# EISENSTEIN CONGRUENCES FOR ODD ORTHOGONAL GROUPS

NEIL DUMMIGAN

Abstract. We work out instances of a general conjecture on congruences between Hecke eigenvalues of induced and cuspidal automorphic representations, modulo divisors of certain critical $L$-values, in the case that the group is a split orthogonal group in an odd number of variables. We also consider functorial lifts to unitary groups, and theta correspondences to metaplectic groups.

## 1. Introduction

Ramanujan discovered the congruence $\tau(p) \equiv 1 + p^{11} \pmod{691}$ (for all primes $p$), where $\Delta = \sum_{n=1}^{\infty} \tau(n)q^n = q\prod_{n=1}^{\infty}(1-q^n)^{24}$. We may view this as being a congruence between Hecke eigenvalues, for $T(p)$ acting on the cusp form $\Delta$ of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$, and on the Eisenstein series $E_{12}$ of weight 12. The modulus 691 comes from a certain $L$-function evaluated at a critical point depending on the weight; specifically it divides the numerator of the rational number $\frac{\zeta(12)}{\pi^{12}}$. Conjecture 4.2 of [BD] is a very wide generalisation of Ramanujan's congruence, to congruences between Hecke eigenvalues of automorphic representations of $G(\mathbb{A})$, where $\mathbb{A}$ is the adele ring and $G/\mathbb{Q}$ is any connected, split reductive group. (The case of a group split over an imaginary quadratic field was dealt with in [Du1].) On one side of the congruence is a cuspidal automorphic representation $\tilde{\Pi}$. On the other is one induced from a cuspidal automorphic representation $\Pi$ of the Levi subgroup $M$ of a maximal parabolic subgroup $P$. The modulus of the congruence comes from a critical value of a certain $L$-function, associated to $\Pi$ and to the adjoint representation of the $L$-group $\hat{M}$ on the Lie algebra $\hat{\mathfrak{n}}$ of the unipotent radical of the maximal parabolic subgroup $\hat{P}$ of $\hat{G}$. Starting from $\Pi$, we conjecture the existence of $\tilde{\Pi}$, satisfying the congruence. Ramanujan's congruence is an instance of the case $G = \mathrm{GL}_2$, $M = \mathrm{GL}_1 \times \mathrm{GL}_1$. Harder's conjecture on congruences between genus-1 and genus-2 (vector-valued) Siegel modular forms is the case $G = \mathrm{GSp}_2$, $P$ the Siegel parabolic, $M \simeq \mathrm{GL}_1 \times \mathrm{GL}_2$. In [BD] we looked at these examples, and others involving $\mathrm{GSp}_3$ and $G_2$.

A main focus of this paper is the case $G = \mathrm{SO}(n+1, n)$, $M \simeq \mathrm{GL}_1 \times \mathrm{SO}(n, n-1)$. This is arguably the most direct generalisation of the congruences of Ramanujan and Harder, which reappear as the cases $n = 1$ and $n = 2$, via the special isomorphisms $\mathrm{PGL}_2 \simeq \mathrm{SO}(2, 1)$ and $\mathrm{PGSp}_2 \simeq \mathrm{SO}(3, 2)$. In the case $n = 1$ the modulus comes from Riemann's zeta function. In the case $n = 2$ it comes from the Hecke $L$-function of a genus-1 cuspidal Hecke eigenform. In the case $n = 3$ it comes from the spinor $L$-function of a genus-2 form. This is quite satisfying, since it was only the standard $L$-function of such a form that appeared in [BD] (in the case

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$G = \mathrm{GSp}_3, M \simeq \mathrm{GL}_1 \times \mathrm{GSp}_2$). The shape of the conjectural congruences is worked out in §2, and the special cases $n = 1, 2, 3$ are examined further in §3. Actually, for each $n$ the conjecture also predicts congruences modulo divisors of Riemann zeta-values. In §4 we see how such congruences are implied by Ramanujan's , combined with a conjectured endoscopic lift from $\mathrm{SO}(2,1) \times \mathrm{SO}(n, n-1)$ to $\mathrm{SO}(n+1, n)$.

In [BD, §§11,12] we observed the compatibility between different instances of the general conjecture, for $G = G_2$ and $G = \mathrm{GSp}_3$, via the Gross-Savin functorial lift from $G_2$ to $\mathrm{GSp}_3$. Likewise here, in §5, we observe a compatibility between the cases $G = \mathrm{SO}(n+1, n)$, $M \simeq \mathrm{GL}_1 \times \mathrm{SO}(n, n-1)$ and $G = U(n, n), M \simeq \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_1 \times U(n-1, n-1)$, via a conjectured functorial lift from $\mathrm{SO}(n+1, n)$ to $U(n, n)$. The interest of this is that the $U(n, n)$ congruences, between Klingen-Eisenstein series and cusp forms, may be easier to prove.

Calculations of Hecke eigenvalues by Faber and van der Geer (using counting of points mod $p$) provided much numerical evidence for cases of Harder's conjecture, in each instance confirming the congruence for $p \leq 37$ [FvdG, vdG]. Ibukiyama's idea for attacking Harder's conjecture [I1] is to apply his conjectural genus-2 Shimura-type correspondence, to convert it to a congruence between a Klingen-Eisenstein series and a cusp form, both of half-integral weight. This can further be converted to a congruence between Jacobi forms. He has proved one instance of such a congruence (for all $p$) [I2]. Inspired by this, in §6 we look at what happens to the conjectured congruences when we apply a theta correspondence from $O(n+1, n)$ to $\widetilde{\mathrm{Sp}}_n$. In particular, for $n = 2$ we arrive at a (conjecturally) automorphic representation of $\widetilde{\mathrm{Sp}}_n(\mathbb{A})$ whose component at $\infty$ is non-holomorphic discrete series, unlike Ibukiyama's. If the integral weight of the cusp form in Harder's conjecture is $\mathrm{Sym}^j \otimes \det^k$ (with $k \geq 3$) then Ibukiyama's conjectural correspondence only applies when $k$ is odd, since otherwise the space of half-integral weight forms turns out to vanish. Using results of Gan and Savin [GS] and Adams and Barbasch [AB] on the theta dichotomy (about which of two extensions of a representation from $\mathrm{SO}(n+1, n)$ to $O(n+1, n)$ participates in a local theta correspondence), we find that our automorphic representation of $\widetilde{\mathrm{Sp}}_n(\mathbb{A})$ would be non-zero precisely in the other case, that $k$ is even. Using Borel's harmonic cusp forms, it seems it should be possible to turn this into a correspondence with a fairly concrete space of functions.

In §7 we look again at $G = \mathrm{SO}(n+1, n)$, but this time with $M \simeq \mathrm{GL}_2 \times \mathrm{SO}(n-1, n-2)$, and work out what the general conjecture says in this instance. The cases $n = 3$ and $n = 4$, involving genus-1 and genus-2 Siegel modular forms, and tensor product $L$-functions, seem especially interesting.

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## 2. $G = \mathrm{SO}(n+1, n)$, $M \simeq \mathrm{GL}_1 \times \mathrm{SO}(n, n-1)$

Let $G = \mathrm{SO}(n+1, n) = \{g \in M_{2n+1} : {}^t gJg = J, \det(g) = 1\}$, with $J = \begin{pmatrix} 0_n & 0 & I_n \\ 0 & 2 & 0 \\ I_n & 0 & 0_n \end{pmatrix}$. This is a connected, reductive (even semi-simple) algebraic group, split over $\mathbb{Q}$. It has a maximal torus $T$ comprising elements of the form $\mathrm{diag}(t_1, \ldots, t_n, 1, t_1^{-1}, \ldots, t_n^{-1})$, which is mapped to $t_i$ by characters $e_i$, for $1 \leq i \leq n$, which span the character group $X^*(T)$. The cocharacter group $X_*(T)$ is spanned by $\{f_1, \ldots, f_n\}$, where $f_1 : t \mapsto \mathrm{diag}(t, 1, \ldots, 1, 1, t^{-1}, 1, \ldots, 1)$, etc. so $\langle e_i, f_j \rangle = \delta_{ij}$, where $\langle , \rangle :$

$X^*(T) \times X_*(T) \to \mathbb{Z}$ is the natural pairing. We can order the roots so that the set of positive roots is $\Phi_G^+ = \{e_i - e_j : i < j\} \cup \{e_i : 1 \leq i \leq n\} \cup \{e_i + e_j : i < j\}$, with simple positive roots $\Delta_G = \{e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n, e_n\}$. The half-sum of the positive roots is $\rho_G = \frac{1}{2}((2n-1)e_1 + (2n-3)e_2 + \cdots + e_n)$. The Weyl group $W_G$ is generated by permutations of the $t_i$ and by inversions swapping $t_i$ with $t_i^{-1}$. The long element $w_0^G$ is the product of all the inversions.

If we choose the simple root $\alpha = e_1 - e_2$, this determines a maximal parabolic subgroup $P = MN$, where $N$ is the unipotent radical and $M$ is the Levi subgroup, characterised by $\Delta_M = \Delta_G - \{\alpha\}$. Then $M \simeq \mathrm{GL}_1 \times \mathrm{SO}(n, n-1)$. The positive roots occurring in the Lie algebra of $N$ are $\Phi_N = \Phi_G^+ - \Phi_M^+ = \{e_1 - e_2, \dots, e_1 - e_n, e_1, e_1 + e_2, \dots, e_1 + e_n\}$, i.e. those positive roots whose expression as a sum of simple roots includes $\alpha$. The half-sum is $\rho_P = \frac{2n-1}{2} e_1$, and $\langle \rho_P, \check{\alpha} \rangle = \frac{2n-1}{2}$, where $\check{\alpha}$ is the coroot associated with $\alpha$. Then $\tilde{\alpha} := \frac{1}{\langle \rho_P, \check{\alpha} \rangle} \rho_P$ is $e_1$.

Let $\hat{G}$ be the Langlands dual group of $G$. (In our particular case, $\hat{G} \simeq \mathrm{Sp}_n$, a symplectic group of $2n$-by-$2n$ matrices. This is explained in more detail in [Du1, §6].) Then $\hat{G}$ has a maximal torus $\hat{T}$ with $X^*(\hat{T}) \simeq X_*(T)$ and $X_*(\hat{T}) \simeq X^*(T)$. Under these isomorphisms, roots of $\hat{G}$ become coroots of $G$, and coroots of $\hat{G}$ become roots of $G$, with $\check{\Delta} := \{\check{\beta} : \beta \in \Delta_G\}$ mapping to a set of simple positive roots for $\hat{G}$. We can define a maximal parabolic subgroup $\hat{P}$ of $\hat{G}$, with Levi subgroup characterised by having set of simple positive roots $\check{\Delta} - \{\check{\alpha}\}$, hence identifiable with $\hat{M}$. Let $\hat{N}$ be the unipotent radical of $\hat{P}$, with Lie algebra $\hat{\mathfrak{n}}$.

