LIFTING CONGRUENCES TO HALF-INTEGRAL WEIGHT

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ABSTRACT. Given a congruence of Hecke eigenvalues between newforms f and g of odd, square free level, and weight $2\kappa-2$, with even $\kappa\geq 6$, we show that, under weak conditions, there is a congruence of Fourier coefficients between corresponding newforms of half-integral weight.

1. Introduction

Shimura [Sh1] gave a way of associating, to a Hecke eigenform of half-integral weight, a Hecke eigenform of integral weight. This is in general many-to-one, but by imposing a condition on the half-integral weight form, Kohnen made it one-to-one. For precise definitions we refer the reader to his paper.

Theorem 1.1. *(Kohnen,* [Ko1]*) Suppose M is odd and squarefree, $\kappa\geq 2$ an integer. For each normalised newform $f\in S_{2\kappa-2}(\Gamma_0(M))$, there is a unique (up to scaling) $\tilde{f}\in S^{+,\mathrm{new}}_{\kappa-(1/2)}(\Gamma_0(4M))$ such that for any fundamental discriminant $(-1)^{\kappa-1}D$ with $D>0$,*

$$L(s-(\kappa-2),\chi_{(-1)^{\kappa-1}D})\sum_{n=1}^{\infty}a_{Dn^2}(\tilde{f})n^{-s}=a_D(\tilde{f})\sum_{n=1}^{\infty}a_n(f)n^{-s},$$

where $f=\sum_{n=1}^{\infty}a_n(f)q^n$ and $\tilde{f}=\sum_{n=1}^{\infty}a_n(\tilde{f})q^n$.

The "+" means that $a_m(\tilde{f})=0$ unless $(-1)^{\kappa-1}m\equiv 0$ or $1 \pmod 4$, and $\chi_{(-1)^{\kappa-1}D}=\left(\frac{(-1)^{\kappa-1}D}{\cdot}\right)$ is the quadratic character associated to the extension $\mathbb{Q}(\sqrt{(-1)^{\kappa-1}D})/\mathbb{Q}$.

The Hecke eigenvalues $a_p(f)$ for $T(p)$ on f are the eigenvalues for half-integral weight Hecke operators $T(p^2)$ on $\tilde{f}$, but it is evident from the above formula that the $a_p(f)$ determine only ratios $a_{Dn^2}(\tilde{f})/a_D(\tilde{f})$ of certain Fourier coefficients for $\tilde{f}$. They say nothing about ratios $a_{D'}(\tilde{f})/a_D(\tilde{f})$ for different fundamental discriminants $(-1)^{\kappa-1}D'\neq(-1)^{\kappa-1}D$. The problem of what the full set of Fourier coefficients tells us is addressed by the following explicit version of a theorem of Waldspurger. (The case $M=1$ was an earlier result of Kohnen and Zagier.)

Theorem 1.2. *(Kohnen,* [Ko2, Corollary 1, Remark]*) With $f,\tilde{f}$ as above, and any fundamental discriminant $(-1)^{\kappa-1}D$ with $D>0$ such that $\chi_{(-1)^{\kappa-1}D}(q)=\epsilon_q(f)$ (the Atkin-Lehner eigenvalue for f) for all primes $q\mid M$,*

$$\frac{a_D(\tilde{f})^2}{\langle\tilde{f},\tilde{f}\rangle}=2^{\omega(M)}\frac{(\kappa-2)!}{\pi^{\kappa-1}}D^{\kappa-(3/2)}\frac{L(\kappa-1,f,\chi_{(-1)^{\kappa-1}D})}{\langle f,f\rangle}.$$

Furthermore, $a_D(\tilde{f})=0$ if $\chi_{(-1)^{\kappa-1}D}(q)=-\epsilon_q(f)$ for some prime $q\mid M$.

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A problem raised by Hida is whether, given a congruence between newforms of integral weight, there is a non-trivial congruence *of Fourier coefficients* between forms of half-integral weight mapping to them via the Shimura lift. Bearing in mind the above, whereas a congruence of Hecke eigenvalues in half-integral weight is a triviality (because they are the same eigenvalues as in integral weight), a congruence of Fourier coefficients is something much stronger, and not at all obvious. Maeda [Ma] proved one instance of this, where the newforms are in $S_8(\Gamma_0(26))$ and the modulus is a divisor of 433. Since here M is even, Theorem 1.1 does not apply. In [DK], under certain hypotheses (including an assumption on the linear independence mod λ of certain ternary theta series arising from quaternion algebras), we proved a fairly general result on lifting congruences from weight 2 to weight 3/2. The main result of this paper uses a completely different method, but does not apply to either of these situations, since the weight has to be at least 10.

