LIFTING PUZZLES AND CONGRUENCES OF IKEDA AND IKEDA-MIYAWAKI LIFTS

NEIL DUMMIGAN

Abstract. We show how many of the congruences between Ikeda lifts and non-Ikeda lifts, proved by Katsurada, can be reduced to congruences involving only forms of genus 1 and 2, using various liftings predicted by Arthur's multiplicity conjecture. Similarly, we show that conjectured congruences between Ikeda-Miyawaki lifts and non-lifts can often be reduced to congruences involving only forms of genus 1, 2 and 3.

1. Introduction

For $k, g \geq 2$ even, let $f \in S_{2k-g}(\mathrm{SL}(2,\mathbb{Z}))$ be a normalised Hecke eigenform. Duke and Imamoglu conjectured the existence of a cuspidal Hecke eigenform $F \in S_k(\mathrm{Sp}_g(\mathbb{Z}))$ (a Siegel modular form of genus g) such that its standard L-function

$$L(s, F, \mathrm{St}) = \zeta(s) \prod_{i=1}^{g} L(f, s + (k-i)).$$

The existence of this F was proved by Ikeda [Ik1], who gave its Fourier expansion, and we call it the Ikeda lift. In the case $g = 2$ it was already known, as the Saito-Kurokawa lift. Katsurada [Ka1] proved that if $k \geq 2g + 4$ and $q > 2k$ is a prime number such that, for some divisor $\mathfrak{q} \mid q$ in a sufficiently large number field,

$$\mathrm{ord}_{\mathfrak{q}}(L_{\mathrm{alg}}(f, k) \prod_{i=1}^{(g/2)-1} L_{\mathrm{alg}}(2i+1, f, \mathrm{St})) > 0,$$

then, under certain weak conditions, there is a congruence mod $\mathfrak{q}$ of Hecke eigenvalues, between F and some Hecke eigenform, in the same space $S_k(\mathrm{Sp}_g(\mathbb{Z}))$, that is not an Ikeda lift. Here the L-values have been normalised by dividing them by particular choices of Deligne periods. This generalises his earlier work on congruences for Saito-Kurokawa lifts (for which only the factor $L(f, k)$ appears), and similarly it uses a pullback formula for an Eisenstein series of genus $2g$ to which a certain differential operator has been applied. The L-values arise as factors in a formula for the Petersson norm of F, which had been proved by Kohnen and Skoruppa for Saito-Kurokawa lifts, and for $g > 2$ was conjectured by Ikeda and proved by Katsurada and Kawamura. For $g = 2$, congruences were proved independently by Brown [Br], who used them to construct elements in Selmer groups supporting the Bloch-Kato conjecture applied to the critical value $L(f, k)$, which for $g = 2$ is immediately to the right of the central point.

Date: November 30th, 2015.
2010 *Mathematics Subject Classification.* 11F46.
Key words and phrases. Congruences of automorphic forms, Ikeda lift, Ikeda-Miyawaki lift.

As g increases, the value $s = k$ migrates further and further to the right in the critical range $1 \leq s \leq 2k - g$. (Of course, we must adjust k if we want to keep the weight $2k - g$ the same to look at a fixed f.) Prime divisors of the algebraic parts of these critical values appear as the moduli of congruences conjectured by Harder [H, vdG], which support the Bloch-Kato conjecture for these critical values. These congruences of Hecke eigenvalues involve vector-valued Siegel modular forms of genus 2, and may be viewed as being congruences of Hecke eigenvalues between cuspidal automorphic representations of $\mathrm{GSp}_2(\mathbb{A})$ and representations induced from the Levi subgroup $\mathrm{GL}_1 \times \mathrm{GL}_2$ of the Siegel parabolic subgroup [BD, §7]. The Hecke eigenvalues of these induced representations involve those of f. Faber and van der Geer [FvdG] computed many Hecke eigenvalues of vector-valued Siegel modular forms of genus 2, providing numerical evidence for many instances of Harder's conjecture. The original example, with $41 \mid L_{\mathrm{alg}}(f, 14)$, for f of weight 22, has been proved by Chenevier and Lannes [CL].

Prime divisors of $L_{\mathrm{alg}}(2i+1, f, \mathrm{St})$ also appear as moduli of conjectural congruences of Hecke eigenvalues involving only genus 2 forms, in general vector-valued, in fact this applies to $L_{\mathrm{alg}}(r, f, \mathrm{St})$ for all odd r from 3 to $2k - g - 1$. The congruences are between cusp forms and Klingen-Eisenstein series, and again may be viewed as being between cuspidal and induced automorphic representations of $\mathrm{GSp}_2(\mathbb{A})$, this time for the Klingen parabolic subgroup [BD, §6]. The first example, for $q = 71$ and f of weight 20, was proved by Kurokawa [Ku], and Mizumoto proved a more general result [Miz]. Their work involved scalar-valued forms of genus 2, and the rightmost critical value of $L(s, f, \mathrm{St})$. One deals with critical values further to the left by increasing the "vector part" j of the weight. Satoh proved a congruence mod 343 in a $j = 2$ case [Sa], and further instances, for other j, were proved in [Du].