Let $\Pi'$ be a unitary, cuspidal, automorphic representation of $\mathrm{SO}(n, n-1)(\mathbb{A})$, $\Pi = 1 \times \Pi'$, which is a unitary, cuspidal, automorphic representation of $M(\mathbb{A})$. Let $\lambda = a_1 e_2 + \cdots a_{n-1} e_n$, with $a_1 \geq a_2 \geq \ldots \geq a_{n-1} \geq 0$ be the infinitesimal character of $\Pi'_\infty$ (or equally of $\Pi_\infty$, up to $W_M$). See [BD, §2] for further explanation. We shall assume that the $a_i$ are all distinct, with $a_{n-1} > 0$, i.e. that $\lambda$ is regular.

For any prime $p$ such that the local component $\Pi_p$ (a representation of $M(\mathbb{Q}_p)$) is unramified (i.e. spherical, with a non-zero $M(\mathbb{Z}_p)$-fixed vector) let $\chi_p = -[\log_p(\beta_1)e_2 + \log_p(\beta_2)e_3 + \cdots \log_p(\beta_{n-1})e_n] \in X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}$ be such that $\Pi_p$ is isomorphic to the (unitarily) parabolically induced representation $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{M(\mathbb{Q}_p)}(|\chi_p|_p)$. Here $B$ is a Borel subgroup of $M$ containing $T$, and for $s \in \mathbb{C}$, $\chi \in X^*(T)$ and a valuation $v$, $|s\chi|_v(t) := |t|_v^s$. The $p$-adic valuation is normalised so that $|p|_p = p^{-1}$. Note that $|\beta_1| = \cdots = |\beta_{n-1}| = 1$ if $\Pi'_p$ is tempered, which will be the case for us. This $\chi_p \in X^*(T) \otimes i\mathbb{R}$ gives rise to $t(\chi_p) \in \hat{T}(\mathbb{C}) \subset \hat{M}(\mathbb{C})$ such that, for any $\mu \in X_*(T) = X^*(\hat{T})$, $\mu(t(\chi_p)) = |\chi_p(\mu(p))|_p$. The conjugacy class of $t(\chi_p)$ in $\hat{M}(\mathbb{C})$ is the Satake parameter of $\Pi_p$, but we shall give $\chi_p$ the same title.

Given a representation $r : \hat{M} \to \mathrm{GL}_d$, we may define a local $L$-factor

$$L_p(s, \Pi_p, r) := \det(I - r(t(\chi_p))p^{-s})^{-1},$$

then an $L$-function (in general incomplete)

$$L_\Sigma(s, \Pi, r) := \prod_{p \notin \Sigma} L_p(s, \Pi_p, r),$$

where $\Sigma$ is a finite set of primes containing all those such that $\Pi_p$ is ramified. In particular, we take for $r$ the adjoint representation of $\hat{M}$ on $\hat{\mathfrak{n}}$, which is a direct sum of subspaces on which $\hat{T}$ acts by those positive roots of $\hat{G}$ that are not roots of $\hat{M}$. These are identified with the coroots $\check{\gamma}$ of $G$, as $\gamma$ runs through $\Phi_N$. It follows that

$$L_p(s,\Pi_p,r)^{-1} = \prod_{\gamma\in\Phi_N}(1-\check{\gamma}(t(\chi_p))p^{-s}) = \prod_{\gamma\in\Phi_N}(1-|\chi_p(\check{\gamma}(p))|_p p^{-s}).$$

Actually, $r$ is a direct sum of irreducible representations $r_i$ for some $1\le i\le m$, where $r_i$ acts on the direct sum $\hat{\mathfrak{n}}_i$ of root spaces for $\Phi_N^i := \{\check{\gamma}\in\Phi_N : \langle\tilde{\alpha},\check{\gamma}\rangle = i\}$. Then

$$L_\Sigma(s,\Pi,r) = \prod_{i=1}^m L_\Sigma(s,\Pi,r_i).$$

In our case $m=2$, with $\Phi_N^1 = \{e_1\pm e_{j+1} : 1\le j\le n-1\}$ and $\Phi_N^2 = \{e_1\}$.

| $\gamma\in\Phi_N$ | $\langle\lambda+s\tilde{\alpha},\check{\gamma}\rangle$ | $\lvert\chi_p(\check{\gamma}(p))\rvert_p$ |
|---|---|---|
| $e_1-e_{j+1}$ $(1\le j\le n-1)$ | $-a_j+s$ | $\beta_j^{-1}$ |
| $e_1+e_{j+1}$ $(1\le j\le n-1)$ | $a_j+s$ | $\beta_j$ |
| $e_1$ | $2s$ | $1$ |

Using the table, $L_p(s,\Pi_p,r_1) = \prod_{i=1}^{n-1}[(1-\beta_i p^{-s})(1-\beta_i^{-1}p^{-s})]$, and $L_\Sigma(s,\Pi,r_1)$ is the $L$-function associated with $\Pi'$ and the standard $(2n-2)$-dimensional representation of $\hat{M} = \mathrm{Sp}_{n-1}$, while $L_p(s,\Pi_p,r_2) = (1-p^{-s})$, so $L_\Sigma(s,\Pi,r_2) = \zeta_\Sigma(s)$.

For $s>0$, we consider a certain parabolically induced representation $\mathrm{Ind}_P^G(\Pi\otimes|s\tilde{\alpha}|)$ of $G(\mathbb{A})$, which has infinitesimal character (at $\infty$) $\lambda+s\tilde{\alpha}$ (up to $W_G$-action). We need $s\in\frac{1}{2}+\mathbb{Z}$ for $L_\Sigma(1+2s,\Pi,r_2)$ to be critical. Then we need all the $a_i$ to be in $\frac{1}{2}+\mathbb{Z}$ for $\lambda+s\tilde{\alpha}$ to be algebraically integral, i.e for $\langle\lambda+s\tilde{\alpha},\check{\beta}\rangle\in\mathbb{Z}$ for all $\beta\in\Phi_G^+$. (This is already true for $\beta\in\Phi_M^+$, and we can check the above table for $\beta=\gamma\in\Phi_N$.)

$$\lambda+s\tilde{\alpha} = se_1+a_1e_2+\cdots a_{n-1}e_n.$$

Then, for the obvious choice of $w\in W_G$, $w(\lambda+s\tilde{\alpha}) = a_1e_1+\cdots a_{n-1}e_{n-1}+se_n$, which is dominant and regular if we add the condition $s<a_{n-1}$ to those already imposed. This coincides with the condition for $L_\Sigma(1+s,\Pi,r_1)$ to be critical. (See the end of [BD, §3] for more on this.) We exclude the smallest value $s=1/2$ from the conjecture below.

Suppose that $q>2\max\langle\lambda,\check{\gamma}\rangle+1 = 2a_1+1$, and let $\mathfrak{q}\mid q$ be a prime divisor of $\mathfrak{q}$ in a number field sufficiently large to accommodate all the Hecke eigenvalues and normalised $L$-values we shall consider. For $1\le i\le m$, dividing $L_\Sigma(1+is,\Pi,r_i)$ by a Deligne period, we get an algebraic number, according to Deligne's conjecture on critical values of $L$-functions [De]. We shall take the Deligne period normalised as in [BD, §4], and call the algebraic number $L_{\mathrm{alg},\Sigma}(1+is,\Pi,r_i)$.

Let $\mathcal{H} = \mathcal{H}(G(\mathbb{Q}_p),G(\mathbb{Z}_p))$ be the Hecke algebra of $\mathbb{C}$-valued, compactly supported, $G(\mathbb{Z}_p)$-bi-invariant functions on $G(\mathbb{Q}_p)$. If $f\in\mathcal{H}$ then $f$ acts on any smooth representation of $G(\mathbb{Q}_p)$ by $v\mapsto\int_{G(\mathbb{Q}_p)}g(v)f(g)\,dg$, where $dg$ is a left- and right-invariant Haar measure, normalised so that $G(\mathbb{Z}_p)$ has volume 1. Then $\mathcal{H}$ is a commutative ring under convolution of functions (which corresponds to composition of operators), and is generated by the characteristic functions $T'_\mu$ of double cosets $G(\mathbb{Z}_p)\mu(p)G(\mathbb{Z}_p)$, where $\mu\in X_*(T)$ is any cocharacter. If the representation is spherical, with $G(\mathbb{Z}_p)$-fixed vector $v_0$, then necessarily $T'_\mu(v_0)$ is also fixed, but since $v_0$ is unique up to scalar multiples, $\mathcal{H}$ acts on $v_0$ by a character. The value of this character on any particular element of $\mathcal{H}$ is a "Hecke eigenvalue".

The main conjecture of [BD] is that if $\mathrm{ord}_{\mathfrak{q}}(L_{\mathrm{alg},\Sigma}(1+is,\Pi,r_i)>0$ then there exists a tempered, unitary, automorphic representation $\tilde{\Pi}$ of $G(\mathbb{A})$, unramified outside $\Sigma$, and with $\tilde{\Pi}_\infty$ of infinitesimal character $w(\lambda+s\tilde{\alpha})$, such that for all $p\notin\Sigma$, and all $\mu\in X_*(T)$, the eigenvalues of $T'_\mu$ on $\tilde{\Pi}_p$ and $\mathrm{Ind}_P^G(\Pi_p\otimes|s\tilde{\alpha}|_p)$ are congruent modulo $\mathfrak{q}$. (Actually, we scale $T'_\mu$ by a certain power of $p$ to make $T_\mu$, see below. Also, for $i=2$ we require only $q>2+2s$).)

The standard representation of $\hat{G}\simeq\mathrm{Sp}_n$ has highest weight $f_1$ (identifying $X^*(\hat{T})$ with $X_*(T)$) and complete set of weights $\{\pm f_1,\pm f_2,\dots,\pm f_n\}$. Given that this is a single $W_G$-orbit, i.e. that $f_1$ is a minuscule weight, we can calculate the "right-hand-side" of the congruence in the following way. The Satake parameter of $\mathrm{Ind}_P^G(\Pi_p\otimes|s\tilde{\alpha}|_p)$ is $\chi_p+s\tilde{\alpha}=-[\log_p(\beta_1)e_2+\log_p(\beta_2)e_3+\cdots\log_p(\beta_{n-1})e_n]+se_1$. Using this,

| $\mu$ | $\lvert(\chi_p+s\tilde{\alpha})(\mu(p))\rvert_p$ |
|---|---|
| $\pm f_1$ | $p^{\pm s}$ |
| $\pm f_{i+1}$ $(1\le i\le n-1)$ | $\beta_i^{\pm 1}$ |

The trace is $p^s+p^{-s}+\sum_{i=1}^{n-1}(\beta_i+\beta_i^{-1})$. We would multiply this by $p^{\langle\rho_G,f_1\rangle}=p^{(2n-1)/2}$ to get the eigenvalue for $T'_{f_1}$, but instead we multiply by $p^{\langle w(\lambda+s\tilde{\alpha}),f_1\rangle}=p^{a_1}$, to get the eigenvalue for $T_{f_1}$:

$$T_{f_1}(\Pi_p\otimes|s\tilde{\alpha}|_p)=p^{a_1+s}+p^{a_1-s}+\sum_{i=1}^{n-1}p^{a_1}(\beta_i+\beta_i^{-1}).$$

## 3. Special cases

3.1. **n = 1.** In the special case $n=1$, $\mathrm{SO}(2,1)\simeq\mathrm{PGL}_2$. This arises from the conjugation action of $\mathrm{PGL}_2$ on the 3-dimensional space of trace-0 matrices, preserving the quadratic form given by the determinant. If $A=\begin{pmatrix}x_2 & x_1\\ x_3 & -x_2\end{pmatrix}$ is such a trace-0 matrix, then $-2\det A=x_1x_3+2x_2^2+x_3x_1$, the quadratic form associated with $J$. Under the isomorphism, $\mathrm{diag}(t_1,t_2)\in\mathrm{PGL}_2$ maps to $\mathrm{diag}(t_1t_2^{-1},1,t_2t_1^{-1})\in\mathrm{SO}(2,1)$, as one readily checks by calculating the conjugation action on $A$. Hence the characters $ae_1$ of (the maximal torus of) $\mathrm{SO}(2,1)$ and $a(e_1'-e_2')$ of $\mathrm{PGL}_2$ correspond, where $e_i:\mathrm{diag}(t_i,t_2)\mapsto t_i$. In particular, looking at the infinitesimal character of $\tilde{\Pi}_\infty$ when $\tilde{\Pi}$ is generated by a cuspidal Hecke eigenform $f$ of weight $k'\ge 2$ and trivial character, $\frac{k'-1}{2}(e_1'-e_2')$ corresponds to $\frac{k'-1}{2}e_1$.