Theorem 1.3. *Let $f, g \in S_{2\kappa-2}(\Gamma_0(M))$, with M odd and squarefree, be normalised newforms, with even $\kappa \geq 6$ (so $2\kappa - 2 \geq 10$, twice an odd number), and $\lambda \mid \ell$ a prime divisor in a number field K containing all the Hecke eigenvalues of f and g. Suppose the following.*

(1) $\overline{\rho}_{f,\lambda}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$ *contains* $\mathrm{SL}_2(\mathbb{F}_\ell)$, *where $\rho_{f,\lambda}$ is the 2-dimensional λ-adic Galois representation attached to f by Deligne* [De2], *and $\overline{\rho}_{f,\lambda}$ is a residual representation. (Thanks to the condition, it is irreducible and therefore well-defined up to isomorphism.)*
(2) $\ell \nmid (2\kappa-2)!M\prod_{\text{prime } q\mid M}(q^2-1)$.
(3) *There exists a fundamental discriminant $-D < 0$ such that $\left(\frac{-D}{p}\right) = \epsilon_p(f)$ for all primes $p \mid M$, and an even character χ of conductor $N > 1$, with $M \mid N$ and $\ell \nmid N$, such that*

$$\mathrm{ord}_\lambda\left(\frac{L^N(3-\kappa,\chi)L^N_{\mathrm{alg}}(1,f,\chi)L^N_{\mathrm{alg}}(2,f,\chi)L(\kappa-1,f,\chi_{-D})}{[\Gamma_0^{(2)}(M):\Gamma_0^{(2)}(N)]}\right) \leq 0.$$

(See below for the definitions of these algebraic parts, and the next section for the definition of $\Gamma_0^{(2)}(M)$. The superscript N on an L-function indicates that Euler factors at $p \mid N$ are omitted.)
(4) $\mathrm{ord}_\lambda(L^M(\kappa,f)/L(\kappa,f)) = 0$.
(5)
$$a_p(f) \equiv a_p(g) \pmod{\lambda} \text{ for all primes } p,$$
and g is the only Hecke eigenform in $S_{2\kappa-2}(\Gamma_0(M))$, not a multiple of f, satisfying this congruence for all $p \nmid M$.

Let $\tilde{f}, \tilde{g} \in S^+_{\kappa-(1/2)}(\Gamma_0(4M))$ be images of f and g respectively under Kohnen's correspondence (Theorem 1.1). Then $\tilde{f}, \tilde{g}$ may be scaled in such a way that

(1) *the Fourier coefficients of $\tilde{f}$ are in K, all integral at λ, but not all divisible by λ, and likewise for $\tilde{g}$.*
(2) *There is a congruence of Fourier coefficients*
$$a_n(\tilde{f}) \equiv a_n(\tilde{g}) \pmod{\lambda}$$
for all $n \geq 1$.

The important condition is of course the last one, the existence of the congruence between f and g. The rest, despite their number and complexity, are fairly weak, as the example in §3 will illustrate.

For integers $1 \leq t \leq 2\kappa - 3$ the algebraic parts are defined by

$$L_{\mathrm{alg}}(t, f) := \frac{L(t, f)}{(2\pi i)^t \omega^{(-1)^t}}, \quad L_{\mathrm{alg}}(t, f, \chi_{-D}) := \frac{L(t, f, \chi_{-D})}{i\sqrt{D}(2\pi i)^t \omega^{(-1)^{t-1}}},$$

where ω^+ and ω^- are canonically scaled Deligne periods as in [Du1, §5] (where I called them Ω^+ and Ω^-). These algebraic parts belong to K. For later, we note that, using [De1, Lemma 5.1.6, (5.1.7)], the relationship between our $\omega^\pm$ and the periods Ω (which depend on r) in [Kato, Proposition 14.21] is that

$$\omega^{(-1)^r} = (2\pi i)^{1-k}\Omega.$$

Hence his

$$(2\pi i)^{r-1}\frac{L(k-r, f)}{\Omega} = (2\pi i)^{k-1}\frac{L(k-r, f)}{(2\pi i)^{k-r}\Omega} = \frac{L(k-r, f)}{(2\pi i)^{k-r}\omega^{(-1)^{k-r}}}$$

is our $L_{\mathrm{alg}}(k-r, f)$. The map per_f referred to in [Kato, Proposition 14.21] is defined in his §§6.3, 4.10, 4.5.

To prove Theorem 1.3, we shall consider Siegel modular forms $\hat{f}$ and $\hat{g}$ (Saito-Kurokawa lifts) of genus 2 and weight κ, for a congruence subgroup $\Gamma_0^{(2)}(M)$. The Fourier coefficients of $\hat{f}$ are intimately related to those of $\tilde{f}$, and it will suffice to prove a congruence of Fourier coefficients between $\hat{f}$ and $\hat{g}$ (with appropriate scaling). To prove the congruence, we find multiples of $\hat{f}(Z)\hat{f}(W)$ and $\hat{g}(Z)\hat{g}(W)$ in a formula for the restriction of a certain genus 4 Eisenstein series from $\mathfrak{H}_4$ to $\mathfrak{H}_2 \times \mathfrak{H}_2$. We need the coefficient of $\hat{f}(Z)\hat{f}(W)$ to have λ in the denominator. For this we use a formula of Agarwal and Brown expressing $\langle \hat{f}, \hat{f} \rangle$ (which naturally appears in the denominator of an expression for the coefficient) as a multiple of $\langle f, f \rangle L(\kappa, f)$. A theorem of Hida and Ribet, that the congruence prime λ appears in the numerator of a ratio of periods $\frac{\langle f, f \rangle}{i\omega^+\omega^-}$, then shows that the first factor $\langle f, f \rangle$ contributes a factor of λ. We also need to apply elements of the Hecke algebra to kill all but the $\hat{f}(Z)\hat{f}(W)$ and $\hat{g}(Z)\hat{g}(W)$ terms without cancelling the λ. For this we use the uniqueness of g to rule out congruences of Hecke eigenvalues between $\hat{f}$ and other Saito-Kurokawa lifts. Any congruences of Hecke eigenvalues between $\hat{f}$ and non-lifts produce elements in a certain Selmer group, by a Ribet-style construction used in [AB1]. By a theorem of Kato, its "order at λ" is bounded by that of $L^M_{\mathrm{alg}}(\kappa, f)$, and any power of λ introduced by killing the non-lift terms is soaked up by the $L(\kappa, f)$ factor in $\langle \hat{f}, \hat{f} \rangle$.