Poor, Ryan and Yuen [PRY] computed the Euler factors at 2 of the standard L-functions of the seven cuspidal Hecke eigenforms in $S_{16}(\mathrm{Sp}_4(\mathbb{Z}))$ (genus 4). Two of these forms are Ikeda lifts, while another two are lifts of pairs of genus 1 forms, of a type conjectured by Miyawaki and proved by Ikeda. The remaining three were more mysterious, but the Euler 2-factors of their standard L-functions factored in such a way as to suggest that they were lifts of some previously unknown kind. A. Mellit suggested to T. Ibukiyama that one of them should be lifted from a vector-valued Siegel modular form of genus 2, whose spinor L-function would appear in the standard L-function of the lift. Ibukiyama [Ib] then made two conjectures on scalar-valued genus 4 lifts of genus 2 vector-valued forms, in whose standard L-functions the spinor and standard L-functions of the lifted form, respectively, would appear. For the "standard" lift, a genus 1 form is also involved. He checked that these conjectures produce precisely the Euler 2-factors computed by Poor, Ryan and Yuen, and generalised the conjectures to predict scalar-valued lifts, to higher genus, of genus 1 and (vector-valued) genus 2 forms.

Reconsidering Katsurada's congruences between Ikeda lifts and non-Ikeda lifts, the occurrence of the same L-values in conjectural congruences involving only genus 1 and genus 2 forms, and the apparent existence of scalar-valued, higher genus lifts of such forms, suggest the question of whether these things are related. Could the non-Ikeda lifts in Katsurada's congruences actually be lifts of the type proposed by Ibukiyama? For $L(f, k)$, Ibukiyama's "standard lift" indeed explains Katsurada's congruence as a "lift" of Harder's. If $4 \mid g$ then for $L((g/2) + 1, f, \mathrm{St})$ (the factor for $i = \frac{g}{4}$), Ibukiyama's "spinor lift" likewise explains Katsurada's congruence as a lift of a congruence of Kurokawa-Mizumoto type. In fact, generalising the spinor lift to lift the genus 1 form as well as a genus 2 form, we may similarly account for congruences involving $L(2i+1, f, \mathrm{St})$, for $\frac{g}{4} \leq i \leq \frac{g}{2} - 1$, i.e. for about half the values of i.

We consider also congruences between Ikeda-Miyawaki lifts and non-Ikeda-Miyawaki lifts, conjectured by Ibukiyama, Katsurada, Poor and Yuen [IKPY]. They could be proved in the same manner as those between Ikeda lifts and non-Ikeda-lifts, if one knew a conjecture of Ikeda on the Petersson norm of an Ikeda-Miyawaki lift. The moduli are large prime divisors of $L_{\mathrm{alg}}(f \otimes \mathrm{Sym}^2 h, 2k+2n) \prod_{i=1}^{n-1} L_{\mathrm{alg}}(2i+1, f, \mathrm{St})$, where f and h are genus 1 forms of weights $2k$ and $k+n+1$ respectively, and the Ikeda-Miyawaki lift is of genus $2n+1$, weight $k+n+1$. Again, it appears that in many cases the non-Ikeda-Miyawaki lift should in fact be some other kind of lift. For $L_{\mathrm{alg}}(f \otimes \mathrm{Sym}^2 h, 2k+2n)$ we "lift" a genus 3 generalisation of Harder's conjecture, worked out by Harder himself in collaboration with the authors of [BFvdG], in which it is Conjecture 10.8. Their computations of genus 3 Hecke eigenvalues, together with L-value approximations by Mellit (subsequently confirmed by exact computations in [IKPY]), provided numerical support for their conjecture in seventeen cases. For $L_{\mathrm{alg}}(2i+1, f, \mathrm{St})$, with $\lceil \frac{n}{2} \rceil \leq i \leq n-1$, we again lift congruences of Kurokawa-Mizumoto type.