We have $M\simeq\mathrm{GL}_1$, and since $n-1=0$, this special case does not quite fit into the above framework, in that $\Phi_N^1$ is empty, so there are no $a_i$, no $\beta_j$, no $L(s,\Pi,r_1)$, and no upper bound on $s$. Since $\Pi$ is the trivial representation of $M(\mathbb{A})$ (with $\lambda=0$), we can take $\Sigma=\emptyset$ and $L(s,\Pi,r_2)$ is still $\zeta(s)$. Letting $k'>2$ be the even integer $1+2s$, $L(s,\Pi,r_2)$ becomes $\zeta(k')$. Though we do not have an $a_1$ when $n=1$, turning to $\mathrm{PGL}_2$ we use the scaling factor $p^{(k'-1)/2}$ and the bound $q>k'$ (as if $a_1=\frac{k'-1}{2}$). So, for a prime $q>k'$ dividing the numerator of the Bernoulli number $B_{k'}$, we predict a cuspidal Hecke eigenform $f$ of weight $k'$ (corresponding to $\lambda + s\tilde{\alpha} = se_1$ with $s = \frac{k'-1}{2}$) and level 1 (because $\Sigma = \emptyset$) such that

$$a_p(f) \equiv p^{k'-1} + 1 \pmod{q}.$$

The right-hand-side is obtained from that in the previous section by omitting all the $\beta_i$ terms and putting $a_1 = s = \frac{k'-1}{2}$. This conjecture is well-known to be true, the case $k' = 12, q = 691$ being Ramanujan's congruence. See [BD, §5] for the same thing arrived at via $G = \mathrm{GL}_2$. The conjecture one obtains by artificially enlarging $\Sigma$ beyond its minimum is also true [DF], as anticipated by Harder [H2].

3.2. $\mathbf{n = 2.}$ In the special case $n = 2$, $\mathrm{SO}(3,2) \simeq \mathrm{PGSp}_2$. This arises from the conjugation action of $\mathrm{PGSp}_2$ on the 5-dimensional space of matrices $A = \begin{pmatrix} x_3 & x_2 & 0 & -x_1 \\ x_5 & -x_3 & x_1 & 0 \\ 0 & x_4 & x_3 & x_5 \\ -x_4 & 0 & x_2 & -x_3 \end{pmatrix}$ such that $AJ = J\,{}^tA$, preserving the quadratic form $(1/2)\mathrm{Tr}(A^2) = x_1x_4 + x_2x_5 + 2x_3^2 + x_4x_1 + x_5x_2$, which is that associated with $J$. Under the isomorphism, $\mathrm{diag}(t_1, t_2, t_0t_1^{-1}, t_0t_2^{-1}) \in \mathrm{PGSp}_2$ maps to $\mathrm{diag}(t_1t_2t_0^{-1}, t_1t_2^{-1}, 1, t_0t_1^{-1}t_2^{-1}, t_2t_1^{-1}) \in \mathrm{SO}(3,2)$, and the characters $ae_1' + be_2' - \frac{1}{2}(a+b)e_0'$ of $\mathrm{PGSp}_2$ and $\frac{a+b}{2}e_1 + \frac{a-b}{2}e_2$ of $\mathrm{SO}(3,2)$ correspond, where $e_i' : \mathrm{diag}(t_1, t_2, t_0t_1^{-1}, t_0t_2^{-1}) \mapsto t_i$. In particular, looking at the infinitesimal character of $\tilde{\Pi}_\infty$ when $\tilde{\Pi}$ is generated by a Siegel modular form $F$ of weight $\mathrm{Sym}^j \det^k$ and trivial character, with $k \geq 3$, $(j+k-1)e_1' + (k-2)e_2' - \frac{j+2k-3}{2}e_0'$ corresponds to $\frac{j+2k-3}{2}e_1 + \frac{j+1}{2}e_2$.

If $\Pi$ comes from a cuspidal Hecke eigenform $f$ of weight $k' > 2$ then $\lambda = \frac{k'-1}{2}e_2$ and $w(\lambda + s\tilde{\alpha}) = \frac{k'-1}{2}e_1 + se_2$. Fixing $j$ and $k$ so that this is $\frac{j+2k-3}{2}e_1 + \frac{j+1}{2}e_2$, the right hand side of the congruence becomes $p^{j+k-1} + p^{k-2} + a_p(f)$. The left hand side will be the Hecke eigenvalue (for the operator usually called "$T(p)$") for a genus-2 cuspidal Hecke eigenform $F$ of weight $\mathrm{Sym}^j \det^k$, level 1 if $f$ is, as long as $\tilde{\Pi}_\infty$ is holomorphic discrete series. The $L$-value $L_\Sigma(1+s, \Pi, r_1)$ is $L_\Sigma(f, 1+s+\frac{k'-1}{2}) = L_\Sigma(f, j+k)$. We recover Harder's conjecture [H1, vdG]. See [BD, §7] for the same conjecture arrived at via $G = \mathrm{GSp}_2$.

3.3. $\mathbf{n = 3.}$ Let $\Pi'$ be a unitary, cuspidal automorphic representation of $\mathrm{PGSp}_2$, generated by $F$ as in the previous subsection, so the infinitesimal character of $\Pi'_\infty$ is $(j+k-1)e_1' + (k-2)e_2' - \frac{j+2k-3}{2}e_0'$, which is $\frac{j+2k-3}{2}e_1 + \frac{j+1}{2}e_2$ as a representation of $\mathrm{SO}(3,2)(\mathbb{A})$, rather $\frac{j+2k-3}{2}e_2 + \frac{j+1}{2}e_3$ when $M \simeq \mathrm{GL}_1 \times \mathrm{SO}(3,2)$ is viewed as a Levi subgroup of $G = \mathrm{SO}(4,3)$. For a prime $p$ at which $\Pi'_p$ is unramified, let $\chi_p = -[\log_p(\alpha_1)e_1' + \log_p(\alpha_2)e_2' + \log_p(\alpha_0)e_0']$ be the Satake parameter, where $\alpha_1\alpha_2\alpha_0^2 = 1$. Viewing $\Pi'$ as a representation of $\mathrm{SO}(3,2)(\mathbb{A})$, this is

$$-\frac{1}{2}[(\log_p(\alpha_1) + \log_p(\alpha_2))e_1 + (\log_p(\alpha_1) - \log_p(\alpha_2))e_2].$$

Since $\alpha_1\alpha_2 = \alpha_0^{-2} = \alpha_1\alpha_2$ and $\alpha_1/\alpha_2 = \alpha_1(\alpha_1\alpha_0^2) = (\alpha_1\alpha_0)^2$, and again looking at $M$ inside $G$, we get

$$\chi_p = -[\log_p(\alpha_0\alpha_1\alpha_2)e_2 + \log_p(\alpha_0\alpha_1)e_3],$$

i.e. $\beta_1 = \alpha_0\alpha_1\alpha_2$ and $\beta_2 = \alpha_0\alpha_1$. Hence

$$L_p(s, \Pi_p, r_1) = \prod_{i=1}^{2}[(1 - \beta_i p^{-s})(1 - \beta_i^{-1}p^{-s})]$$

$$= (1-\alpha_0\alpha_1\alpha_2p^{-s})(1-\alpha_0p^{-s})(1-\alpha_0\alpha_1p^{-s})(1-\alpha_0\alpha_2p^{-s}),$$

and we see that $L_{\Sigma}(s,\Pi,r_1)$ is the spinor $L$-function $L_{\Sigma}(s,F,\mathrm{spin})$.

The conjecture predicts congruences modulo $\mathfrak{q}$ such that $q>j+2k-2$ and $\mathrm{ord}_{\mathfrak{q}}L_{\mathrm{alg},\Sigma}(1+s,F,\mathrm{spin})>0$, where $s\in\frac{1}{2}+\mathbb{Z}$ and $0<s<\frac{j+1}{2}$, excluding $s=1/2$. The infinitesimal character of $\tilde{\Pi}_{\infty}$ is $\frac{j+2k-3}{2}e_1+\frac{j+1}{2}e_2+se_3$, and the right-hand-side of the congruence (for the Hecke eigenvalues of $T_{f_1}$) is

$$p^{((j+2k-3)/2))+s}+p^{((j+2k-3)/2)-s}+p^{(j+2k-3)/2}(\alpha_0+\alpha_0\alpha_1+\alpha_0\alpha_2+\alpha_0\alpha_1\alpha_2)$$
$$=p^{((j+2k-3)/2))+s}+p^{((j+2k-3)/2)-s}+T(p)(F),$$

where $T(p)(F)$ denotes the eigenvalue for $T(p)$ acting on $F$.