We make much use of the work of Agarwal and Brown [AB1], [AB2], but they do not prove congruences of Fourier coefficients between Hecke eigenforms. Their concern is to prove congruences of Hecke eigenvalues between Saito-Kurokawa lifts and non-lifts (and hence construct elements in Selmer groups), by limiting those between different Saito-Kurokawa lifts. By contrast, ours is to limit congruences of Hecke eigenvalues between lifts and non-lifts (using Kato's theorem to bound Selmer groups) enough to allow the deduction of congruences of Fourier coefficients between different Saito-Kurokawa lifts. The way in which the congruence between f and g implies that between $\hat{f}$ and $\hat{g}$ is, as outlined above, quite subtle, starting with the Hida-Ribet theorem about congruence primes appearing in Petersson norms, which depends on congruences being cohomological.

In [Du2] we looked at congruences between newforms of different weights, in a Hida family, and showed that sometimes they can be lifted to half-integral weight. Using Theorem 1.2, it follows that when one twisted L-value vanishes, the other has algebraic part divisible by λ. We made an application to the Bloch-Kato conjecture, especially in the case when the smaller weight is 2. This theme was further developed by McGraw and Ono [MO]. The theorem in this paper is not so suitable for such applications, since heuristics from random matrix theory suggest that for f of weight $2\kappa-2\geq 6$, at most finitely many of the twisted L-values will vanish [CKRS].

Questions by Tobias Berger and Narasimha Kumar during a seminar led to improvements to an earlier version of this paper.

2. Congruences between Saito-Kurokawa lifts

Let $\mathrm{Sp}_2(\mathbb{Z}):=\left\{g\in M_4(\mathbb{Z}):\ g^t\begin{pmatrix}0&-I\\I&0\end{pmatrix}g=\begin{pmatrix}0&-I\\I&0\end{pmatrix}\right\}$, and

$$\Gamma_0^{(2)}(M):=\left\{\begin{pmatrix}A&B\\C&D\end{pmatrix}\in\mathrm{Sp}_2(\mathbb{Z}):\ C\in M\,M_2(\mathbb{Z})\right\}.$$

Consider any Siegel cusp form $F\in S_\kappa(\Gamma_0^{(2)}(M))$, so F is holomorphic,

$$F((AZ+B)(CZ+D)^{-1})=\det(CZ+D)^\kappa F(Z)$$

for all $Z\in\mathfrak{H}_2:=\{Z\in M_2(\mathbb{C}):\ {}^tZ=Z,\,\mathrm{Im}(Z)>0\}$ and $\begin{pmatrix}A&B\\C&D\end{pmatrix}\in\Gamma_0^{(2)}(M)$, with a vanishing condition at cusps. There is a Fourier expansion

$$F(Z)=\sum_S a(F,S)e^{2\pi i\,\mathrm{tr}(SZ)},$$

where $S=\begin{pmatrix}a&b/2\\b/2&c\end{pmatrix}$, with $a,b,c\in\mathbb{Z}$, $a>0$, $\mathrm{disc}(S):=b^2-4ac<0$.

Following Agarwal and Brown [AB2, §3] we summarise how one obtains (along the lines of Manickam, Ramakrishnan and Vasudevan [MRV]) a Saito-Kurokawa lift $\hat{f}\in S_\kappa(\Gamma_0^{(2)}(M))$ of the normalised newform $f\in S_{2\kappa-2}(\Gamma_0(M)$. (Note that at the bottom of [AB2, p. 646], "$(2n-j^2)$" should be "$(2n-j)^2$".) First one takes an $\tilde{f}\in S^+_{\kappa-(1/2)}(\Gamma_0(4M))$, determined only up to scaling, using Kohnen's correspondence (Theorem 1.1). Next one applies an isomorphism

$$\mathcal{J}:S^+_{\kappa-(1/2)}(\Gamma_0(4M))\to J^c_{\kappa,1}(\Gamma_0(M)^J)$$

to a space of Jacobi cusp forms of weight κ and index 1. Then

$$\hat{f}(Z)=\hat{f}\left(\begin{pmatrix}\tau&z\\z&\tau'\end{pmatrix}\right):=\sum_{m\geq 1}V_m(\mathcal{J}(\tilde{f}))e^{2\pi i m\tau'},$$

where $V_m:J^c_{\kappa,1}(\Gamma_0(M)^J)\to J^c_{\kappa,m}(\Gamma_0(M)^J)$ are certain index-shifting operators.