We may now appear to have a proliferation of unsupported conjectures on the existence of various lifts. But we show how they all fit into Arthur's endoscopic classification of the discrete spectrum of $\mathrm{Sp}_g(\mathbb{Q})\backslash\mathrm{Sp}_g(\mathbb{A})$, and would be consequences of his conjectural multiplicity formula. Actually, for certain groups including Sp_g, Arthur has proved a version of his multiplicity formula [A, Theorem 1.5.2]. But its equivalence to the version applied here is dependent on an as-yet unproved equivalence between two ways of defining and parametrising an L-packet at ∞, as explained following [CR, Conjecture 3.23].

After preliminaries on Arthur's endoscopic classification and multiplicity formula, in Sections 3 and 4, we apply them in Section 5 to obtain all the various lifts (including those of Ikeda and Ikeda-Miyawaki), conditional on the as yet unproved multiplicity formula. The compatibility of the Ikeda lift with Arthur's conjecture was already mentioned in [Ik1, §14], and Ibukiyama looked at the Arthur parameters of his proposed lifts in [Ib, §3.4], without checking the multiplicity formula. In Section 6 we look at the congruences between Ikeda lifts and non-Ikeda lifts proved by Katsurada, and those between Ikeda-Miyawaki lifts and non-Ikeda-Miyawaki lifts conjectured in [IKPY]. Finally, in Section 7 we describe in more detail how some of these congruences can be accounted for in the manner indicated above.

The Hecke algebra for Siegel modular forms of genus g is generated by Hecke operators for each prime p, traditionally denoted $T(p)$ and $T_i(p^2)$ for $1 \leq i \leq g$. Strictly speaking, our approach only accounts for congruences between Hecke eigenvalues for the $T_i(p^2)$, not the $T(p)$. This is because we produce Arthur parameters for $G = \mathrm{Sp}_g$ (with $\hat{G} = \mathrm{SO}(g+1, g)$) rather than for $G = \mathrm{GSp}_g$ (with $\hat{G} = \mathrm{Spin}(g+1, g)$). The Siegel modular forms we consider are all eigenforms for the $T(p)$ as well as the $T_i(p^2)$, but we cannot deduce from this the congruence of the $T(p)$ Hecke eigenvalues.

2. Symplectic and special orthogonal groups

Let $G = \mathrm{Sp}_g = \{h \in M_{2g} :\ {}^t hJh = J\}$, where

$$J_{i,2g+1-i} = \begin{cases} 1 & \text{if } 1 \leq i \leq g; \\ -1 & \text{if } g+1 \leq i \leq 2g, \end{cases}$$

and all other entries are 0. It has a maximal torus T comprising elements of the form $\mathrm{diag}(t_1, \ldots, t_g, t_g^{-1}, \ldots, t_1^{-1})$, which is mapped to t_i by characters e_i, for $1 \leq i \leq g$, which span the character group $X^*(T)$. The cocharacter group $X_*(T)$ is spanned by $\{f_1, \ldots, f_g\}$, where $f_1 : t \mapsto \mathrm{diag}(t, 1, \ldots, 1, t^{-1})$, etc. so $\langle e_i, f_j\rangle = \delta_{ij}$. We can order the roots so that the positive roots are $\Phi_G^+ = \{e_i - e_j : i < j\} \cup \{2e_i : 1 \leq i \leq g\} \cup \{e_i + e_j : i < j\}$, and the simple roots $\Delta_G = \{e_1 - e_2, e_2 - e_3, \ldots, e_{g-1} - e_g, 2e_g\}$. The simple coroots (in order) are $\{f_1 - f_2, \ldots, f_{g-1} - f_g, f_g\}$.

Let $\hat{G} = \mathrm{SO}(g+1, g) = \{h \in M_{2g+1} :\ {}^t h\tilde{J}h = \tilde{J},\ \det(h) = 1\}$, with

$$\tilde{J}_{i,2g+2-i} = \begin{cases} 1 & \text{if } i \neq g+1; \\ 2 & \text{if } i = g+1, \end{cases}$$

and all other entries 0. It has a maximal torus $\hat{T}$ comprising elements of the form $\mathrm{diag}(t_1, \ldots, t_g, 1, t_g^{-1}, \ldots, t_1^{-1})$, which is mapped to t_i by characters $\tilde{e}_i$, for $1 \leq i \leq g$, which span $X^*(\hat{T})$. The cocharacter group $X_*(\hat{T})$ is spanned by $\{\tilde{f}_1, \ldots, \tilde{f}_g\}$, where $\tilde{f}_1 : t \mapsto \mathrm{diag}(t, 1, \ldots, 1, t^{-1})$, etc. so $\langle \tilde{e}_i, \tilde{f}_j\rangle = \delta_{ij}$. We can order the roots so that $\Phi_{\hat{G}}^+ = \{\tilde{e}_i - \tilde{e}_j : i < j\} \cup \{\tilde{e}_i : 1 \leq i \leq g\} \cup \{\tilde{e}_i + \tilde{e}_j : i < j\}$, and $\Delta_{\hat{G}} = \{\tilde{e}_1 - \tilde{e}_2, \tilde{e}_2 - \tilde{e}_3, \ldots, \tilde{e}_{g-1} - \tilde{e}_g, \tilde{e}_g\}$. The simple coroots (in order) are $\{\tilde{f}_1 - \tilde{f}_2, \ldots, \tilde{f}_{g-1} - \tilde{f}_g, 2\tilde{f}_g\}$. Note that for any root β with coroot $\check{\beta}$, we have $\langle \beta, \check{\beta}\rangle = 2$.

