and 4 is the same as swapping columns 3 and 4, while swapping rows 1 and 2 is the same as multiplying columns 2 and 4 by $-1$, then swapping them. Thus we may completely disregard the left action of $S_4$, and concentrate on that of $(\mu_6)^4$. Multiplying the top row of $A$ by a $6^{\text{th}}$ root of unity is the same as multiplying each column by the same, then dividing rows 2, 3 and 4 by the same. So we need consider only the subgroup $(\mu_6)^3$ acting on rows 2, 3 and 4. Each element in the subgroup $\{(1,1,1), (-1,-1,1), (-1,1,-1), (1,-1,-1)\}$, is equivalent to some permutation of the columns by an element of the Klein 4 subgroup. It is easy to see that this is the full stabiliser of the coset $A\Gamma$ under the left action of $(\mu_6)^3$. So the number of right $\Gamma$-equivalence classes of left $\Gamma$-translates of $A$ is indeed $6^3/4 = 54$.

Now we look at $B$. Multiplying a row containing 2 by a $6^{\text{th}}$ root of unity is the same as multiplying the column containing the same 2 by the same. Multiplying the second row by a $6^{\text{th}}$ root of unity is the same as multiplying columns 1 and 2 by the same, then dividing row 1 by the same. Swapping rows 1 and 2 is the same as swapping columns 1 and 2. All that is left of the left $S_4$ action is $\binom{4}{2} = 6$ ways of deciding which two rows contain two non-zero elements, and all that is left of the $(\mu_6)^4$ action is $(\mu_6)$ on the top row. So the number of right $\Gamma$-equivalence classes of left $\Gamma$-translates of $B$ is indeed $6 \times 6 = 36$.

One might object that we know $A, B \in \mathrm{GL}_4(\mathbb{Z}_2)\, r\mathrm{GL}_4(\mathbb{Z}_2)$, but how do we know that $A, B \in K_2\, rK_2$? A little trial and error shows that the only way to find four mutually orthogonal columns of norm 4, with entries in $O_F$, and to get a matrix with invariant factors $1, 2, 2, 2^2$ from them, is to do what we have done. Given that we know that there exist 90 rational representatives (integral away from 2) of the left cosets $r_i K_2$ into which $K_2\, rK_2$ decomposes, the 90 elements we have found must be them. □

## 5. The computation and its results

Having found the $\gamma_i$, we must now evaluate $\mathrm{tr}(T) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma, \gamma_i} \mathrm{ch}_V(\gamma_i \gamma)$. (Recall that $V$ is of highest weight $[5, 1, -1, -5]$.) The character values may be calculated as described above, and it is not difficult to write a Maple program to work through the different elements of $\Gamma$ and the $\gamma_i$. I got it to work through $\Gamma$ for each $\gamma_i$, and to display 90 subsums before the grand total. This took three and a half days of computer time, but need only have taken a couple of hours, since there were only two different subsums, one taken 54 times, the other 36 times. With hindsight, the reason is that for $\gamma, \gamma' \in \Gamma$, $\mathrm{ch}_V(\gamma A \gamma') = \mathrm{ch}_V(A\gamma'\gamma)$, so for left $\Gamma$-equivalent $\gamma_i$, the subsum is simply rearranged. The subsums computed (via floating point approximations) were clearly close approximations to an integer and a half integer, and it appears that

$$\mathrm{tr}(T) = \frac{54(-2430) + 36(9355.5)}{31104} = \frac{423}{2^6}.$$

I am indebted to J. Tilouine for asking, at the end of a talk, whether there is an integral structure that explains the integrality away from 2. Indeed there is. The representation with highest weight $[5, 1, -1, -5]$ is $\det^{-5}$ times that with highest weight $[10, 6, 4, 0]$, which can be realised as a subspace of $W^{\otimes(10+6+4)} = W^{\otimes 20}$, where $W$ is the 4-dimensional standard representation, as in II.13 of [Carter, Segal and Macdonald 95]. Choosing the standard basis for $W$ imposes an integral structure on $W$ and on all its tensor powers, and that for $W^{\otimes 20}$ induces one on the subspace $V$, and its subspace $S = V^{\Gamma}$. Each $\gamma_i$ is $1/2$ times an integral matrix, so acts by $(1/2^{20})$ times an integral matrix, with respect to an integral basis of $S$, and likewise for $T$. So we know, independent of computation, that $2^{20}\mathrm{tr}(T)$ is an integer.