As in [AB2, Theorem 3.2, Corollary 3.4], if the Fourier coefficients of $\tilde{f}$ are in K, all integral at λ, then it is immediate from the explicit formulas defining $\mathcal{J}$ and the V_m that the same is true of $\mathcal{J}(\tilde{f})$ and $\hat{f}$. We actually need to go in the opposite direction. Choosing a scaling of $\tilde{f}$ and choosing a scaling of $\hat{f}$ are equivalent. By a theorem of Shimura [Sh2], $S_\kappa(\Gamma_0^{(2)}(M))$ has a basis comprising forms with rational Fourier coefficients. For each prime $p \nmid M$ the Hecke operator $T(p)$ on $S_\kappa(\Gamma_0^{(2)}(M))$ preserves rationality of Fourier coefficients. Its eigenvalue on $\hat{f}$ is $a_p(f) + p^{\kappa-2} + p^{\kappa-1}$. Taking the intersection of the kernels of $T(p) - (a_p(f) + p^{\kappa-2} + p^{\kappa-1})$ for sufficiently many primes p, we arrive at a 1-dimensional space spanned by an $\hat{f}$ with coefficients in K. If $-D < 0$ is a fundamental discriminant then the formulas show that for any $r, a \in \mathbb{Z}$ with $a \geq 1$ and $r^2 - 4a = -D$ (one can always choose either $r = 0$ or $r = 1$), the coefficient of $e^{2\pi i(a\tau+\tau'+rz)}$ in $\hat{f}$ is $c(D)$, i.e. $a\left(\hat{f}, \begin{pmatrix} a & r/2 \\ r/2 & 1 \end{pmatrix}\right) = c(D)$, where $\tilde{f} = \sum_{n\geq 1} c(n)q^n$. Hence $c(D) \in K$ for all such D.

By the formula in Theorem 1.1, all the $c(n)$ are determined by the $c(D)$ for fundamental $-D$, in such a way that if we scale $\tilde{f}$ so that the minimum of $\mathrm{ord}_\lambda(c(D))$ (with $-D$ fundamental) is 0, then all the $c(n)$ belong to K, all integral at λ, not all divisible by λ. Clearly all the Fourier coefficients $a(\hat{f}, S)$ are integral at λ, not all divisible by λ. Furthermore, given how the $c(n)$ can be recovered from the $a(\hat{f}, S)$ and the $a_m(f)$, to prove Theorem 1.3 it now suffices to prove the following.

Proposition 2.1. *Let $f, g \in S_{2\kappa-2}(\Gamma_0(M))$, with M odd and squarefree, be normalised newforms, with even $\kappa \geq 6$, and $\lambda \mid \ell$ a prime divisor in a number field K containing all the Hecke eigenvalues of f and g. Suppose the following.*

(1) $\overline{\rho}_{f,\lambda}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$ *contains* $\mathrm{SL}_2(\mathbb{F}_\ell)$.
(2) $\ell \nmid (2\kappa - 2)!M \prod_{\text{prime } q|M}(q^2 - 1)$.
(3) *There exists a fundamental discriminant $-D < 0$ such that $\left(\frac{-D}{p}\right) = \epsilon_p(f)$ for all primes $p \mid M$, and an even character χ of conductor $N > 1$, with $M \mid N$ and $\ell \nmid N$, such that*
$$\mathrm{ord}_\lambda\left(\frac{L^N(3-\kappa,\chi)L^N_{\mathrm{alg}}(1,f,\chi)L^N_{\mathrm{alg}}(2,f,\chi)L(\kappa-1,f,\chi_{-D})}{[\Gamma_0^{(2)}(M) : \Gamma_0^{(2)}(N)]}\right) \leq 0.$$
(4) $\mathrm{ord}_\lambda(L^M(\kappa, f)/L(\kappa, f)) = 0$.
(5)
$$a_p(f) \equiv a_p(g) \pmod{\lambda} \text{ for all primes } p,$$
and g is the only Hecke eigenform in $S_{2\kappa-2}(\Gamma_0(M)$, not a multiple of f, satisfying this congruence for all $p \nmid M$.

Let $\hat{f}, \hat{g} \in S_\kappa(\Gamma_0^{(2)}(M))$ be Saito-Kurokawa lifts of f and g respectively. Then $\hat{f}, \hat{g}$ may be scaled in such a way that

(1) *the Fourier coefficients of $\hat{f}$ are in K, all integral at λ, but not all divisible by λ, and likewise for $\hat{g}$.*
(2) *There is a congruence of Fourier coefficients*
$$a(\hat{f}, S) \equiv a(\hat{g}, S) \pmod{\lambda}$$
for all S.

Proof. By [AB1, Lemma 6.3],
$$\mathcal{E}_M(Z, W) = \sum_{i=1}^{m+r} c_i F_i(Z) F_i^c(W),$$

for $Z, W \in \mathfrak{H}_2$. Here $\mathcal{E}_M(Z,W)$, the restriction to $\mathfrak{H}_2 \times \mathfrak{H}_2$ of some Eisenstein series of weight κ on $\mathfrak{H}_4$ (for which we need $\kappa \geq 6$) has rational Fourier coefficients, integral at ℓ (using $\ell \geq 5$ and $\ell \nmid M$), $\{F_1, \ldots, F_{m+r}\}$ is an orthogonal basis for $S_\kappa(\Gamma_0^{(2)}(M))$, all Hecke eigenforms (for all $T(p)$ with $p \nmid M$), and $F^c(W) := \overline{F(-\overline{W})}$. For $1 \leq i \leq m$ (and only for those i), F_i belongs to the Saito-Kurokawa subspace, meaning that

$$T(p)(F_i) = (a_p(h_i) + p^{\kappa-2} + p^{\kappa-1})F_i, \text{ for all primes } p \nmid M,$$

for a newform h_i of weight $2\kappa - 2$ and level $\Gamma_0(M')$ for some $M' \mid M$, cf. [AB1, Definition 5.3]. Note that $F_i^c = F_i$ for all $1 \leq i \leq m$. We choose $h_1 = f$ and $h_2 = g$.