We see then that the root systems of G and $\hat{G}$ are dual to each other, so $\hat{G}$ is, as the notation indicates, the Langlands dual of G. The isomorphisms $X^*(\hat{T}) \simeq X_*(T)$ and $X^*(T) \simeq X_*(\hat{T})$ are such that $\tilde{e}_i \leftrightarrow f_i$ and $e_i \leftrightarrow \tilde{f}_i$, respectively.

Let $\mathfrak{H}_g$ be the Siegel upper half space of g by g complex symmetric matrices with positive-definite imaginary part. For $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}_g(\mathbb{Z})$ and $Z \in \mathfrak{H}_g$, let $M\langle Z\rangle := (AZ + B)(CZ + D)^{-1}$ and $J(M, Z) := CZ + D$. Let V be the space of a representation ρ of $\mathrm{GL}(g, \mathbb{C})$. A holomorphic function $f : \mathfrak{H}_g \to V$ is said to belong to the space $M_\rho(\mathrm{Sp}_g(\mathbb{Z}))$ of Siegel modular forms of genus g and weight ρ if

$$f(M\langle Z\rangle) = \rho(J(M, Z))f(Z)\ \forall M \in \mathrm{Sp}_g(\mathbb{Z}), Z \in \mathfrak{H}_g,$$

and, in the case $g = 1$, if it is holomorphic at the cusps. If $g > 1$, the Siegel operator Φ on $M_\rho(\mathrm{Sp}_g(\mathbb{Z}))$ is defined by

$$\Phi f(z) = \lim_{t \to \infty} f\left(\begin{bmatrix} z & 0 \\ 0 & it \end{bmatrix}\right) \quad \text{for } z \in \mathfrak{H}_{g-1}, t \in \mathbb{R}.$$

The kernel of Φ, denoted $S_\rho(\mathrm{Sp}_g(\mathbb{Z}))$, is the space of Siegel cusp forms of genus g and weight ρ. When $\rho = \det^k$, the forms are scalar valued, of weight k, and $S_\rho(\mathrm{Sp}_g(\mathbb{Z}))$ is denoted $S_k(\mathrm{Sp}_g(\mathbb{Z}))$.

3. Arthur's endoscopic classification

Let $G = \mathrm{Sp}_g$ as above, so $\hat{G} = \mathrm{SO}(g+1, g)$. Let $\mathrm{St} : \hat{G} \to \mathrm{SL}(2g+1)$ be the standard inclusion homomorphism. Let $\mathcal{X}(\hat{G})$ be the set of (c_v), indexed by places v of $\mathbb{Q}$, such that for finite p, c_p is a semisimple conjugacy class in $\hat{G}(\mathbb{C})$, and c_∞ is a semisimple conjugacy class in $\mathrm{Lie}(\hat{G}(\mathbb{C}))$. Let $\Pi(G)$ be the set of irreducible representations π of $G(\mathbb{A})$ such that π_∞ is unitary and each π_p, for finite primes p, is smooth and unramified, i.e. has a non-zero $G(\mathbb{Z}_p)$-fixed vector. Let $\Pi_{\mathrm{disc}}(G)$ be the subset of those occurring discretely in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. Given $\pi \in \Pi_{\mathrm{disc}}(G)$, let $c(\pi) = (c_v(\pi))$, where for finite p, $c_p(\pi)$ is the Satake parameter of π_p, and $c_\infty(\pi)$ is the infinitesimal character of π_∞. We may do something similar with G replaced by $\mathrm{PGL}(m)$ and $\hat{G}$ by $\widehat{\mathrm{PGL}(m)} = \mathrm{SL}(m)$, or with G replaced by $\mathrm{SO}(g+1, g)$ and $\hat{G}$ by Sp_g, $\mathrm{St} : \mathrm{Sp}_g \to \mathrm{SL}(2g)$, or with G and $\hat{G}$ both replaced by $\mathrm{SO}(g, g)$, $\mathrm{St} : \mathrm{SO}(g, g) \to \mathrm{SL}(2g)$.