This crude bound can be improved to $2^{10}\mathrm{tr}(T) \in \mathbb{Z}$, by decomposing into weight spaces as in [Harder 10], immediately preceding Section 2.4.1. (I am grateful to Harder for directing me to this section.) As in the previous paragraph, we may choose an $O_{F,2}$-module $V$ on which $\mathrm{GL}_4(O_{F,2})$ acts via the representation with highest weight $[5, 1, -1, 5]$. Adjoining to $O_{F,2}$ a root of unity of sufficiently large odd order $m$ that all the weights occurring are distinct on the diagonal subgroup $\mu_m^4$, we may decompose $V$, integrally at 2, into weight spaces. Then we just need to look at the eigenvalues by which the element $\tilde{r} = \mathrm{diag}(2, 1, 1/2, 1)$ acts on each weight space, and see what the largest power of 2 occurring in the denominator is. For the weights $[-5, 1, 5, -1]$ or $[-5, -1, 5, 1]$ it attains its maximum $2^{10}$.

Now recall that for $v \in V^{\Gamma}$, $T(v) = \sum_i \gamma_i v$. Hence $T^2(v) = \sum_{i,j} \gamma_i\gamma_j v$, and $\mathrm{tr}(T^2) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma,i,j} \mathrm{ch}_V(\gamma_i\gamma_j\gamma)$. Left $\Gamma$-equivalences between the $\gamma_i$ may be exploited as before, but because of the summation over $j$, the computation of $\mathrm{tr}(T^2)$ is necessarily much longer than that of $\mathrm{tr}(T)$, and it took a week of computer time (on a standard desktop computer). The result was multiplied by 2 repeatedly until it got close to an integer. When multiplied by $2^{12}$ it became 1629476.912, so it appears that

$$\mathrm{tr}(T^2) = \frac{1629477}{2^{12}}.$$

Since we are only seeking experimental evidence for a conjecture, rather than to prove anything, we assume from now on that the given values for $\mathrm{tr}(T)$ and $\mathrm{tr}(T^2)$ are correct. (See also the comment at the end.)

Now if $a$ and $b$ are the eigenvalues of $T$ on $S$, then $a+b = \frac{423}{2^6}$, $a^2+b^2 = \frac{1629477}{2^{12}}$, and $ab = (1/2)((a+b)^2 - (a^2+b^2)) = -\frac{362637}{2^{11}}$. Solving the quadratic equation for $a$ and $b$, we find the following.

**Proposition 5.1.** *Let $V$ and $S$ be as in Proposition 2.2. Let $T = T_{K_2 r K_2}$ with $r = \theta_2(\mathrm{diag}(2, 1, 1/2, 1))$ as in §4. Assuming that the traces of $T$ and $T^2$ on $S$ are as above, the eigenvalues for $T$ acting on $S$ are $\frac{1089}{2^6}$ and $-\frac{333}{2^5}$.*

## 6. Motivation for the computation

More details may be found in [Dummigan 11]. Let $F=\mathbb{Q}(\sqrt{-D})$, $k>1$ an odd integer, and let $\sum_{n=1}^{\infty} a_n(f)q^n = f \in S_k(\Gamma_0(D),\chi_{-D})$ be a non-CM, normalised Hecke eigenform, where $\chi_{-D}$ is the quadratic character associated to $F/\mathbb{Q}$. Let $E$ be a number field containing $\mathbb{Q}(\{a_n\})$. Take an integer $j$ such that $0\leq j<(k-3)/2$. The ratio $\frac{L(\mathrm{Sym}^2 f,2k-2-2j)}{(2\pi i)^{3k-3-4j}\,\langle f,f\rangle}$ is known to belong to $E$. Suppose that $\lambda \mid \ell$ is a prime divisor of $E$ such that $\mathrm{ord}_\lambda\left(\frac{L(\mathrm{Sym}^2 f,2k-2-2j)}{(2\pi i)^{3k-3-4j}\,\langle f,f\rangle}\right)>0$, with $\ell>2k$ and $\ell\nmid D$.