First we have to eliminate the possibility that $h_i = f$ or g for some $3 \leq i \leq m$, by checking the proof of [AB1, Theorem 5.4]. As in [1, §3.1], we may assume that the adelization of F_i generates an irreducible automorphic representation of $\mathrm{GSp}_2(\mathbb{A})$, type IIb at primes $p \nmid N$ and type VIb at $p \mid N$. (That it is necessarily non-spherical, hence VIb rather than IIb, at $p \mid N$, follows from [P-S, Theorem 2.4(2)].) Now that the local components of the automorphic representation are uniquely determined, it then follows from [Sc, Theorem 5.2(ii)] that F_i is a scalar multiple of F_1 or F_2, which is a contradiction.

By assumption (the uniqueness in (5)), for each $3 \leq i \leq m$ there exists a prime $q_i \nmid M$ such that $a_{q_i}(h_i) \not\equiv a_{q_i}(f) \pmod{\lambda}$. (We temporarily extend K to contain all the Hecke eigenvalues for $F_1, \ldots, F_{m+r}$.) It follows that if $\mu_p(F_i)$ denotes the eigenvalue of $T(p)$ acting on F_i then

$$\mu_{q_i}(F_i) \not\equiv \mu_{q_i}(F_1) \pmod{\lambda}, \quad \text{for } 3 \leq i \leq m.$$

Let $\mathbb{T}$ be an algebra of Hecke operators, with coefficients in the localisation $\mathcal{O}_{K,(\lambda)}$, acting on $S_\kappa(\Gamma_0^{(2)}(M))$, cf. [AB1, §4.2, §7.3]. (Though they use more, $T(p)$ for $p \nmid M$ would suffice.) Let $\mathbb{T}^X$ and $\mathbb{T}^Y$ be the quotients through which $\mathbb{T}$ acts on the subspaces $X := \mathbb{C}\hat{f}$ and $Y := \langle F_{m+1}, \ldots, F_{m+r} \rangle_{\mathbb{C}}$, with surjective restriction homomorphisms $\pi_X : \mathbb{T} \to \mathbb{T}^X$ and $\pi_Y : \mathbb{T} \to \mathbb{T}^Y$, kernels I_X and I_Y respectively. Using the elementary isomorphisms

$$\frac{\mathbb{T}^X}{\pi_X(I_Y)} \simeq \frac{\mathbb{T}}{I_Y + I_X} \simeq \frac{\mathbb{T}^Y}{\pi_Y(I_X)},$$

there exists an element $t \in \mathbb{T}$ such that $t(F_i) = 0$ for all $m+1 \leq i \leq m+r$, and $t(\hat{f}) = \alpha \hat{f}$, where $\mathrm{ord}_\lambda(\alpha) = \mathrm{ord}_\lambda\left(\mathrm{Fitt}\left(\frac{\mathbb{T}^Y}{\pi_Y(I_X)}\right)\right)$.

For every $3 \leq i \leq m$, there exists a prime $q_i \nmid M$ such that

$$\mu_{q_i}(F_i) \not\equiv \mu_{q_i}(F_1) \pmod{\lambda}.$$

Recall that

$$\mathcal{E}_M(Z,W) = \sum_{i=1}^{m+r} c_i F_i(Z) F_i^c(W).$$

Now apply $t \prod_{i=3}^m (T(q_i) - \mu_{q_i}(F_i))$ to both sides (in the variable Z). This kills all the terms for $i \geq 3$ on the right hand side. If we further take a partial Fourier coefficient of $e^{2\pi i \mathrm{Tr}(SW)}$, with $\mathrm{disc}(S) = -D$ a fundamental discriminant, we get

$$\mathcal{F}(Z) = b_1 \hat{f}(Z) + b_2 \hat{g}(Z),$$

where the Fourier coefficients of $\mathcal{F}$ are integral at λ and

$$b_1 = c_1 c(D)\alpha \prod_{i=3}^{m} (\mu_{q_i}(F_1) - \mu_{q_i}(F_i)),$$

so that if we choose D with $\mathrm{ord}_\lambda(c(D)) = 0$ then

$$\mathrm{ord}_\lambda(b_1) = \mathrm{ord}_\lambda(c_1) + \mathrm{ord}_\lambda\left(\mathrm{Fitt}\left(\frac{\mathbb{T}^Y}{\pi_Y(I_X)}\right)\right).$$

