CRITICAL VALUES, CONGRUENCES AND MOVING BETWEEN SELMER GROUPS

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ABSTRACT. We look at various related constructions of elements in Selmer groups, which confirm predictions of the Bloch-Kato conjecture, or which, in conjunction with the Bloch-Kato conjecture, yield predictions that can be verified. We begin with a particular critical value for the tensor product L-function associated to a pair of cusp forms of different weights.

1. A critical value of the tensor product L-function

In this section we review parts of [Du1], in particular Theorem 14.2. Let $f \in S_{k'} := S_{k'}(\mathrm{SL}_2(\mathbb{Z})), g \in S_k$, with $k' > k$, be normalised eigenforms. If $f = \sum_{n=1}^{\infty} a_n q^n$ then

$$L_f(s) = \sum_{n=1}^{\infty} a_n n^{-s} = \prod_{p \text{ prime}} L_{f,p}(s),$$

where $L_{f,p}(s) = (1 - a_p p^{-s} + p^{k'-1-2s})^{-1}$. The series converges for $\Re s$ sufficiently large, but there is an analytic continuation to the whole of $\mathbb{C}$.

Let K be any number field containing $\mathbb{Q}(\{a_n\})$, and λ any prime of O_K, say $\lambda \mid \ell$. By a theorem of Deligne, there exists a continuous linear representation

$$\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(V'_\lambda)$$

(where V'_λ is a 2-dimensional K'_λ-vector space) such that, for any prime p, and any λ such that $\ell \neq p$,

$$L_{f,p}(s) = \det(I - \rho_f(\mathrm{Frob}_p^{-1})p^{-s})^{-1}. \tag{1}$$

Here Frob_p is an element of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ lifting the automorphism $x \mapsto x^p$ of $\mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F}_p)$.

Let $K = \mathbb{Q}(\{a_n, b_n\})$, where $g = \sum_{n=1}^{\infty} b_n q^n$. We also have $\rho_g : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(V_\lambda)$, and define in the natural way $\rho_{f\otimes g} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(V'_\lambda \otimes V_\lambda)$. Substituting $\rho_{f\otimes g}$ for ρ_f in (1), we obtain

$$L(s) := L_{f\otimes g}(s) := \prod_p L_{f\otimes g,p}(s).$$

This, like $L_f(s)$, is an example of a motivic L-function, and its critical values are $L(t)$ for $k \leq t \leq k'-1$. It is easy to show that

$$L_{f\otimes g}(s) = \zeta(2s + 2 - k - k')D(s, f, g), \tag{2}$$

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with $D(s,f,g) := \sum_{n=1}^{\infty} a_n b_n n^{-s}$. Shimura [Sh] proved the following formula for the critical values:

$$D(k'-1-r,f,g) = c\pi^{k'-1}(f, g\delta^{(r)}_{k'-k-2r}E_{k'-k-2r}) \tag{3}$$

(Petersson inner product), where $c = \frac{(k'-k-2r-1)!}{(k'-2-r)!(k'-k-r-1)!}(-1)^r 4^{k'-1}$,

$$\delta_\kappa = \frac{1}{2\pi i}\left(\frac{\kappa}{2iy} + \frac{\partial}{\partial z}\right), \quad \delta_\kappa^{(r)} = \delta_{\kappa+2r-2}\dots\delta_{\kappa+2}\delta_\kappa,$$

and $E_\kappa = 1 - \frac{2\kappa}{B_\kappa}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$. Note that $g\delta^{(r)}_{k'-k-2r}E_{k'-k-2r}$ satisfies the same transformation properties as a modular form of weight k', but in general is not holomorphic.

If $\mathrm{ord}_\ell(B_k/2k) > 0$ then there should exist $\lambda \mid \ell$ such that

$$b_n \equiv \sigma_{k-1}(n) \pmod{\lambda}\ \forall n \geq 1. \tag{4}$$

The case $k = 12, \ell = 691$ is Ramanujan's famous congruence. We shall assume that such a congruence holds. It is certainly true whenever, as expected, g and its Galois conjugates span S_k.