As an example, if π_f is the cuspidal automorphic representation of $\mathrm{PGL}(2)(\mathbb{A})$ associated with a normalised, cuspidal Hecke eigenform $f = \sum_{n=1}^{\infty} a_n q^n$ of weight k for $\mathrm{SL}(2, \mathbb{Z})$, then $c_p(\pi_f) = \mathrm{diag}(\alpha_p, \alpha_p^{-1})$, where $a_p = p^{(k-1)/2}(\alpha_p + \alpha_p^{-1})$, and $c_\infty(\pi_f) = \mathrm{diag}((k-1)/2, -(k-1)/2)$. We have $L(f, s + \frac{k-1}{2}) = \prod_p \det(I - c_p(\pi_f)p^{-s})^{-1}$. In this example we may also think of $\mathrm{PGL}(2)$ as $\mathrm{SO}(2,1)$, and $\mathrm{SL}(2)$ as $\widehat{\mathrm{SO}(2,1)} = \mathrm{Sp}_1$. If instead we consider the cuspidal automorphic representation π_f^{st} of $\mathrm{Sp}_1(\mathbb{A}) = \mathrm{SL}_2(\mathbb{A})$ associated with f then $c_p(\pi_f^{\mathrm{st}}) = \mathrm{diag}(\alpha_p^2, 1, \alpha_p^{-2}) \in \mathrm{SO}(2,1)(\mathbb{C})$, and $\prod_p \det(I - \mathrm{St}(c_p(\pi_f^{\mathrm{st}}))p^{-s})^{-1}$ is the standard L-function $L(s, f, \mathrm{St}) = L(s + (k-1), \mathrm{Sym}^2 f)$, while $c_\infty(\pi_f^{\mathrm{st}}) = \mathrm{diag}(k-1, 0, 1-k)$, which can be thought of as $(k-1)e_1$.

By Arthur's symplectic-orthogonal alternative [CR, Theorem* 3.9], given any $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}(m))$ (the subset of cuspidal representations in $\Pi_{\mathrm{disc}}(\mathrm{PGL}(m))$), there is a

$$G^\pi = \begin{cases} \mathrm{Sp}_{(m-1)/2} & \text{if } m \text{ is odd;} \\ \mathrm{SO}(m/2, m/2) \text{ or } \mathrm{SO}((m/2)+1, m/2) & \text{if } m \text{ is even,} \end{cases}$$

and $\pi' \in \pi_{\mathrm{disc}}(G^\pi)$ such that $c(\pi) = \mathrm{St}(c(\pi'))$.

Following [CR, §3.11] (where more generally G is a classical semisimple group over $\mathbb{Z}$), let $\Psi_{\mathrm{glob}}(G)$ be the set of quadruples $(k, (n_i), (d_i), (\pi_i))$, where $1 \le k \le 2g+1$, k an integer, $n_i \ge 1$ are integers with $\sum_{i=1}^{k} n_i = 2g+1$, $d_i \mid n_i$ and each $\pi_i \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}(n_i/d_i))$ is a self-dual, cuspidal, automorphic representation of $\mathrm{PGL}(n_i/d_i)(\mathbb{A})$. There are two conditions:

(1) if $(n_i, d_i) = (n_j, d_j)$ with $i \neq j$, then $\pi_i \neq \pi_j$;
(2) d_i is odd if $\widehat{G^{\pi_i}}$ is orthogonal, while d_i is even if $\widehat{G^{\pi_i}}$ is symplectic.

An element $\psi \in \Psi_{\mathrm{glob}}(G)$ is called a global Arthur parameter. We write

$$\underline{\psi} = \pi_1[d_1] \oplus \pi_2[d_2] \oplus \ldots \oplus \pi_k[d_k],$$

where there is an equivalence relation, such that for the equivalence class $\underline{\psi}$ of ψ the order of the summands is unimportant. If π_i is the trivial representation we just write $[d_i]$ for $\pi_i[d_i]$, and we just write π_i for $\pi_i[1]$.

To a global Arthur parameter $\psi \in \Psi_{\mathrm{glob}}(G)$, we associate a homomorphism

$$\rho_\psi : \prod_{i=1}^k (\mathrm{SL}(n_i/d_i) \times \mathrm{SL}(2)) \to \mathrm{SL}_{2g+1},$$

well-defined up to conjugation in $\mathrm{SL}_{2g+1}(\mathbb{C})$, namely $\bigoplus_{i=1}^k (\mathbb{C}^{n_i/d_i} \otimes \mathrm{Sym}^{d_i-1}(\mathbb{C}^2))$. Hence we get a map

$$\rho_\psi : \prod_{i=1}^k (\mathcal{X}(\mathrm{SL}(n_i/d_i)) \times \mathcal{X}(\mathrm{SL}(2))) \to \mathcal{X}(\mathrm{SL}_{2g+1}).$$

Let $e = c(1) \in \mathcal{X}(\mathrm{SL}(2))$, where $1 \in \Pi_{\mathrm{disc}}(\mathrm{PGL}(2))$ is the trivial representation. Then $e_p = \mathrm{diag}(p^{1/2}, p^{-1/2})$ and $e_\infty = (1/2, -1/2)$.