We aim now to show that $\mathrm{ord}_\lambda(b_1) < 0$. According to [AB1, Theorem 6.2] (and with a less peculiar normalisation of the standard L-function),

$$c_1 = \mathcal{B}_{\kappa,M} \frac{L^M(3-\kappa, \hat{f}, \mathrm{st}, \chi)}{\pi^3 \langle \hat{f}, \hat{f}\rangle},$$

with $\mathcal{B}_{\kappa,M} = \frac{\pm 2^{2\kappa-3}}{3[\mathrm{Sp}_2(\mathbb{Z}):\Gamma_0^{(2)}(N)]}$. By [AB1, Theorem 5.8], which is also [AB2, Corollary 4.7],

$$\langle \hat{f}, \hat{f}\rangle = \mathcal{A}_{\kappa,M} \frac{c(D)^2}{D^{\kappa-3/2}} \frac{L(\kappa, f)}{\pi L(\kappa-1, f, \chi_{-D})} \langle f, f\rangle,$$

with $\mathcal{A}_{\kappa,M} = \frac{M^\kappa \zeta^M(4)\zeta^M(1)^2(\kappa-1)\prod_{p|M}(1+p^2)(1+p^{-1})}{2^{\omega(M)+3}[\Gamma_0(M):\Gamma_0(4M)][\mathrm{Sp}_2(\mathbb{Z}):\Gamma_0^{(2)}(M)]}$. Here, looking at [AB2, Theorem 4.1], $-D < 0$ is a fundamental discriminant such that $\left(\frac{-D}{p}\right) = \epsilon_p(f)$ for all primes $p \mid M$ (which therefore ought to be a condition in [AB1, Theorems 5.8, 6.5]). Note that in their citation of [Ko2, Corollary 1], it is not necessary to view $\tilde{f}$ as a Shintani lift.

Using the conditions on ℓ, the fact that $[\Gamma_0(M) : \Gamma_0(4M)] = 6$, and that $L(s, \hat{f}, \mathrm{st}) = \zeta(s)L(s+(\kappa-2), f)L(s+(\kappa-1), f)$, we need to show that

$$\mathrm{ord}_\lambda\left(\frac{\alpha D^{\kappa-3/2}}{c(D)^2} \frac{L^N(3-\kappa,\chi)L^N(1,f,\chi)L^N(2,f,\chi)L(\kappa-1,f,\chi_{-D})}{\pi^2 L(\kappa,f)\langle f,f\rangle[\Gamma_0^{(2)}(M):\Gamma_0^{(2)}(N)]}\right) < 0.$$

Multiplying both the numerator and the denominator by $(2\pi i)^{\kappa+2}\sqrt{-D}\omega^-(\omega^+)^2$, and using the hypothesis (3) of the proposition, it is good enough to show that

$$\mathrm{ord}_\lambda\left(\frac{\langle f,f\rangle}{i\omega^+\omega^-}\right) + \mathrm{ord}_\lambda\left(\frac{L_{\mathrm{alg}}(\kappa,f)}{\alpha}\right) > 0.$$

As in [Du1, (4)], using work of Hida [Hi1, §6], the ratio $\frac{\langle f,f\rangle}{i\omega^+\omega^-}$ is, up to S-units (where S is the set of primes dividing $(2\kappa-2)!M$), an integral cohomological congruence ideal η_f. A good additional reference is [Hi2, (5.18)]. The $\langle \zeta_+, \zeta_-\rangle$ in [Hi2, Theorem 5.16] is our η_f. It follows from a theorem of Ribet [Ri2, Theorems 1.3, 1.4] (which removes an ordinarity assumption from an earlier theorem of Hida) that, since λ is a "congruence prime" for f (and $\ell \nmid k!N$), λ divides η_f, as required. (Although Hida and Ribet worked with rational coefficients, combining Galois orbits of newforms, this is not necessary.)

To obtain $\mathrm{ord}_\lambda(b_1) < 0$, it remains to show that

$$\mathrm{ord}_\lambda\left(\mathrm{Fitt}\left(\frac{\mathbb{T}^Y}{\pi_Y(I_X)}\right)\right) \le \mathrm{ord}_\lambda(L_{\mathrm{alg}}(\kappa,f)).$$

The left hand side measures mod λ congruences of Hecke eigenvalues between $\hat{f}$ and the non-lifts $F_{m+1},\dots,F_{m+r}$. Suppose that $m+1\leq j\leq m+r$ and that

$$\mu_q(F_j)\equiv\mu_q(F_1)\pmod{\lambda},\quad\text{for all primes } q\nmid M.$$

By [AB1, Theorem 7.3, Theorem 7.4, Corollary 7.5], using that

$$\ell\nmid(2\kappa-2)!M\prod_{\text{prime } q\mid M}(q^2-1),$$

F_j is not a weak endoscopic lift, and since also it does not belong to the Saito-Kurokawa subspace, the 4-dimensional λ-adic representation $\rho_{F_j,\lambda}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ associated with F_j by Weissauer [We] must be irreducible, cf. [AB1, beginning of §7].

The congruence of Hecke eigenvalues (viewed as traces of Frobenius elements) implies that a residual representation $\overline{\rho}_{F_j,\lambda}$ has composition factors $\overline{\rho}_{f,\lambda}$ and the Tate twists $\mathbb{F}_\lambda(1-\kappa),\mathbb{F}_\lambda(2-\kappa)$ of the trivial representation. Using the irreducibility of $\rho_{F_j,\lambda}$, and adapting an argument used by Ribet [Ri1] it is possible to choose a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant lattice for $\rho_{F_j,\lambda}$ whose reduction provides a non-split extension of $\mathbb{F}_\lambda(2-\kappa)$ by $\overline{\rho}_{f,\lambda}$, hence of $\mathbb{F}_\lambda$ by $\overline{\rho}_{f,\lambda}(\kappa-2)$. As in the proof of [AB1, Theorem 8.8], one can show that this gives a non-zero class in $H^1(\mathbb{Q},W_{f,\lambda}(\kappa-2))$, where $\rho_{f,\lambda}$ is on a space $V_{f,\lambda}$, with $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant lattice $T_{f,\lambda}$, and $W_{f,\lambda}:=V_{f,\lambda}/T_{f,\lambda}$. Furthermore, this class satisfies the Bloch-Kato local conditions at all primes $p\nmid M$, including $p=\ell$. In the notation of [AB1], it gives us a non-zero element of the Selmer group $\mathrm{Sel}_{\{p\mid\ell M\}}(\{p\mid M\},W_{f,\lambda}(\kappa-2))$.