Theorem 1.1. *Suppose that $\ell > k'-2$, (4) holds and that λ is not a congruence prime for f in $S_{k'}$. Suppose also that $k' > 2k$ and that $k'/2$ is odd. Then*

$$\mathrm{ord}_\lambda\left(\frac{L((k'/2)+k-1)}{\pi^{k'+k-1}(f,f)}\right) > 0.$$

The condition $k' > 2k$ guarantees that $(k'/2)+k-1$ lies in the critical range.

Proof. Note that $r = (k'/2)-k$ is odd, $k'-k-2r = k$ and $2s+2-k-k' = k$. By (2) and (3),

$$L((k'/2)+k-1) = \zeta(k)c.\pi^{k'-1}(f, g\delta_k^{(r)}E_k).$$

Hence it suffices to show that $\mathrm{ord}_\lambda\left(\frac{-B_k}{2k}(f, g\delta_k^{(r)}E_k)/(f,f)\right) > 0$. Note that $(f, g\delta_k^{(r)}E_k) = (f, \mathrm{Hol}(g\delta_k^{(r)}E_k))$, where the holomorphic projection operator is such that, term-by-term, the constant term disappears, and for $m > 0$,

$$\mathrm{Hol} : y^{-j}q^m \mapsto \frac{(k'-2-j)!}{(k'-2)!}(4\pi m)^j q^m. \tag{5}$$

(See [St] or pp. 288–290 of [GZ].) A lemma of Hida (Lemma 5.3 of [Hi]) says that $\mathrm{Hol}(g\delta_\kappa^{(r)}g') = (-1)^r\mathrm{Hol}(g'\delta_\kappa^{(r)}g)$, for any $g' \in M_\kappa$. Letting $g' = g$, and recalling that r is odd, we find that $\mathrm{Hol}(g\delta_k^{(r)}g) = 0$. Letting $h = g + \frac{B_k}{2k}E_k$, it then suffices to show that $\mathrm{ord}_\lambda\left(\frac{(f, g\delta_k^{(r)}h)}{(f,f)}\right) > 0$. The congruence (4) implies that λ divides all the Fourier coefficients of h. Then the fact that ℓ is too large to divide the denominator in (5) implies that λ also divides all the Fourier coefficients of $h' := \mathrm{Hol}(g\delta_k^{(r)}h)$. Let $\{f_1,\dots,f_d\}$ be a basis of eigenforms for $S_{k'}$, with $f_1 = f$. If $h' = \sum \alpha_i f_i$, then we need $\lambda \mid \alpha_1$. But this follows easily from $\lambda \mid h'$ and the fact that λ is not a congruence prime for f. □

There is a "natural" way to choose $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant O_λ-lattices T'_λ in V'_λ and T_λ in V_λ, as in 1.6 of [DFG]. Let $W'_\lambda := V'_\lambda/T'_\lambda$, $W'[\lambda] = W'_\lambda[\lambda]$, etc. Let $V''_\lambda := V'_\lambda \otimes V_\lambda$. Note that $T'_\lambda(k'/2)$ is analogous to the ℓ-adic Tate module of an elliptic curve.

One can show that, given Theorem 1.1, a special case of the general Bloch-Kato conjecture on special values of L-functions [BK, Fo] demands the existence of a non-zero element of the Selmer group $H^1_f(\mathbb{Q}, W''_\lambda((k'/2)+k-1))$. This is a subgroup of H^1 defined by conditions on the local restrictions. (A Tate twist has been applied to the coefficient module, corresponding the the point at which the L-function is evaluated.) It is analogous to the Shafarevich-Tate group appearing in the rank-zero case of the Birch and Swinnerton-Dyer leading-term conjecture. We can support the Bloch-Kato conjecture by constructing such an element as follows.