Theorem 3.1. *(Arthur's Endoscopic Classification* [CR, Theorem* 3.12],[A, Theorem 1.5.2]*) . Given $\pi \in \Pi_{\mathrm{disc}}(G)$, there is $\psi(\pi) \in \Psi_{\mathrm{glob}}(G)$ (the global Arthur parameter of π) such that*

$$\mathrm{St}(c(\pi)) = \rho_{\psi(\pi)}(\prod_{i=1}^k c(\pi_i) \times e).$$

As an example, if π_f is the cuspidal automorphic representation of $\mathrm{PGL}(2)(\mathbb{A})$ associated with a normalised, cuspidal Hecke eigenform $f = \sum_{n=1}^\infty a_n q^n$ of weight $2k-2$ for $\mathrm{SL}(2,\mathbb{Z})$, with k even, if F, a cusp form of weight k for $\mathrm{Sp}_2(\mathbb{Z})$, is the Saito-Kurokawa lift of f, and if π_F is the associated cuspidal automorphic representation of $\mathrm{Sp}_2(\mathbb{A})$, then $\psi(\pi_F) = \pi_f[2] \oplus [1]$, with $c_\infty(\pi_F) = \mathrm{diag}(k-1, k-2, 0, 2-k, 1-k)$, $c_p(\pi_F) = \mathrm{diag}(\alpha_p p^{1/2}, \alpha_p p^{-1/2}, 1, \alpha_p^{-1} p^{1/2}, \alpha_p^{-1} p^{-1/2})$ and standard L-function $L(s, F, \mathrm{St}) = \prod_p (\det(I - \mathrm{St}(c_p(\pi_F)) p^{-s}))^{-1} = \zeta(s) L(f, s+(k-1)) L(f, s+(k-2))$.

At this point we should say a little more about the relation between Siegel modular forms and automorphic representations. Asgari and Schmidt [AS] describe how to get a cuspidal automorphic representation π'_F of $\mathrm{PGSp}_g(\mathbb{A})$, holomorphic discrete series at ∞, from a Hecke eigenform F in $S_k(\mathrm{Sp}_g(\mathbb{Z}))$, with $k \geq g+1$, and something similar works for vector-valued forms [T, §5.2]. From this π'_F one can get a cuspidal automorphic representation π_F of $\mathrm{Sp}_g(\mathbb{A})$, whose Satake parameters are obtained from those of π'_F by applying the 2-to-1 covering map from $\mathrm{Spin}(g+1, g)$ to $\mathrm{SO}(g+1, g)$. Conversely, given $\pi \in \Pi_{\mathrm{disc}}(\mathrm{Sp}_g)$ with $c_\infty(\pi) = \mathrm{diag}(k-1, \ldots, k-g, 0, g-k, \ldots, 1-k)$ and π_∞ holomorphic discrete series, it comes from some $\pi' \in \Pi_{\mathrm{disc}}(\mathrm{PGSp}_g(\mathbb{A}))$ (by [CR, Proposition 4.7]), which is actually in $\Pi_{\mathrm{cusp}}(\mathrm{PGSp}_g(\mathbb{A}))$ (by [T, Remark 5.2.3]). This is then of the form π'_F for some Hecke eigenform (for the $T(p)$ as well as the $T_i(p^2)$) $F \in S_k(\mathrm{Sp}_g(\mathbb{Z}))$, as explained in [T, §5.2].

4. Arthur's multiplicity formula

Closely related to ρ_ψ above is

$$r_\psi : \prod_{i=1}^k (\widehat{G^{\pi_i}} \times \mathrm{SL}(2)) \to \widehat{G} = \mathrm{SO}(g+1, g).$$

Then $\mathrm{St} \circ r_\psi$ is a direct sum $\oplus_{i=1}^k V_i$, where V_i is an irreducible n_i-dimensional representation of $\widehat{G^{\pi_i}} \times \mathrm{SL}(2)$. Following [CR, §3.20], let C_ψ be the centraliser of $\mathrm{im}(r_\psi)$ in $\hat{G}$. This is an elementary abelian 2-group generated by $Z(\hat{G})$ and elements s_i for those i such that n_i is even, where $\mathrm{St}(s_i)$ acts as -1 on V_i, and as $+1$ on V_j for all $j \neq i$.