In fact, Theorem 8.8 in [AB1] gives us something stronger, that

$$\mathrm{ord}_\lambda\left(\mathrm{Fitt}\left(\frac{\mathbb{T}^Y}{\pi_Y(I_X)}\right)\right)\leq\mathrm{ord}_\lambda\left(\mathrm{Fitt}\left(\mathrm{Sel}_{\{p\mid\ell M\}}(\{p\mid M\},W_{f,\lambda}(\kappa-2))\right)\right).$$

Letting $k=2\kappa-2, r=\kappa-2, k-r=\kappa, T=T_{f,\lambda}(\kappa-2)$ in [Kato, Proposition 14.21(2)], (where $\mathcal{S}(T(r))$ should be $\mathcal{S}(T)$ on the left hand side), it would say that $\mathrm{ord}_\lambda\left(L_{\mathrm{alg}}(\kappa,f)\right)$ is what the Bloch-Kato conjecture [BK] predicts it should be, as long as (in his notation) $\mu=1$, cf. the end of [Kato, §14.5]. (Recall that our Deligne period ω^+ is $(2\pi i)^{1-k}\Omega$ in [Kato, Proposition 14.21].) Using the condition that $\overline{\rho}_{f,\lambda}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$ contains $\mathrm{SL}_2(\mathbb{F}_\ell)$ (which implies the condition [Kato, (12.5.2)] by [Se, §3.4, Lemma 3]), it follows from [Kato, Theorem 14.5(3)] and its proof that $\mathrm{ord}_\lambda(\mu)\geq 0$. Hence [Kato, Proposition 14.21(2)] says that $\mathrm{ord}_\lambda\left(L_{\mathrm{alg}}(\kappa,f)\right)$ is at least what the Bloch-Kato conjecture predicts it should be. Since the truth of the Bloch-Kato conjecture is invariant under relaxing local conditions and dropping Euler factors at a finite set of primes, this implies that

$$\mathrm{ord}_\lambda\left(\mathrm{Fitt}\left(\mathrm{Sel}_{\{p\mid\ell M\}}(\{p\mid M\},W_{f,\lambda}(\kappa-2))\right)\right)\leq\mathrm{ord}_\lambda\left(L^M_{\mathrm{alg}}(\kappa,f)\right).$$

Using hypothesis (4), this gives us the desired

$$\mathrm{ord}_\lambda\left(\mathrm{Fitt}\left(\frac{\mathbb{T}^Y}{\pi_Y(I_X)}\right)\right)\leq\mathrm{ord}_\lambda(L_{\mathrm{alg}}(\kappa,f)).$$

To complete the proof, recall the equation

$$\mathcal{F}(Z)=b_1\hat{f}(Z)+b_2\hat{g}(Z),$$

where $\mathcal{F}(Z)$ has integral Fourier coefficients, and we now know that $\mathrm{ord}_\lambda(b_1)<0$. Dividing both sides of the equation by b_1, we see that there is a congruence of

Fourier coefficients of $\hat{f}$ and the re-scaled $(b_2/b_1)\hat{g}$. Note that $\hat{f}$ is scaled as in the statement of the proposition, and the congruence forces $(b_2/b_1)\hat{g}$ to be likewise. □

Remark 2.2. More generally, if f and g are congruent modulo λ^s with $s > 0$, one may prove similarly a congruence mod λ^s between $\tilde{f}$ and $\tilde{g}$.

3. An example

The 34-dimensional space $S_{10}(\Gamma_0(35))$ contains normalised newforms

$$f = q + 28q^2 - 116q^3 + 272q^4 + 625q^5 + \dots$$

and

$$g = q + (-12 + \sqrt{2})q^2 + (-87 + 108\sqrt{2})q^3 + (-360 - 48\sqrt{2})q^4 + 625q^5 + \dots,$$

among other newforms with coefficient fields of degrees 4, 5 and 6. According to the computer algebra package Magma [BCP], the q-expansions of f and g are congruent modulo $\lambda = (199, \sqrt{2} - 20)$, with $K = \mathbb{Q}(\sqrt{2})$ and $\ell = 199$, at least as far as the coefficients of q^{100}. We check that all the conditions of Theorem 1.3 are satisfied by this example.