Since $k'/2$ is odd and the sign in the functional equation of $L_f(s)$ is $(-1)^{k'/2}$, $L_f(k'/2) = 0$. (Note that $s = k'/2$ is the centre of symmetry of the functional equation.) An analogue of the Birch and Swinnerton-Dyer conjecture predicts that the dimension of $H^1_f(\mathbb{Q}, V'_\lambda(k'/2))$ equals the order of vanishing of $L_f(s)$ at $s = k'/2$. If we assume that $\lambda \nmid a_\ell$ (i.e. that f is "ordinary" at λ), then, given the oddness of the order of vanishing, theorems of Skinner-Urban[SU] or Nekovář[N] (either will do) give us that $H^1_f(\mathbb{Q}, V'_\lambda(k'/2)) \neq 0$. Scaling to land in $H^1_f(\mathbb{Q}, T'_\lambda(k'/2))$, then reducing $\pmod{\lambda}$, yields a non-zero element of $H^1(\mathbb{Q}, W'[\lambda](k'/2))$.

By (4) we have, for all primes p, $b_p \equiv 1 + p^{k-1} \pmod{\lambda}$. Using the fact that $b_p = \mathrm{Tr}(\rho_g(\mathrm{Frob}_p^{-1}))$, this implies that the composition factors of $W[\lambda]$ (i.e. of $\overline{\rho}_g$) are $\mathbb{F}_\lambda$ and $\mathbb{F}_\lambda(1-k)$. With our natural choices of lattices, it is possible to show that $\mathbb{F}_\lambda(1-k)$ is a submodule (Theorem 7.3 of [Du3]). Hence $W[\lambda](k-1)$ has a trivial submodule $\mathbb{F}_\lambda$, so $W'[\lambda] \otimes W[\lambda](k-1+(k'/2))$ has a submodule isomorphic to $W'[\lambda](k'/2)$.

Hence our non-zero element of $H^1(\mathbb{Q}, W'[\lambda](k'/2))$ produces elements of $H^1(\mathbb{Q}, W''[\lambda]((k'/2)+k-1)$, then of $H^1(\mathbb{Q}, W''_\lambda((k'/2)+k-1)$. It is possible to show that this latter element is non-zero, and satisfies the Bloch-Kato local conditions.

2. Applications of related constructions

(1) Essentially the same construction produces a non-zero element of λ-torsion in a Selmer group attached to $L(\mathrm{Sym}^2 g, (k/2)+k-1)$, when $(k/2)$ is odd, where λ is the modulus of (4). Then, working backwards, the Bloch-Kato conjecture predicts (in the case that λ is not a congruence prime for g in S_k) that λ divides $\frac{L(\mathrm{Sym}^2 g,(k/2)+k-1)}{\pi^{4k-3}(g,g)}$. This divisibility may be observed experimentally [Du1], but it appears to be an open problem to prove it in general, in contrast to the tensor-product case. For some experimental evidence in the Hilbert modular case, see also [Du2].

(2) The best understood critical value of $L(\mathrm{Sym}^2 g, s)$ is at $s = k$, where one gets a simple multiple of (g, g) ([P], see also (2.5) of [Sh]). Only in the case $k = 2$ is $(k/2) + k - 1 = k$. The above construction may be applied in the case of $g \in S_2(\Gamma_0(N))$ attached to an elliptic curve $E/\mathbb{Q}$. We must have $E(\mathbb{Q})$ of positive rank (to get the analogue of $H^1_f(\mathbb{Q}, V'_\lambda(k'/2)) \neq 0$), and also a rational point of order ℓ (to get factors $\mathbb{F}_\ell$ and $\mathbb{F}_\ell(-1)$ for $\overline{\rho}_g \simeq E[\ell](-1)$). The ratio of (g, g) to the canonical Deligne period is essentially the degree of the modular parametrisation $\phi : X_0(N) \to E$. (Let's suppose that E is chosen to be optimal in its isogeny class, i.e. ϕ has minimal degree.) This leads to predictions about the modular degree, which can be proved. Specifically, the following is Theorem 1.3 of [Du3].