Arthur [A] defined a character $\epsilon_\psi : C_\psi \to \{\pm 1\}$, where ϵ_ψ is trivial on $Z(\hat{G})$ and

$$\epsilon_\psi(s_i) = \prod_{j \neq i} \epsilon(\pi_i \times \pi_j)^{\min(d_i, d_j)},$$

$\epsilon(\pi_i \times \pi_j) = \pm 1$ being the global epsilon factor appearing in the functional equation of $L(s, \pi_i \times \pi_j)$, which in our case, where $\pi_i \times \pi_j$ will be unramified at all finite primes, is just the local factor $\epsilon_\infty(\pi_i \times \pi_j)$.

Given $\pi \in \Pi(G)$ such that $c(\pi) = \psi \in \Psi_{\mathrm{alg}}$ (a certain subset of $\Psi_{\mathrm{glob}}(G)$, see [CR, Definition 3.15]), we can ask whether π actually occurs in $\Pi_{\mathrm{disc}}(G)$. Arthur's multiplicity conjecture answers this question. The answer depends on comparing ϵ_ψ with another character which depends on how all the π_p and π_∞ sit in their L-packets. Since all the π_p are unramified, their L-packets are trivial, i.e. they are uniquely determined up to isomorphism by their $c_p(\pi)$. Therefore we only need consider π_∞, which we want to be the holomorphic discrete series representation within its L-packet. There is an associated Shelstad parameter $\chi_{\mathrm{hol}} : C_{\psi_\infty} \to \mathbb{C}^\times$, where C_{ψ_∞} is a certain group which can be viewed as a 2-torsion subgroup of $\hat{T}$, such that $C_\psi \subseteq C_{\psi_\infty}$, and the requirement of Arthur's multiplicity formula is that $\chi_{hol}|_{C_\psi} = \epsilon_\psi$. By [CR, Lemma 9.3], χ_{hol} is the restriction of either $\sum_{\mathrm{odd}\ i=1}^{g} \tilde{e}_i$ or $\sum_{\mathrm{even}\ i=1}^{g} \tilde{e}_i \in X^*(\hat{T})$, and the restrictions to C_ψ are the same [CR, Lemma 9.5], so we act as if $\chi_{\mathrm{hol}} = \sum_{\mathrm{odd}\ i=1}^{g} \tilde{e}_i$. Note that although C_ψ and C_{ψ_∞} are only well-defined up to conjugacy, there is a natural way of viewing one inside the other, compatible with the above view of C_{ψ_∞} inside $\hat{T}[2]$, and the explicit realisation in $\hat{T}[2]$ of the various $s_i \in C_\psi$ in the proofs in the next section.

5. Application to various lifts

All the propositions in this section are conditional upon Arthur's multiplicity conjecture.

5.1. **Ikeda lifts.** For $k, g \geq 2$ even, and $f \in S_{2k-g}(\mathrm{SL}(2,\mathbb{Z}))$ a Hecke eigenform, let π_f be the associated cuspidal, automorphic representation of $\mathrm{PGL}(2)(\mathbb{A})$, and consider $\pi_f[g] \oplus [1] \in \Psi_{\mathrm{alg}}$.

Proposition 5.1. *There exists $\pi \in \Pi_{\mathrm{disc}}(\mathrm{Sp}_g)$ such that $\psi(\pi) = \pi_f[g] \oplus [1]$.*

Proof. Since $n_1 = 2g$ is even, but $n_2 = 1$ is odd, C_ψ is generated by $Z(\hat{G})$ and $s_1 =: s_f$. We have $\epsilon_\psi(s_f) = \epsilon_\infty(\pi_f \times 1)^1 = \epsilon_\infty(\pi_f)$. Note that $c_\infty(\pi_f) = \mathrm{diag}(\frac{2k-g-1}{2}, \frac{1-g-2k}{2})$. The associated motive (twisted to have weight 0) would have Hodge type $\{(p,q),(q,p)\}$, with $p = \frac{1-g-2k}{2}$ and $q = \frac{2k-g-1}{2}$. Putting this in the formula i^{q-p+1} in the table in [De, §5.3], we recover the well-known $\epsilon_\infty(\pi_f) = i^{2k-g} = (-1)^{k-(g/2)} = (-1)^{g/2}$. Of course we would have to make a half-integral twist to really have a motive, with integral Hodge weights, but since we are only interested in the difference $q - p$, we can ignore this.