## 4. The case $i=2$

For any $n$, we have $L_{\Sigma}(s,\Pi,r_2)=\zeta_{\Sigma}(s)$. We have already seen what happens for $n=1$, so we assume now that $n\geq 2$, for which we have so far considered only $i=1$. $\mathrm{SO}(2,1)\times\mathrm{SO}(n,n-1)$ is an endoscopic group of $\mathrm{SO}(n+1,n)$, and there should be a functorial lift from $\mathrm{SO}(2,1)(\mathbb{A})\times\mathrm{SO}(n,n-1)(\mathbb{A})$ to $\mathrm{SO}(n+1,n)(\mathbb{A})$, coming from the obvious homomorphism of $L$-groups $\theta:\mathrm{Sp}_1\times\mathrm{Sp}_{n-1}\to\mathrm{Sp}_n$. As in the case $n=1$, let $s=\frac{k'-1}{2}$, and suppose that $q>k'$ with $\mathrm{ord}_q(\zeta_{\mathrm{alg},\Sigma}(k'))>0$. Then we know there exists a cuspidal automorphic representation $\Pi''$ of $\mathrm{SO}(2,1)(\mathbb{A})$, unramified outside $\Sigma$, satisfying a congruence as above. Recalling that $\Pi=1\times\Pi'$, where $\Pi'$ is on $\mathrm{SO}(n,n-1)(\mathbb{A})$, we need to let $\tilde{\Pi}$ be the functorial lift of $\Pi''\times\Pi'$. To see this, let $t(\Pi''_p)\in\mathrm{Sp}_1(\mathbb{C})$, $t(\Pi'_p)\in\mathrm{Sp}_{n-1}(\mathbb{C})$ and $t(\tilde{\Pi}_p)\in\mathrm{Sp}_n(\mathbb{C})$ be the Satake parameters at a prime $p\notin\Sigma$. Then $t(\tilde{\Pi}_p)=\theta(t(\Pi''_p),t(\Pi'_p))$, so $\mathrm{tr}(t(\tilde{\Pi}_p)=\mathrm{tr}(t(\Pi''_p)+\mathrm{tr}(t(\Pi'_p)$. Scaling by $p^{a_1}$, and bearing in mind the congruence satisfied by $\Pi''$, we see that

$$T_{f_1}(\tilde{\Pi}_p)\equiv p^{a_1+s}+p^{a_1-s}+T_{f_1}(\Pi'_p)\pmod{\mathfrak{q}},$$

as required, where the second $T_{f_1}$ is for $\mathrm{SO}(n,n-1)$. Similar reasoning using the Satake isomorphism works for any $T_{\mu}$.

Note that the automorphic representation $\tilde{\Pi}$ might not have non-zero holomorphic vectors. For example if $n=2$, $\Sigma=\emptyset$ and $\Pi'$, $\Pi''$ come from cuspidal Hecke eigenforms $f$ and $g$ of level 1, then there is no holomorphic Yoshida lift, but the automorphic representation still exists.

## 5. $G=U(n,n)$, $M\simeq\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_1)\times U(n-1,n-1)$

Let $F/\mathbb{Q}$ be an imaginary quadratic extension. For any integer $n\geq 1$, let

$$G=U(n,n)=\{A\in\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_{2n}\mid AJ'\overline{A}^t=J'\},$$

where $J=\begin{bmatrix}0_n & -I_n\\ I_n & 0_n\end{bmatrix}$, be the unitary group associated to an Hermitian form of signature $(n,n)$. (If $\mathrm{tr}_{F/\mathbb{Q}}(\eta)=0$ then $\eta J'$ is an Hermitian matrix.)

The algebraic group $G/\mathbb{Q}$ has a maximal (non-split) torus $T$, with $T(\mathbb{Q})=\{\mathrm{diag}(a_1,\ldots,a_n,a_1^*,\ldots,a_n^*)\,:\,a_1,\ldots,a_n\in F^{\times}\}$, where $a^*:=\overline{a}^{-1}$, containing a maximal split torus $T_d$ such that $T_d(\mathbb{Q})=\{\mathrm{diag}(a_1,\ldots,a_n,a_1^{-1},\ldots,a_n^{-1}):a_1,\ldots,a_n\in\mathbb{Q}\}$. In the complexification $G(\mathbb{C})\simeq\mathrm{GL}_{2n}(\mathbb{C})$, $T(\mathbb{C})$ becomes the standard diagonal torus with characters $e'_i(\mathrm{diag}(t_1,\ldots,t_{2n})):=t_i$ for $1\leq i\leq 2n$. In a little more detail, $F\otimes\mathbb{C}\simeq\mathbb{C}\times\mathbb{C}$, with $F$ embedded via $z\mapsto(z,\overline{z})$, and $T(\mathbb{C})=$

$$\{\mathrm{diag}((t_1,t_{n+1}^{-1}),(t_2,t_{n+2}^{-1}),\ldots,(t_n,t_{2n}^{-1}),(t_{n+1},t_1^{-1}),\ldots,(t_{2n},t_n^{-1})):t_1,\ldots,t_{2n}\in\mathbb{C}^{\times}\},$$

$G(\mathbb{C}) = \{(A, J\,{}^tA^{-1}J^{-1}) : A \in \mathrm{GL}_{2n}(\mathbb{C})\}$, which we can identify with $\mathrm{GL}_{2n}(\mathbb{C})$ by looking only at the first entry. The natural action of $\mathrm{Gal}(F/\mathbb{Q})$ (with complex conjugation swapping the entries of an ordered pair) induces an action on the character group of $T(\mathbb{C})$, with the non-trivial element switching $e'_i$ with $-e'_{n+i}$, for $1 \leq i \leq n$.

We may choose a system of simple positive roots $e'_1 - e'_2, \ldots, e'_{n-1} - e'_n, e'_n - e'_{2n}, e'_{n+2} - e'_{n+1}, \ldots, e'_{2n} - e'_{2n-1}$. Then $\mathrm{Gal}(F/\mathbb{Q})$ fixes $e'_n - e'_{2n}$ and switches $e'_i - e'_{i+1}$ with $e'_{n+i+1} - e'_{n+i}$, for $1 \leq i < n$. The set of positive roots is then $\Phi^+ = \{e'_i - e'_j, e'_{n+j} - e'_{n+i} : 1 \leq i < j \leq n\} \cup \{e'_i - e'_j : 1 \leq i \leq n,\ n+1 \leq j \leq 2n\}$. The half-sum of positive roots is given by $\rho = \frac{1}{2}((2n-1)(e'_1 - e'_{n+1}) + (2n-3)(e'_2 - e'_{n+2}) + \cdots + (e'_n - e'_{2n}))$.

The absolute Weyl group $W_G \simeq S_{2n}$, the symmetric group on $2n$ letters, permuting the $t_i$, while the relative Weyl group ${}_{\mathbb{Q}}W$ is a subgroup isomorphic to $S_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$, where $S_n$ permutes the $t_{n+i}$ the same way it permutes the $t_i$, for $1 \leq i \leq n$, and $(\mathbb{Z}/2\mathbb{Z})^n$ is generated by involutions that swap $t_i$ and $t_{n+i}$ (hence $a_i$ and $a_i^{-1}$ in the entries of an element of $T_d(\mathbb{Q})$). The product of all these involutions is the long element $w_0^G$.

We choose $\alpha = e'_1 - e'_2$, whose $\mathrm{Gal}((F/\mathbb{Q})$-orbit is $\Delta_\alpha = \{e'_1 - e'_2, e'_5 - e'_4\}$. There is a maximal $\mathbb{Q}$-rational parabolic subgroup $P = MN$ characterised by $\Delta_M = \Delta_G - \Delta_\alpha$. The Levi subgroup $M \simeq \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_1) \times U(n-1, n-1)$, with $\left(a, \begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) \mapsto \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & \overline{a}^{-1} & 0 \\ 0 & C & 0 & D \end{pmatrix}$. We have $\Phi_N = \{e'_1 - e'_j, e'_j - e'_{n+1} : 1 \leq j \leq 2n,\ j \neq 1, n+1\} \cup \{e'_1 - e'_{n+1}\}$, $\rho_P = \frac{2n-1}{2}(e'_1 - e'_{n+1})$, $\langle \rho_P, \check{\alpha}\rangle = \frac{2n-1}{2}$, $\tilde{\alpha} = e'_1 - e'_{n+1}$.

Let $\Pi'$ be a unitary, irreducible, cuspidal automorphic representation of $U(n-1, n-1)(\mathbb{A})$, and let $\Pi = 1 \times \Pi'$ on $M(\mathbb{A})$. Suppose for simplicity that $\Pi'$ has trivial central character. Let's say, for an unramified prime $p$, $\Pi_p$ has

$$\chi_p = -[\log_p(\beta_1)(e'_2 - e'_{n+2}) + \log_p(\beta_2)(e'_3 - e'_{n+3}) + \cdots \log_p(\beta_{n-1})(e'_n - e'_{2n})],$$

and $\Pi_\infty$ has infinitesimal character

$$\lambda = a_1(e'_2 - e'_{n+2}) + \cdots a_{n-1}(e'_n - e'_{2n}),$$

with $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$ and the $a_i$ all in $\mathbb{Z}$ or all in $\frac{1}{2} + \mathbb{Z}$ (so that $\lambda$ is algebraically integral and dominant for $M$). In fact we shall assume that $a_1 > a_2 > \ldots > a_{n-1} > 0$.

In the table below, we list together pairs of roots in the same $\mathrm{Gal}(F/\mathbb{Q})$-orbit.

| $\gamma \in \Phi_N$ | $\langle \lambda + s\tilde{\alpha}, \check{\gamma}\rangle$ | $\lvert\chi_p(\check{\gamma}(p))\rvert_p$ |
|---|---|---|
| $\{e'_1 - e'_{j+1}, e'_{n+j+1} - e'_{n+1}\}$ $(1 \leq j \leq n-1)$ | $-a_j + s$ | $\beta_j^{-1}$ |
| $\{e'_1 - e'_{n+j+1}, e'_{j+1} - e'_{n+1}\}$ $(1 \leq j \leq n-1)$ | $a_j + s$ | $\beta_j$ |
| $e'_1 - e'_{n+1}$ | $2s$ | $1$ |

Using the table, $m = 2$ in $r = \oplus_{i=1}^m r_i$. Since $G$ is now split only over the extension $F$, we must proceed a little differently from above, but using [Du1, Lemma 3.1],

$$L_p(s, \Pi_p, r_1) = \begin{cases} \prod_{i=1}^{n-1}[(1 - \beta_i^2 p^{-s})(1 - \beta_i^{-2} p^{-s})] & \text{if } p \text{ is inert;} \\ \prod_{i=1}^{n-1}[(1 - \beta_i p^{-s})^2(1 - \beta_i^{-1} p^{-s})^2] & \text{if } p \text{ is split.} \end{cases}$$

Consequently, $L_\Sigma(s,\Pi,r_1) = L_\Sigma(s,\Pi,\mathrm{st})$, the "standard" $L$-function attached to the irreducible $4n$-dimensional representation of ${}^LU(n,n)$, while $L_\Sigma(s,\Pi,r_2) = \zeta_\Sigma(s)$. For $s>0$, we need $s\in\frac{1}{2}+\mathbb{Z}$ for $L_\Sigma(1+2s,\Pi,r_2)$ to be critical. Then we need all the $a_i$ to be in $\frac{1}{2}+\mathbb{Z}$ for $\lambda+s\tilde{\alpha}$ to be algebraically integral. For $L_\Sigma(1+s,\Pi,r_1)$ to be critical, we also need $s<a_{n-1}$, and we exclude $s=1/2$.

$\lambda+s\tilde{\alpha} = s(e'_1-e'_{n+1})+a_1(e'_2-e'_{n+2})+\cdots a_{n-1}(e'_n-e'_{2n})$. Then, for suitable $w\in W_G$, $w(\lambda+s\tilde{\alpha}) = a_1(e'_1-e'_{n+1})+\cdots+a_{n-1}(e'_{n-1}-e'_{2n-1})+s(e'_n-e'_{2n})$, which is dominant and regular.