Conjecture 6.1 of [Dummigan 11], specialised to an inert prime $p$, and the Hecke operator $\tilde{T}$ for $U(2,2)(\mathbb{Z}_p)\,\tilde{r}U(2,2)(\mathbb{Z}_p)$, where $\tilde{r}=\mathrm{diag}(p,1,1/p,1)$, predicts that there exists a cuspidal automorphic representation $\Pi$ of $U(2,2)(\mathbb{A})$, (the same $\Pi$ for different $p$) with the property that the local component $\Pi_p$ is unramified, and such that the eigenvalue $\mu_{\tilde{T}}(\Pi_p)$ for $\tilde{T}$ on a spherical vector satisfies the congruence

$$\mu_{\tilde{T}}(\Pi_p)\equiv p^{2(j+3-k)}(1+p^{2(k-2-2j)})(a_p(f)^2+2p^{k-1})+(p^3-p^2+p-1)\pmod{\lambda'},$$

where $\lambda'$ is a divisor of $\lambda$ in a sufficiently large number field.

There should be a functorial transfer to an automorphic representation of $G=U(4)$, which is isomorphic to $U(2,2)$ at all finite places but compact at infinity. Hoping that there is a $K$-fixed vector, it should show up in $A(G,K,V)$, and the conjectured congruence should be satisfied by the eigenvalue of the Hecke operator $T$ associated to $K_p\,\theta_p(\tilde{r})K_p$. The nature of the $U(2,2)(\mathbb{R})$-component of $\Pi$ (suppressed above) is such that we should expect $V$ to be of highest weight $\Lambda=[k-3-j,j,-j,j+3-k]$.

**Example.** $S_9(\Gamma_0(3),\chi_{-3})$ is two-dimensional, spanned by a normalised newform $f$ with coefficients in $\mathbb{Q}(\sqrt{-14})$, and its conjugate [Mod].

$$f=q+6\sqrt{-14}\,q^2+(45-18\sqrt{-14})\,q^3-248\,q^4-60\sqrt{-14}\,q^5+\dots.$$

Using Theorem 4.3 in [Katsurada 10], one finds

$$\frac{-i\Gamma(8)\Gamma(14)\Gamma(7)}{\Gamma(3)}\frac{L(\mathrm{Sym}^2 f,14)}{2^{26}\pi^{20}\langle f,f\rangle}=\frac{9992960}{6561}-\frac{23680}{45927}\sqrt{-14},$$

which has norm $\frac{2^{15}\cdot 5^3\cdot 19\cdot 37^2}{3^8\cdot 7}$. We can take $\lambda$ then to be an appropriate divisor of $\ell=19$ (split in $\mathbb{Q}(\sqrt{-14})$) or 37 (inert in $\mathbb{Q}(\sqrt{-14})$). For this example, $k=9$ and $j=1$, so $\Lambda$ would be $[5,1,-1,-5]$.

The right hand side of the conjectured congruence, evaluated for $p=2$ and this choice of $f$ (and with $k=9, j=1$), becomes $1665/2^7$. The differences between this and the two eigenvalues for $T$ which were computed above (as in Proposition 5.1), are $\frac{3^3\cdot 19}{2^7}$ and $\frac{3^4\cdot 37}{2^7}$. The prominent appearance of the same factors 19 and 37 provides strong support for the conjecture. To meet the possible objection that the 1629476.912 in the previous section is not especially close to an integer (presumably because of accumulated errors), I should point out that there were not any lower powers of 2 producing anything closer to an integer, so there is nothing I had earlier rejected for not producing the desired 19 and 37. Moreover, given that $\mathrm{tr}(T)$ has denominator $2^6$, it is not surprising that $\mathrm{tr}(T^2)$ should have denominator $2^{12}$. Furthermore, it seems unlikely that the rationality of the eigenvalues of $T$ would have happened by accident if the calculation were incorrect.

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University of Sheffield, School of Mathematics and Statistics, Hicks Building, Hounsfield Road, Sheffield, S3 7RH, U.K.

*E-mail address*: `n.p.dummigan@shef.ac.uk`