(1) We apply a theorem of Billerey and Dieulefait [BD, Introduction, Square-free level case]. Since $199 \nmid 35$, $199 > 4(10) - 3$, and none of $5^8, 5^{10}$, 7^8 or 7^{10} is congruent to 1 modulo 199, $\overline{\rho}_{f,199}$ is irreducible and has image of order divisible by 199. By a theorem of Dickson[Di], the image of $\overline{\rho}_{f,199}$ in $\mathrm{PGL}_2(\mathbb{F}_{199})$ contains $\mathrm{PSL}_2(\mathbb{F}_{199})$. If the image of $\overline{\rho}_{f,199}$ (in $\mathrm{GL}_2(\mathbb{F}_{199})$) does not contain $\mathrm{diag}(-1, -1)$ then every element of $\mathrm{SL}_2(\mathbb{F}_{199})$ is uniquely ± 1 times something in this image, giving a well-defined character from $\mathrm{SL}_2(\mathbb{F}_{199})$ to $\{\pm 1\}$. Since $\mathrm{SL}_2(\mathbb{F}_{199})$ has no non-trivial abelian character (the smallest degree of an irreducible character being $\frac{199-1}{2}$), it follows that the image of $\overline{\rho}_{f,199}$ contains $\mathrm{SL}_2(\mathbb{F}_{199})$. (This argument was inspired by the proof of (3.1) in [Ri3].)

(2) $199 \nmid 10!(35)(5^2 - 1)(7^2 - 1)$, whose prime divisors are $2, 3, 5$ and 7.

(3) We take $N = M$, and χ quadratic of conductor 5. It is easy to check that $\mathrm{ord}_{199}(L^N(-3, \chi)) = 0$, using Bernoulli polynomials. Using the Magma command $\mathrm{LRatio}(f, 6)$, where

$$f := \mathrm{NewformDecomposition}(\mathrm{CuspidalSubspace}(\mathrm{ModularSymbols}(35, 10)))[1],$$

we find $\mathrm{LRatio}(6, f) = 24843/2$, which factorises as $3 \cdot 7^2 \cdot 13^2/2$, implying that $\mathrm{ord}_{199}(L_{\mathrm{alg}}(6, f)) = 0$. We don't really need that, but it shows that in this example there are no congruences of Hecke eigenvalues between $\hat{f}$ and non-lifts.

Since $\left(\frac{-8}{5}\right) = \left(\frac{-8}{7}\right) = \epsilon_5(f) = \epsilon_7(f) = -1$, we let $-D = -8$, and aim to show that $\mathrm{ord}_{199}(L_{\mathrm{alg}}(5, f, \chi_{-8})) = 0$. Letting f be a 2-dimensional space of modular symbols created in Magma as above, and $\phi := \mathrm{IntegralMapping}(f)$ a projection into this space, if we apply ϕ to the winding element $X^5Y^3\{0, \infty\}$ then we get $(24843, 0)$. The 24843 recovers what we obtained earlier using $\mathrm{LRatio}(6, f)$. Applying ϕ instead to a twisted winding element

$$(8X + Y)^4Y^4\{-1/8, \infty\} + (8X + 3Y)^4Y^4\{-3/8, \infty\}$$
$$-(8X - 3Y)^4Y^4\{3/8, \infty\} - (8X - Y)^4Y^4\{1/8, \infty\},$$

we get $(-13829760, 0)$. The 0 is a check on the correctness of the computation, and up to small prime factors, the $-13829760 = -2^7 \cdot 3^2 \cdot 5 \cdot 7^4$ gives us $L_{\mathrm{alg}}(5, f, \chi_{-8})$, by [MTT, (8.6), §3(i)].

If we plug in $X^a Y^{8-a}\{0, \infty\}$ with $0 \leq a \leq 8$ and a even, we always get a multiple of $(-781, 1)$. So the $\pm$-parts under the natural complex conjugation action must be spanned by $v^+ := (1, 0)$ and $v^- := (-781, 1)$. Applying ϕ to the twisted winding elements

$$Y^8\{-1/5, \infty\} + Y^8\{-4/5, \infty\} - Y^8\{-2/5, \infty\} - Y^8\{-3/5, \infty\}$$

and

$$(5X + Y)Y^7\{-1/5, \infty\} + (5X + 4Y)Y^7\{-4/5, \infty\}$$
$$-(5X + 2Y)Y^7\{-2/5, \infty\} - (5X + 3Y)Y^7\{-3/5, \infty\},$$

we obtain $2^{11} \cdot 3^2 \cdot 5^4 \cdot 7^4 \cdot 11 v^+$ and $2^3 \cdot 5^5 \cdot 7^4 \cdot 13 \cdot 1511 v^-$, respectively. This shows that $\mathrm{ord}_{199}(L_{\mathrm{alg}}(1, f, \chi)) = \mathrm{ord}_{199}(L_{\mathrm{alg}}(2, f, \chi)) = 0$. The factors by which these are multiplied to get $L^N_{\mathrm{alg}}(1, f, \chi)$ and $L^N_{\mathrm{alg}}(2, f, \chi)$ are $1 + 7^4 7^{-1} = 2^3 \cdot 43$ and $1 + 7^4 7^{-2} = 2 \cdot 5^2$.

(4) The ratio $L^M_{\mathrm{alg}}(6, f)/L_{\mathrm{alg}}(6, f)$ is a product of factors $(1 - 5^{-2}) = -\frac{2^3 \cdot 3}{5^2}$ and $(1 - 7^{-2}) = -\frac{2^4 \cdot 3}{7^2}$.

(5) Since the Sturm bound [St] is $\frac{10}{12} \cdot 35 \cdot \left(1 + \frac{1}{5}\right)\left(1 + \frac{1}{7}\right) = 40$, the congruence already observed experimentally between f and g actually holds for all coefficients. The uniqueness of g is easily verified using the command Reductions$(g, 199)$ in Magma.

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