The main conjecture of [Du1] is that if $\mathrm{ord}_{\mathfrak{q}}(L_{\Sigma,\mathrm{alg}}(1+is,\Pi,r_i)>0$ then there exists a tempered, unitary, automorphic representation $\tilde{\Pi}$ of $G(\mathbb{A})$, unramified outside $\Sigma$, and with $\tilde{\Pi}_\infty$ of infinitesimal character $w(\lambda+s\tilde{\alpha})$, such that for all $p\notin\Sigma$, and all $\mu\in X_*(T_\#)$ (where $T_\# = \begin{cases} T \text{ if } p \text{ splits;} \\ T_d \text{ if } p \text{ is inert}\end{cases}$ is a maximal split torus of $T/\mathbb{Q}_p$), the eigenvalues of $T'_\mu$ on $\tilde{\Pi}_p$ and $\mathrm{Ind}_P^G(\Pi_p\otimes|s\tilde{\alpha}|_p)$ are congruent modulo $\mathfrak{q}$. (Again, we scale $T'_\mu$ by a certain power of $p$ to make $T_\mu$.) We require $q>2a_1+1$ for $i=1$, $q>2+2s$ for $i=2$.

In the case that $p$ splits in $F$ (so $G$ splits over $\mathbb{Q}_p$), consider $\mu=f_1\in X_*(T)$, so $\mu(p) = \mathrm{diag}((p,1),(1,1),\ldots,(1,1),(1,p^{-1}),(1,1),\ldots,(1,1))$, where $\langle e_i,f_j\rangle=\delta_{ij}$. As a character of $\hat{T}$, $f_1$ is the highest weight of the standard representation of $\hat{G}\simeq\mathrm{GL}_{2n}$, with weights $\{f_1,f_2,\ldots,f_{2n}\}$. Using $\chi_p+s\tilde{\alpha} = -[\log_p(\beta_1)(e'_2-e'_{n+2})+\log_p(\beta_2)(e'_3-e'_{n+3})+\cdots\log_p(\beta_{n-1})(e'_n-e'_{2n})]+s(e'_1-e'_{n+1})$, we find

| $\mu$ | $\lvert(\chi_p+s\tilde{\alpha})(\mu(p))\rvert_p$ |
|---|---|
| $f_1$ | $p^{-s}$ |
| $f_{n+1}$ | $p^{-s}$ |
| $f_{j+1}$ $(1\le j\le n-1)$ | $\beta_j$ |
| $f_{n+j+1}$ $(1\le j\le n-1)$ | $\beta_j^{-1}$ |

The trace is $p^s+p^{-s}+\sum_{i=1}^{n-1}(\beta_i+\beta_i^{-1})$. Multiplying by $p^{\langle w(\lambda+s\tilde{\alpha}),f_1\rangle}=p^{a_1}$, we find that

$$T_{f_1}(\Pi_p\otimes|s\tilde{\alpha}|_p) = p^{a_1+s}+p^{a_1-s}+\sum_{i=1}^{n-1}p^{a_1}(\beta_i+\beta_i^{-1}).$$

By now it should be obvious that what we see here is very closely related to what happened for $G=\mathrm{SO}(n+1,n)$, $M\simeq\mathrm{GL}_1\times\mathrm{SO}(n,n-1)$. The main difference we see is that if $L_\Sigma(s,\Pi,r_1)$ is the $L$-function we see there, then the one we see here is $L_\Sigma(s,\Pi,r_1)L_\Sigma(s,\Pi\otimes(\psi\circ|\tilde{\alpha}|),r_1)$, where $\psi$ is the quadratic character associated with $F/\mathbb{Q}$. As explained in detail in [Du1, §6], there is conjecturally a Langlands functorial lift from automorphic representations of $\mathrm{SO}(n+1,n)(\mathbb{A})$ to automorphic representations of $U(n,n)(\mathbb{A})$. The case $n=2$ was already considered in [Du1, §8].

If $q>1+2a_1$ and $\mathrm{ord}_{\mathfrak{q}}(L_{\mathrm{alg},\Sigma}(1+s,\Pi,r_1))>0$, if the Hecke eigenvalues of $\tilde{\Pi}$ satisfy the mod $\mathfrak{q}$ congruences predicted in §2, then the functorial lift of $\tilde{\Pi}$ satisfies the conjecture here. If $\mathrm{ord}_{\mathfrak{q}}(L_{\mathrm{alg},\Sigma}(s,\Pi\otimes(\psi\circ|\tilde{\alpha}|),r_1)>0$, one applies the same reasoning with $\Pi\otimes(\psi\circ|\tilde{\alpha}|)$ substituted for $\Pi$.

## 6. The theta correspondence

Recall from §3 the conjectured cuspidal automorphic representation $\tilde{\Pi}$ of $G(\mathbb{A}) = \mathrm{SO}(n+1,n)(\mathbb{A})$, such that $\tilde{\Pi}_\infty$ has infinitesimal character $w(\lambda+s\tilde{\alpha}) = a_1e_1+a_2e_2+\cdots+a_{n-1}e_{n-1}+se_n$, with $a_1 > a_2 > \ldots > s > 0$. We might also write $s = a_n$. Each $e_i \in X^*(T)$, but can be identified with its derivative, a character of the Lie algebra of $T(\mathbb{R})$, whose elements are of the form $\mathrm{diag}(D,0,-D)$, with $D = \mathrm{diag}(d_1,d_2,\ldots,d_n)$, all $d_i \in \mathbb{R}$. Then $e_i : \mathrm{diag}(D,0,-D) \mapsto d_i$. We have defined $\mathrm{SO}(n+1,n)$ to be the group preserving the quadratic form $x_1x_{n+2}+x_2x_{n+3}+\cdots+x_nx_{2n+1}+x_{n+1}^2$. If we make the change of variables $X_1 = (x_1+x_{n+2})/2$, $X_2 = (x_2+x_{n+3})/2, \ldots, X_n = (x_n+x_{2n+1})/2$, $X_{n+1} = x_{n+1}$, $X_{n+2} = (x_1-x_{n+2})/2, \ldots, X_{2n+1} = (x_n-x_{2n+1})/2$, then the form becomes $X_1^2+X_2^2+\cdots+X_n^2+X_{n+1}^2-X_{n+2}^2-\cdots-X_{2n+1}^2$. Let $G'$ (isomorphic to $G$) be the group preserving this form. Under the change of variables, $\mathrm{diag}(D,0,-D)$ gets conjugated to $\begin{pmatrix} 0_n & 0 & D \\ 0 & 0 & 0 \\ D & 0 & 0_n \end{pmatrix}$, the general element of the Lie algebra of a non-compact torus in $G'(\mathbb{R})$. (When exponentiated, it produces matrices involving hyperbolic functions.) This can be conjugated by a well-chosen element of $G'(\mathbb{C})$ to $-iD'$, where $D'$, an element of a maximal compact Cartan subalgebra $\mathfrak{h}$, is

$$\begin{cases} \mathrm{diag}\left(\begin{pmatrix} 0 & -d_1 \\ d_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -d_3 \\ d_3 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & -d_{n-1} \\ d_{n-1} & 0 \end{pmatrix}, 0, \begin{pmatrix} 0 & -d_2 \\ d_2 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & -d_n \\ d_n & 0 \end{pmatrix}\right) & n \text{ even;} \\ \mathrm{diag}\left(\begin{pmatrix} 0 & -d_1 \\ d_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -d_3 \\ d_3 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & -d_n \\ d_n & 0 \end{pmatrix}, \begin{pmatrix} 0 & -d_2 \\ d_2 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & -d_{n-1} \\ d_{n-1} & 0 \end{pmatrix}, 0\right) & n \text{ odd.} \end{cases}$$

This is all inside the Lie algebra of the diagonally embedded maximal compact subgroup $K_\infty \simeq \mathrm{SO}(n+1)(\mathbb{R})\times\mathrm{SO}(n)(\mathbb{R})$. Let $e'_j$ be the element of $i\mathfrak{h}^*$ sending $D'$ to $id_j$. An admissible representation of $G(\mathbb{R})$ having infinitesimal character $\lambda = \sum a_ie_i$ (relative to $T$, up to $W_G$ of course) is isomorphic to a representation of $G'(\mathbb{R})$ whose infinitesimal character relative to $\mathfrak{h}$ is $\lambda' = \sum a_ie'_i$. An example of such a representation is the discrete series representation with Harish-Chandra parameter $\lambda'$. We suppose, for the moment, that $\tilde{\Pi}_\infty$ is isomorphic to this representation.

We have a half-sum of positive roots (realised on $\mathfrak{g}'_{\mathbb{C}}$) $\rho_{G'} = \frac{1}{2}[(2n-1)e'_1+(2n-3)e'_2+\cdots+e'_n]$, where we keep the ordering such that the simple positive roots are $e'_1-e'_2,\ldots,e'_{n-1}-e'_n, e'_n$. The sum of the coefficients is $n^2/2$. We also have a half-sum $\rho_c$ of "compact" positive roots, for the Lie algebra of $K_\infty$.

$$2\rho_c = \begin{cases} ((n-1)e'_1+(n-3)e'_3+\cdots+e'_{n-1})+((n-2)e'_2+(n-4)e'_4+\cdots+2e'_{n-2}) & n \text{ even;} \\ ((n-1)e'_1+(n-3)e'_3+\cdots+2e'_{n-2})+((n-2)e'_2+(n-4)e'_4+\cdots+e'_{n-1}) & n \text{ odd.} \end{cases}$$

The sum of the coefficients is $n(n-1)/2$.

The lowest $K_\infty$-type of $\tilde{\Pi}_\infty$ has highest weight $\Lambda = \lambda'+\rho_{G'}-2\rho_c$. The sum of the coefficients is $\frac{n}{2}+\sum a_i$. There are two extensions $\tilde{\Pi}_\infty^{\pm 1}$ of $\tilde{\Pi}_\infty$ from $\mathrm{SO}(n+1,n)(\mathbb{R})$ to $O(n+1,n)(\mathbb{R})$, and it follows from [AB, §§2,3] that $-1$ acts on $\tilde{\Pi}_\infty^{\epsilon}$ as $(-1)^{(n/2)+\sum a_i}\epsilon$. Let $\widetilde{\mathrm{Sp}}(n)$ be the metaplectic double cover of the genus-$n$ symplectic group $\mathrm{Sp}(n)$, for which a convenient reference is [GS]. According to a theorem of Adams and Barbasch [AB, Theorem 3.3], only $\tilde{\Pi}_\infty^{+}$ has a non-zero local theta lift to $\widetilde{\mathrm{Sp}}_n(\mathbb{R})$, and it is a discrete series representation with Harish-Chandra parameter

$$\tilde{\lambda} = \begin{cases} (a_1\tilde{e}_1 + a_3\tilde{e}_2 + \cdots + a_{n-1}\tilde{e}_{n/2}) - (a_2\tilde{e}_{(n+2)/2} + \cdots + a_n\tilde{e}_n) & n \text{ even;} \\ (a_1\tilde{e}_1 + a_3\tilde{e}_2 + \cdots + a_n\tilde{e}_{(n+1)/2}) - (a_2\tilde{e}_{(n+3)/2} + \cdots + a_{n-1}\tilde{e}_n) & n \text{ odd.} \end{cases}$$

Here $\tilde{e}_j$ is essentially $e'_j$, using the maximal compact Cartan subalgebra of the Lie algebra of $\widetilde{\mathrm{Sp}}_n(\mathbb{R})$ isomorphic to $\mathfrak{h}$, as explained in [AB, §1].