Theorem 2.1. *Let $E/\mathbb{Q}$ be an optimal elliptic curve of conductor N. Suppose that E has a rational point of prime order $\ell = 5$ or 7. Suppose also that E has a prime p of split multiplicative reduction such that $p \not\equiv 1 \pmod{\ell}$. If $L(E, 1) = 0$ then $\ell \mid \deg(\phi)$.*

This work has been refined and generalised to modular abelian varieties of higher dimension, by my student Ian Young.

(3) The construction used above depends on having a map from one Galois module to another, which can be used to carry an element from the cohomology of one to the cohomology of the other, providing a candidate for an element of a Selmer group. For example, in the above case, multiplication by the rational point of order ℓ gives a map from $E[\ell]$ to $\mathrm{Sym}^2 E[\ell]$. 7.1(4) of [DW] is a numerical example in which multiplication by a rational point of order 7 is likewise used to get from $\mathrm{Sym}^5 E[7]$ to $\mathrm{Sym}^6 E[7]$.

In [Du4] a different map is used, namely the squaring map from $E[2]$ to $\mathrm{Sym}^2 E[2]$, to try to explain Watkins' conjecture that 2^R divides $\deg(\phi)$, where R is the rank of $E(\mathbb{Q})$. 7.2(2) of [DW] is a numerical example in which the cubing map is used to get from $\mathrm{Sym}^2 E[3]$ to $\mathrm{Sym}^6 E[3]$.

(4) In [CM], the two Galois modules are isomorphic: $E[\ell] \simeq E'[\ell]$, and the cohomology class coming from a rational point of infinite order on E is used to produce an element of order ℓ in the Shafarevich-Tate group of E', in examples where the Birch and Swinnerton-Dyer conjecture predicts the latter. In fact, the same congruence of modular forms resulting from $E[\ell] \simeq E'[\ell]$ also shows how vanishing of $L(E, 1)$ leads to divisibility by ℓ of $L_{\mathrm{alg}}(E', 1)$, as explained in [DSW], which contains a generalisation to higher weight cusp forms.

(5) In [DIK], we have a cuspidal Hecke eigenform f of weight $j+2k-2$, a cuspidal Hecke eigenform F of genus 2 and type $\mathrm{Sym}^j \otimes \det^k$ (vector valued when $j > 0$), and a congruence of Hecke eigenvalues (for all p) $\mu_F(p) \equiv a_p(f) + p^{k-2} + p^{j+k-1} \pmod{\lambda}$, where λ is a large prime divisor of $L_{\mathrm{alg}}(f, j+k)$ and $\mu_G(p)$ is the eigenvalue of a genus-2 Hecke operator $T(p)$ acting on F. In the case $j = 0$, F is a non-lift congruent to the Saito-Kurokawa lift of f, while in the case $j > 0$ the congruence is predicted by Harder's conjecture [Ha, vdG]. If ρ_F is a λ-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ attached to F, then the congruence implies that the composition factors of $\overline{\rho}_F$ are $\overline{\rho}_f$, $\mathbb{F}_\lambda(2-k)$ and $\mathbb{F}_\lambda(1-j-k)$. We may arrange for $\mathbb{F}_\lambda(2-k)$ to be a submodule, $\mathbb{F}_\lambda(1-j-k)$ to be a quotient, with $\overline{\rho}_f$ in the middle. Then $\overline{\rho}_f(2-k)$ is a submodule of $\wedge^2 \overline{\rho}_F$. This can be used to move an element of order λ in a Selmer group associated to $\lambda \mid L_{\mathrm{alg}}(f, j+k)$ [Br], to one in a Selmer group associated to a certain critical value of the standard L-function of F. Bloch-Kato then predicts the divisibility by λ of a certain ratio of standard L-values for F. This divisibility may be proved in the case $j = 0$, and confirmed by computation in examples for which $j > 0$.

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