For any finite prime $p$, there are extensions $\tilde{\Pi}_p^{\epsilon}$ of $\tilde{\Pi}_p$ from $\mathrm{SO}(n+1,n)(\mathbb{Q}_p)$ to $O(n+1,n)(\mathbb{Q}_p)$, where $\epsilon = \pm 1$ is now the scalar by which $-1$ acts. By a theorem of Gan and Savin [GS, Theorem 1.4(i)], $\tilde{\Pi}_p^{\epsilon}$ has a non-zero local theta lift to $\widetilde{\mathrm{Sp}}_n(\mathbb{Q}_p)$ if and only if $\epsilon = \epsilon(V)\epsilon(1/2, \tilde{\Pi}_p)$. Here $\epsilon(V) = +1, -1$ according as the quadratic space $V/\mathbb{Q}_p$ has maximal isotropic subspaces of dimension $n$ or $n-1$, respectively. For us it is always $+1$. The factor $\epsilon(1/2, \tilde{\Pi}_p)$ is defined by Lapid and Rallis [LR]. It is the local contribution at $p$ to the sign in the functional equation for the $L$-function of $\tilde{\Pi}$ associated with the standard $2n$-dimensional representation of ${}^LG \simeq \mathrm{Sp}_n(\mathbb{C})$. We shall let $\epsilon_p$ denote $\epsilon = \epsilon(1/2, \tilde{\Pi}_p)$.

For $\tilde{\Pi}_\infty^+ \otimes (\otimes_p \tilde{\Pi}_p^{\epsilon_p})$ to be an automorphic representation of $O(n+1,n)(\mathbb{A})$, it is necessary and sufficient that $-1 \in O(n+1,n)(\mathbb{A})$ acts trivially, i.e. that $(-1)^{(n/2)+\sum a_j} = \prod_p \epsilon_p$. According to the table in [De, §5.3], the local sign at $\infty$ for a Hodge type $\{(p,q),(q,p)\}$ should be $i^{q-p+1}$. For the conjectural motive whose $L$-function is the standard $L$-function of $\tilde{\Pi}$, the Hodge type should be $\oplus_{j=1}^n \{(a_j, -a_j), (-a_j, a_j)\}$, so we find that we can interpret $(-1)^{(n/2)+\sum a_j}$ as the local sign at $\infty$, and our global condition is simply that the sign in the functional equation is $+1$. If it is satisfied, then we should expect $\theta_\infty(\tilde{\Pi}_\infty^+) \otimes (\otimes_p \theta_p(\tilde{\Pi}_p^{\epsilon_p}))$ to be an automorphic representation of $\widetilde{\mathrm{Sp}}_n(\mathbb{A})$, by Howe's lifting conjecture. (Strictly speaking, it appears not to be known that $\theta_2(\tilde{\Pi}_2)$ is irreducible. See the condition in [GS, Theroem 3.1(iii)].)

The other discrete series representations of $G'(\mathbb{R})$ with the same infinitesimal character, which $\tilde{\Pi}_\infty$ might be, have Harish-Chandra parameters in the same $W_{G'}$-orbit, of the form $\sum a_{\sigma(i)} e'_i$, where $\sigma(1) > \sigma(3) > \ldots > \sigma(2[(n+1)/2]-1)$ and $\sigma(2) > \sigma(4) > \ldots > \sigma(2[n/2])$ [AB, Lemma 3.1]. The latter sequence is any subset of size $[n/2]$ of $\{1, 2, \ldots, n\}$, in descending order, so the number of these representations is $\binom{n}{[n/2]}$. Though changing the Harish-Chandra parameters in this way changes $\theta_\infty(\tilde{\pi}_\infty)$, $\rho_{G'}$ and $\rho_c$, it does not change the sum of coefficients, and therefore does not change the criterion for non-vanishing of $\theta_\infty(\tilde{\Pi}_\infty^+) \otimes (\otimes_p \theta_p(\tilde{\Pi}_p^{\epsilon_p}))$.

At a finite prime $p \neq 2$, the metaplectic covering

$$0 \longrightarrow \{\pm 1\} \longrightarrow \widetilde{\mathrm{Sp}}_n(\mathbb{Q}_p) \longrightarrow \mathrm{Sp}_n(\mathbb{Q}_p)$$

splits over the hyperspecial maximal compact subgroup $K_p := \mathrm{Sp}_n(\mathbb{Z}_p)$ [GS, 2.6], so one may consider the action of Hecke operators $K_p\widetilde{\mu(p)}K_p$ (for $\mu \in X_*(T)$) on $\theta_p(\tilde{\Pi}_p)$. (Here we use the obvious isomorphism between the maximal torus $T$ in $\mathrm{SO}(n+1,n)$ and a maximal torus of $\mathrm{Sp}_n$, and $\widetilde{\mu(p)}$ is either of the two liftings to $\widetilde{\mathrm{Sp}}_n(\mathbb{Q}_p)$.) It follows from a theorem of Gan and Savin [GS, Theorem 1.3(iii)] that the Hecke eigenvalues (for $\mu$) will be the same for $\theta_p(\tilde{\Pi}_p)$ as for $\tilde{\Pi}_p$.

6.1. $\mathbf{n = 1.}$ $a_1 = s = \frac{k-1}{2}$, so $a_1 + \frac{n}{2} = k/2$, and if $\tilde{\Pi}$ is unramified at all finite primes (so that all $\epsilon_p = +1$) then we require $k/2$ to be even. Since (letting $H = \widetilde{\mathrm{Sp}}_n$)

$\rho_H = \tilde{e}_1$ and $\rho_{H,c} = 0$, we have lowest $\tilde{K}_\infty$-type of highest weight $\tilde{\Lambda} = \tilde{\lambda} + \rho_H - 2\rho_{H,c} = \frac{k-1}{2}\tilde{e}_1 + \tilde{e}_1 = \frac{k+1}{2}\tilde{e}_1$. This $\frac{k+1}{2}$ is the familiar expression for the weight of the Shintani lift of a form of integral weight $k$.

6.2. $\mathbf{n = 2.}$ In the notation of §3.2, $a_1 = \frac{j+2k-3}{2}$ and $a_2 = \frac{j+1}{2}$, so if $\tilde{\Pi}$ is unramified at all finite primes (i.e. $F$ is of level 1) then the condition is that $\frac{j+2k-3}{2} + \frac{j+1}{2} + \frac{2}{2} = j + k$ is even. Since $j$ is already even, this is equivalent to requiring that $k$ be even. Now $\tilde{\lambda} = \frac{j+2k-3}{2}\tilde{e}_1 - \frac{j+1}{2}\tilde{e}_2$, which is positive in the ordering such that the positive roots are $\tilde{e}_1 - \tilde{e}_2, \tilde{e}_1 + \tilde{e}_2, 2\tilde{e}_1$ and $-2\tilde{e}_2$, with $\tilde{e}_1 - \tilde{e}_2$ compact. Hence $\rho_H = 2\tilde{e}_1 - \tilde{e}_2$ and $2\rho_c = \tilde{e}_1 - \tilde{e}_2$, so

$$\tilde{\Lambda} = \frac{j+2k-3}{2}\tilde{e}_1 - \frac{j+1}{2}\tilde{e}_2 + \tilde{e}_1 = \frac{j+2k-1}{2}\tilde{e}_1 - \frac{j+1}{2}\tilde{e}_2,$$

which is the highest weight of $\mathrm{Sym}^{j+k}\det^{-(j+1)/2}$.

The anti-holomorphic discrete series representation with the same infinitesimal character has Harish-Chandra parameter $\tilde{\lambda} = -\frac{j+1}{2}\tilde{e}_1 - \frac{j+2k-3}{2}\tilde{e}_2$, for which $\rho_H = -\tilde{e}_1 - 2\tilde{e}_2$, $2\rho_c = \tilde{e}_1 - \tilde{e}_2$ and $\tilde{\Lambda} = -\frac{j+5}{2}\tilde{e}_1 - \frac{j+2k-1}{2}\tilde{e}_2$, the highest weight of the representation *dual to* $\mathrm{Sym}^{k-3}\det^{(j+5)/2}$. (A form with values in a representation contributes an embedding of the dual representation into the space of automorphic forms on the adele group.) We do not see this representation using the theta lift from $\mathrm{SO}(3,2)$ (as opposed to $\mathrm{SO}(5)$). However, it appears in Ibukiyama's conjectural Shimura-type correspondence between integral and half-integral weight Siegel modular forms of genus 2 [I1, I2]. This involves vector-valued holomorphic modular forms of weight $\mathrm{Sym}^{k-3}\det^{(j+5)/2}$, but only applies when $k$ is odd. (The important thing is that it should be an even symmetric power, so that $-I$ can act trivially without forcing the space of forms to vanish.) When this condition does apply, Harder's conjecture is converted to half-integral weight, to a congruence between a cusp form and a Klingen-Eisenstein series. Further applying an Eichler-Zagier type correspondence, one arrives at a conjectural congruence between vector-valued, genus-2 Jacobi forms, holomorphic or skew-holomorphic. One instance of such a congruence (mod 43, with $k = 5$ and $j = 18$) is proved in [I2].

The question raised by Ibukiyama in [I1, §1.3], of what to do when $k$ is even, appears to be provided with an answer by the above, but it is desirable to see what a vector in the lowest $\tilde{K}_\infty$-type really is. It seems reasonable to suppose that one makes the obvious modifications to the integral weight case, which is dealt with by Harris and Kudla using Borel's harmonic cusp forms [HK]. Our $\tilde{\lambda} = \frac{j+2k-3}{2}\tilde{e}_1 - \frac{j+1}{2}\tilde{e}_2$ must be replaced by its "dual" $\tilde{\lambda}^* = \frac{j+1}{2}\tilde{e}_1 - \frac{j+2k-3}{2}\tilde{e}_2$. This is "$\Lambda + \rho$" in the notation of [HK, Theorem 1.2]. Their "$q$" is $\#\{2\tilde{e}_1\} = 1$, where we are counting the non-compact positive roots $\beta$ (for the standard ordering) such that $\langle \tilde{\lambda}^*, \check{\beta} \rangle > 0$. Their "$\sigma$" is the representation of highest weight $\tilde{\lambda}^* - \rho = \frac{j-3}{2}\tilde{e}_1 - \frac{j+2k-1}{2}\tilde{e}_2$. According to [HK, Theorem 1.2(b)(ii)] (or rather assuming a half-integral weight modification of it), we should look in $\mathcal{H}^q_{\mathrm{cusp},\sigma}$, which, looking at [Harr, §3.5], is a space of $\sigma$-valued 1-forms in the anti-holomorphic variables (i.e. a linear combination of $d\overline{Z_1}, d\overline{Z_2}, d\overline{Z_3}$, with coefficients $C^\infty$ $\sigma$-valued functions of $\begin{pmatrix} Z_1 & Z_3 \\ Z_3 & Z_2 \end{pmatrix} \in \mathcal{H}_2$), killed by the operator $\overline{\partial}$ and by its adjoint with respect to a certain Hermitian metric. Had we applied this to the situation in the previous paragraph, we would have had $\tilde{\lambda}^* = \frac{j+2k-3}{2}\tilde{e}_1 + \frac{j+1}{2}\tilde{e}_2$, $q = 3$, $\sigma$ of highest weight $\tilde{\lambda}^*-\rho=\frac{j+2k-7}{2}\tilde{e}_1+\frac{j-1}{2}$. Then we would look at $\sigma$-valued 3-forms in the anti-holomorphic variables, of the form $\omega=f(Z)\,d\overline{Z_1}\,d\overline{Z_2}\,d\overline{Z_3}$, killed by $\overline{\partial}$ and its adjoint. The annihilation by $\overline{\partial}$ is automatic, but if we look at $*\omega=g(Z)\,dZ_1\,dZ_2\,dZ_3$, this must also be killed by $\overline{\partial}$, with the result that $g$ is a holomorphic function of $Z$. It only transforms according to $\sigma$ when multiplied by $dZ_1\,dZ_2\,dZ_3$, which transforms according to $\wedge^3(\mathfrak{p}^+)^*$, of highest weight $-[(e_1+e_2)+2e_1+2e_2]=-3(e_1+e_2)$. We must subtract this from the weight of $\sigma$, to find that $g$ is a holomorphic $\sigma'$-valued form, with $\sigma'$ of highest weight $\frac{j+2k-1}{2}\tilde{e}_1+\frac{j+5}{2}$, i.e. $\mathrm{Sym}^{k-3}\det^{(j+5)/2}$, as expected.

Observe also, recalling §3.2, and the paragraph immediately preceding §6.1, that Hecke eigenvalues for $\mathrm{PGSp}_2(\mathbb{Z}_p)\mathrm{diag}(p,p,1,1)\mathrm{PGSp}_2(\mathbb{Z}_p)$ will agree with those of $\widetilde{K_p\mathrm{diag}(p,1,p^{-1},1)K_p}$ in half-integral weight.

6.3. $\mathbf{n=3}$. With $\lambda'=\frac{j+2k-3}{2}e_1'+\frac{j+1}{2}e_2'+se_3'$ the parity condition (when $\tilde{\Pi}$) is unramified at all finite $p$) is that $\frac{j+2k-3}{2}+\frac{j+1}{2}+s+\frac{3}{2}$ is even, i.e. that $s+\frac{1}{2}\equiv k \pmod 2$. We find also $\tilde{\lambda}=\frac{j+2k-3}{2}\tilde{e}_1+s\tilde{e}_2-\frac{j+1}{2}\tilde{e}_3$, or $\frac{j+2k-3}{2}\tilde{e}_1+\frac{j+1}{2}\tilde{e}_2-s\tilde{e}_3$. Then $\rho_H=3\tilde{e}_1-\tilde{e}_3$ or $3\tilde{e}_1+2\tilde{e}_2+\tilde{e}_3$, $2\rho_{H,c}=2(\tilde{e}_1-\tilde{e}_2)$ or $2(\tilde{e}_1-\tilde{e}_3)$ and $\tilde{\Lambda}=\frac{j+2k-1}{2}\tilde{e}_1+(s+2)\tilde{e}_2-\frac{j+3}{2}\tilde{e}_3$ or $\frac{j+2k-1}{2}\tilde{e}_1+\frac{j+5}{2}\tilde{e}_2-(s-3)\tilde{e}_3$, respectively.

## 7. $G=\mathrm{SO}(n+1,n)$, $M\simeq\mathrm{GL}_2\times\mathrm{SO}(n-1,n-2)$

Assuming $n\geq 2$, we now choose the simple root $\alpha=e_2-e_3$, determining a maximal parabolic subgroup $P=MN$, where $N$ is the unipotent radical and $M$ is the Levi subgroup, characterised by $\Delta_M=\Delta_G-\{\alpha\}$. Then $M\simeq\mathrm{GL}_2\times\mathrm{SO}(n-1,n-2)$ (or just $\mathrm{GL}_2$ if $n=2$). The positive roots occurring in the Lie algebra of $N$ are $\Phi_N=\Phi_G^+-\Phi_M^+=\{e_1-e_3,\dots,e_1-e_n,e_2-e_3,\dots,e_2-e_n,e_1,e_2,e_1+e_2,\dots,e_1+e_n,e_2+e_3,\dots,e_2+e_n\}$, i.e. those positive roots whose expression as a sum of simple roots includes $\alpha$. The half-sum is $\rho_P=(n-1)(e_1+e_2)$, and $\langle\rho_P,\check{\alpha}\rangle=n-1$, where $\check{\alpha}$ is the coroot associated with $\alpha$. Then $\tilde{\alpha}:=\frac{1}{\langle\rho_P,\check{\alpha}\rangle}\rho_P$ is $e_1+e_2$.

Let $\Pi_f$ be the unitary, cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ associated to a cuspidal Hecke eigenform $f$ of weight $k'$ and trivial character, and let $\Pi'$ be a unitary, cuspidal, automorphic representation of $\mathrm{SO}(n-1,n-2)(\mathbb{A})$. Let $\Pi=\Pi_f\times\Pi'$, which is a unitary, cuspidal, automorphic representation of $M(\mathbb{A})$. Let $\lambda=\frac{k'-1}{2}(e_1-e_2)+a_1e_3+\cdots a_{n-2}e_n$, with $a_1\geq a_2\geq\ldots\geq a_{n-2}\geq 0$, be the infinitesimal character of $\Pi_\infty$, up to $W_M$. See [BD, §2] for further explanation. We shall assume that the $a_i$ are all distinct, from each other and from $(k'-1)/2$, with $a_{n-2}>0$, i.e. that $\lambda$ is regular.

For any prime $p$ such that the local component $\Pi_p$ is unramified, let $\chi_p=-[\log_p(\alpha_p)(e_1-e_2)+\log_p(\beta_1)e_3+\log_p(\beta_2)e_3+\cdots\log_p(\beta_{n-2})e_n]\in X^*(T)\otimes_{\mathbb{Z}}\mathbb{C}$ be such that $\Pi_p$ is isomorphic to the (unitarily) parabolically induced representation $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{M(\mathbb{Q}_p)}(|\chi_p|_p)$, where $B$ is a Borel subgroup of $M$ containing $T$. Note that $|\beta_1|=\cdots=|\beta_{n-1}|=1$ if $\Pi_p'$ is tempered, which will be the case for us, and also that $p^{(k'-1)/2}(\alpha_p+\alpha_p^{-1})=a_p(f)$, where $f=\sum_{i=1}^{\infty}a_n(f)q^n$, with $a_1(f)=1$. This $\chi_p\in X^*(T)\otimes i\mathbb{R}$ gives rise to a Satake parameter $t(\chi_p)\in\hat{T}(\mathbb{C})\subset\hat{M}(\mathbb{C})$, as before.

| $\gamma \in \Phi_N$ | $\langle \lambda + s\tilde{\alpha}, \check{\gamma} \rangle$ | $\lvert \chi_p(\check{\gamma}(p)) \rvert_p$ |
|---|---|---|
| $e_1 - e_{j+2}$ $(1 \leq j \leq n-2)$ | $\frac{k'-1}{2} - a_j + s$ | $\alpha_p \beta_j^{-1}$ |
| $e_2 - e_{j+2}$ $(1 \leq j \leq n-2)$ | $-\frac{k'-1}{2} - a_j + s$ | $\alpha_p^{-1} \beta_j^{-1}$ |
| $e_1 + e_{j+2}$ $(1 \leq j \leq n-2)$ | $\frac{k'-1}{2} + a_j + s$ | $\alpha_p \beta_j$ |
| $e_1 + e_{j+2}$ $(1 \leq j \leq n-2)$ | $-\frac{k'-1}{2} + a_j + s$ | $\alpha_p^{-1} \beta_j$ |
| $e_1$ | $(k'-1) + 2s$ | $\alpha_p^2$ |
| $e_2$ | $-(k'-1) + 2s$ | $\alpha_p^{-2}$ |
| $e_1 + e_2$ | $2s$ | $1$ |

Using the table, $L_\Sigma(s, \Pi, r_1) = L_\Sigma(f \otimes \mathrm{st}(\Pi'), s + \frac{k'-1}{2})$, while $L_\Sigma(s, \Pi, r_2) = L(\mathrm{Sym}^2 f, s + (k'-1))$.

For $s > 0$, the representation $\mathrm{Ind}_P^G(\Pi \otimes |s\tilde{\alpha}|)$ of $G(\mathbb{A})$ has infinitesimal character (at $\infty$) $\lambda + s\tilde{\alpha}$ (up to $W_G$-action). We need $s \in \mathbb{Z}$ for $L_\Sigma(1+2s, \Pi, r_2)$ to be critical, then for all the $a_i$ to be in $\frac{1}{2} + \mathbb{Z}$, for $\lambda + s\tilde{\alpha}$ to be algebraically integral.

Let $1 \leq t \leq n-2$ be such that $\frac{k'-1}{2}$ is in between $a_t$ and $a_{t+1}$ (or $\frac{k'-1}{2} > a_1$ if $t = 1$, $\frac{k'-1}{2} < a_{n-2}$ if $t = n-2$).

$$\lambda + s\tilde{\alpha} = \left( \frac{k'-1}{2} + s \right) e_1 + \left( -\frac{k'-1}{2} + s \right) e_2 + a_1 e_3 + \cdots a_{n-2} e_n.$$

Then, for the obvious choice of $w \in W_G$, $w(\lambda + s\tilde{\alpha}) = a_1 e_1 + \cdots + a_t e_t + \left( \frac{k'-1}{2} + s \right) e_{t+1} + \left( \frac{k'-1}{2} - s \right) e_{t+2} + a_{t+1} e_{t+3} + \cdots + a_{n-2} e_n$, which is dominant and regular if we add the condition $s < \min\{a_t - \frac{k'-1}{2}, \frac{k'-1}{2} - a_t\}$ to those already imposed. This coincides with the condition for $L_\Sigma(1+s, \Pi, r_1)$ to be critical. We exclude the smallest value $s = 1$ from the conjecture below.

Suppose that $q > 2\max\langle \lambda, \check{\gamma} \rangle + 1 = 2a_1 + 1$, and let $\mathfrak{q} \mid q$ be a prime divisor of $\mathfrak{q}$ in a number field sufficiently large to accommodate all the Hecke eigenvalues and normalised $L$-values we shall consider.

The main conjecture of [BD] is that if $\mathrm{ord}_{\mathfrak{q}}(L_{\mathrm{alg},\Sigma}(1 + is, \Pi, r_i) > 0$ then there exists a tempered, unitary, automorphic representation $\tilde{\Pi}$ of $G(\mathbb{A})$, unramified outside $\Sigma$, and with $\tilde{\Pi}_\infty$ of infinitesimal character $w(\lambda + s\tilde{\alpha})$, such that for all $p \notin \Sigma$, and all $\mu \in X_*(T)$, the eigenvalues of $T_\mu$ on $\tilde{\Pi}_p$ and $\mathrm{Ind}_P^G(\Pi_p \otimes |s\tilde{\alpha}|_p)$ are congruent modulo $\mathfrak{q}$.

The standard representation of $\hat{G} \simeq \mathrm{Sp}_n$ has highest weight $f_1$ (identifying $X^*(\hat{T})$ with $X_*(T)$) and complete set of weights $\{\pm f_1, \pm f_2, \ldots, \pm f_n\}$. Given that this is a single $W_G$-orbit, i.e. that $f_1$ is a minuscule weight, we can calculate the "right-hand-side" of the congruence in the following way. The Satake parameter of $\mathrm{Ind}_P^G(\Pi_p \otimes |s\tilde{\alpha}|_p)$ is $-[\log_p(\alpha_p)(e_1 - e_2) + \log_p(\beta_1)e_3 + \log_p(\beta_2)e_3 + \cdots \log_p(\beta_{n-2})e_n] + s(e_1 + e_2)$. Using this,

| $\mu$ | $\lvert(\chi_p + s\tilde{\alpha})(\mu(p))\rvert_p$ |
|---|---|
| $f_1$ | $\alpha_p p^{-s}$ |
| $f_2$ | $\alpha_p^{-1} p^{-s}$ |
| $f_{i+2}$ $(1 \leq i \leq n-2)$ | $\beta_i$ |

The trace is $(\alpha_p+\alpha_p^{-1})(p^s+p^{-s})+\sum_{i=1}^{n-2}(\beta_i+\beta_i^{-1})$. We multiply by $p^{\langle w(\lambda+s\tilde{\alpha}),f_1\rangle}$ to get the eigenvalue for $T_{f_1}$:

$$T_{f_1}(\Pi_p\otimes|s\tilde{\alpha}|_p)=\begin{cases}a_p(f)(1+p^{2s})+\sum_{i=1}^{n-2}p^{(k'-1)/2+s}(\beta_i+\beta_i^{-1}) & \text{if } \frac{k'-1}{2}>a_1;\\ (p^{(a_1-(k'-1)/2)+s}+p^{(a_1-(k'-1)/2)-s})a_p(f)+\sum_{i=1}^{n-2}p^{a_1}(\beta_i+\beta_i^{-1}) & \text{if } \frac{k'-1}{2}<a_1.\end{cases}$$

7.1. $\mathbf{n=3}$. In this case $\mathrm{SO}(n-1,n-2)=\mathrm{SO}(2,1)\simeq\mathrm{PGL}_2$, and we may take for $\Pi'$ the unitary, cuspidal automorphic representation attached to a cuspidal Hecke eigenform $g$, of weight $\ell$ and trivial character. Then, if $\beta_p:=\beta_1$, $p^{(\ell-1)/2}(\beta_p+\beta_p^{-1})=b_p(g)$, the Hecke eigenvalue at $p$ for $g$. We have $a_1=\frac{\ell-1}{2}$, and $s<\frac{|k'-\ell|}{2}$, with $L_\Sigma(1+s,r_1,\Pi)=L_\Sigma(1+s+\frac{k'+\ell-2}{2},f\otimes g)$, and $L_\Sigma(1+2s,\Pi,r_2)=L_\Sigma(1+2s+(k'-1),\mathrm{Sym}^2 f)$. The right-hand-side of the congruence is

$$\begin{cases}a_p(f)(1+p^{2s})+p^{(k'-\ell)/2+s}b_p(g) & \text{if } k'>\ell;\\ (p^{(\ell-k')/2+s)}+p^{(\ell-k')/2-s})a_p(f)+b_p(g) & \text{if } \ell>k'.\end{cases}$$

The infinitesimal character of the predicted $\tilde{\Pi}$ would be

$$\begin{cases}\left(\frac{k'-1}{2}+s\right)e_1+\left(\frac{k'-1}{2}-s\right)e_2+\frac{\ell-1}{2}e_3 & \text{if } k'>\ell;\\ \frac{\ell-1}{2}e_1+\left(\frac{k'-1}{2}+s\right)e_2+\left(\frac{k'-1}{2}-s\right)e_3 & \text{if } \ell>k'.\end{cases}$$

Notice that, keeping the same $L$-value, we can exchange $f$ and $g$ to switch from one case to the other. Letting $\ell=26$, $k'=12$, $g$ and $f=\Delta$ the unique normalised cusp forms of level one with these weights, and $s=5$, we know that $691\mid L_{\mathrm{alg}}(24,f\otimes g)$, by [Du2, Theorem 14.2]. The infinitesimal character of the predicted $\tilde{\Pi}$ would be $\frac{1}{2}(25e_1+21e_2+e_3)$, but calculations of Chenevier and Renard [CR, Table 7] show that there is no such $\tilde{\Pi}$. The way out of this is to realise that so far in this paper we have neglected the last condition in the statement of [BD, Conjecture 4.2], which would require that the $\mathfrak{q}$-adic Galois representation giving rise to $L(s,f\otimes g)$ remains irreducible mod $\mathfrak{q}$. In this case it is not satisfied, because of the reducibility of the 2-dimensional mod 691 representation attached to $\Delta$ (Ramanujan's congruence). One way to think of this is that the construction of [Du2, §§8,11] already provides the element of order 691 in a Selmer group required by the Bloch-Kato conjecture, so the congruence is not necessary.

If $\mathrm{ord}_{\mathfrak{q}}(L_{\mathrm{alg},\Sigma}(1+2s+(k'-1),\mathrm{Sym}^2 f)>0$ (for $q>2a_1+1$) then $\mathfrak{q}$ is the modulus of a conjectural congruence (proved in some cases) between vector-valued, genus-2, Siegel modular Hecke eigenforms, a cusp form and a Klingen-Eisenstein series, as discussed in [BD, §6]. It is easy to check that if $\tilde{\Pi}'$ satisfies that instance of the general conjecture ($G=\mathrm{PGSp}_2$, $P$ the Klingen parabolic) then an endoscopic lift of $\tilde{\Pi}'\times\Pi_g$ from $\mathrm{SO}(3,2)\times\mathrm{SO}(2,1)$ to $\mathrm{SO}(4,3)$, via the homomorphism $\mathrm{Sp}_2\times\mathrm{Sp}_1\to\mathrm{Sp}_3$ of $L$-groups, would satisfy the conjecture here.

If $\mathrm{ord}_{\mathfrak{q}}(L_{\mathrm{alg},\Sigma}(1+is,\Pi,r_i)>0$, for either $i=1$ or $i=2$, and if $\tilde{\Pi}$ satisfies the conjecture here, then a functorial lift of $\tilde{\pi}$ from $\mathrm{SO}(4,3)$ to $U(3,3)$ would satisfy an instance of the main conjecture in [Du1] (for $G=U(3,3)$, $M\simeq\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_2)\times U(1,1)$).

When $i=1$, the near-central critical value $L(g\otimes f,\frac{k'+\ell}{2})$, which appears in [BDS, Corollary 9.2] and [BFvdG, Conjecture 10.7], corresponds to the inadmissible value $s=0$.

7.2. $\mathbf{n=4.}$ In this case $\mathrm{SO}(n-1,n-2)=\mathrm{SO}(3,2)\simeq\mathrm{PGSp}_2$, and we may take for $\Pi'$ the unitary, cuspidal automorphic representation attached to a cuspidal Hecke eigenform $F$, of weight $\mathrm{Sym}^j\otimes\det^k$ and trivial character. Then, as we saw in §3.2, $p^{(j+2k-3)/2}(\beta_1+\beta_2+\beta_1^{-1}+\beta_2^{-1})$ is the Hecke eigenvalue for $T(p)$ on $F$. We have $a_1=\frac{j+2k-3}{2}, a_2=\frac{j+1}{2}$, with $L_\Sigma(1+s,r_1,\Pi)=L_\Sigma(1+s+\frac{k'+j+2k-4}{2},f\otimes F)$, and $L_\Sigma(1+2s,\Pi,r_2)=L_\Sigma(1+2s+(k'-1),\mathrm{Sym}^2 f)$. The right-hand-side of the congruence is

$$\begin{cases} a_p(f)(1+p^{2s})+p^{(k'-(j+2k-2))/2+s}T(p)(F) & \text{if } k'>j+2k-2;\\ (p^{((j+2k-2-k')/2)+s}+p^{((j+2k-2-k')/2)-s})a_p(f)+T(p)(F) & \text{if } k'<j+2k-2,\end{cases}$$

where $T(p)(F)$ denotes the eigenvalue for the usual Hecke operator $T(p)$ (in genus 2) acting on $F$. The infinitesimal character of the predicted $\tilde{\Pi}$ would be

$$\begin{cases} \left(\frac{k'-1}{2}+s\right)e_1+\left(\frac{k'-1}{2}-s\right)e_2+\frac{j+2k-3}{2}e_3+\frac{j+1}{2}e_4 & \text{if } k'>j+2k-2;\\ \frac{j+2k-3}{2}e_1+\left(\frac{k'-1}{2}+s\right)e_2+\left(\frac{k'-1}{2}-s\right)e_3+\frac{j+1}{2}e_4 & \text{if } j+2<k'<j+2k-2;\\ \frac{j+2k-3}{2}e_1+\frac{j+1}{2}e_2+\left(\frac{k'-1}{2}+s\right)e_3+\left(\frac{k'-1}{2}-s\right)e_4 & \text{if } k'<j+2.\end{cases}$$

The value of $s$ must be small enough that the coefficients are in descending order.

If $\mathrm{ord}_{\mathfrak{q}}(L_{\mathrm{alg},\Sigma}(1+2s+(k'-1),\mathrm{Sym}^2 f)>0$ (for $q>2a_1+1$) then again it is easy to check that if $\tilde{\Pi}'$ (as in the previous subsection) satisfies the instance $G=\mathrm{PGSp}_2$, $P$ the Klingen parabolic, of the general conjecture, then an endoscopic lift of $\tilde{\Pi}'\times\Pi_F$ from $\mathrm{SO}(3,2)\times\mathrm{SO}(3,2)$ to $\mathrm{SO}(5,4)$, via the homomorphism $\mathrm{Sp}_2\times\mathrm{Sp}_2\to\mathrm{Sp}_4$ of $L$-groups, would satisfy the conjecture here.

7.3. $\mathbf{n=2.}$ Here, $G=\mathrm{SO}(3,2)\simeq\mathrm{PGSp}_2$, $L_\Sigma(1+s,r_1,\Pi)$ disappears, and $L_\Sigma(1+2s,\Pi,r_2)=L_\Sigma(1+2s+(k'-1),\mathrm{Sym}^2 f)$. We recover congruences between genus-2 cusp forms and Klingen-Eisenstein series ($G=\mathrm{PGSp}_2$, $P$ the Klingen parabolic, [BD, §6]).

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University of Sheffield, School of Mathematics and Statistics, Hicks Building, Hounsfield Road, Sheffield, S3 7RH, U.K.

*E-mail address*: n.p.dummigan@shef.ac.uk