On the other hand $\chi_{\mathrm{hol}} = \tilde{e}_1 + \ldots + \tilde{e}_{g-1}$ (odd subscripts), which has $\frac{g}{2}$ terms, and $s_f = \mathrm{diag}(\underbrace{-1,\ldots,-1}_{g \text{ times}}, 1, \underbrace{-1,\ldots,-1}_{g \text{ times}})$, so $\chi_{\mathrm{hol}}(s_f) = (-1)^{g/2}$. Since this is the same as $\epsilon_\psi(s_f)$, π exists. ∎

Note that $c_\infty(\pi) = \mathrm{diag}(k-1, k-2, \ldots, k-g, 0, g-k, \ldots, 2-k, 1-k)$ matches $c_\infty(\pi_F)$, where π_F is the automorphic representation of $\mathrm{Sp}_g(\mathbb{A})$ associated with a cuspidal Hecke eigenform $F \in S_k(\mathrm{Sp}_g(\mathbb{Z}))$, and since π_∞ is holomorphic discrete series, π is of the form π_F. From $\psi(\pi_F)$ we can read off the standard L-function $L(s, F, \mathrm{St}) = \zeta(s) \prod_{i=1}^{g} L(f, s + (k-i))$, and we recognise F as the Ikeda lift of f [Ik1].

5.2. **Standard lifts.** Let k, g, f be as in the previous section, and let F be a cuspidal Hecke eigenform for $\mathrm{Sp}_2(\mathbb{Z})$, of weight $\det^\kappa \otimes \mathrm{Sym}^j(\mathbb{C}^2)$, with $(\kappa, j) = (k-g+2, g-2)$ (so we must impose $k > g-2$). To F we associate an automorphic representation π_F^{st} of $\mathrm{Sp}_2(\mathbb{A})$, with $c_\infty(\pi_F) = \mathrm{diag}(j+\kappa-1, \kappa-2, 0, 2-\kappa, 1-j-\kappa) = \mathrm{diag}(k-1, k-g, 0, g-k, 1-k)$. To get $\mathrm{diag}(k-1, k-2, \ldots, k-g, 0, g-k, \ldots, 2-k, 1-k)$ (seen in the previous section) from $\mathrm{diag}(k-1, k-g, 0, g-k, 1-k)$, we need to fill in the gaps using $(g-2)$ copies of $c_\infty(\pi_f) = \mathrm{diag}(\frac{2k-g-1}{2}, \frac{1-g-2k}{2})$, shifted to left and right. So we consider $\psi = \pi_F^{\mathrm{st}} \oplus \pi_f[g-2] \in \Psi_{\mathrm{alg}}$. Note that we have abused notation somewhat; π_F^{st} is a representation of $\mathrm{Sp}_2(\mathbb{A})$, but we are using the same notation for its lift to $\mathrm{PGL}(5)(\mathbb{A})$, via $\mathrm{St} : \mathrm{SO}(3,2) \to \mathrm{SL}(5)$. We must insist that we are in a situation where this lift is cuspidal, so we must exclude the case where $g = 2$ and F is a Saito-Kurokawa lift. (Similar remarks apply in subsequent sections.) In fact, we may as well exclude the case $g = 2$, in which F is already scalar-valued, and π below would be just the same as π_F^{st}.

Proposition 5.2. *There exists $\pi \in \Pi_{\mathrm{disc}}(\mathrm{Sp}_g)$ such that $\psi(\pi) = \pi_F^{\mathrm{st}} \oplus \pi_f[g-2]$.*

Proof. Since $n_1 = 5$ is odd, but $n_2 = 2(g-2)$ is even, C_ψ is generated by $Z(\hat{G})$ and $s_2 =: s_f$. We have $\epsilon_\psi(s_f) = \epsilon_\infty(\pi_f \times \pi_F^{\mathrm{st}})^1 = \epsilon_\infty(\pi_f \times \pi_F^{\mathrm{st}})$. Since $c_\infty(\pi_f) = \mathrm{diag}(\frac{2k-g-1}{2}, \frac{1-g-2k}{2})$ and $c_\infty(\pi_F) = \mathrm{diag}(k-1, k-g, 0, g-k, 1-k)$, the associated motive (twisted to have weight 0) would have Hodge type a union of $\{(-q, q), (q, -q)\}$, where $2q$ runs through $\{2k-g-1+2(k-1) = 4k-g-3, 4k-3g-1, 2k-g-1, g-1, g-1\}$. Putting this in the formula $i^{q-p+1} = i^{2q+1}$, we find that

$$\epsilon_\infty(\pi_f \times \pi_F^{\mathrm{st}}) = i^{4k-g-2+4k-3g+2k-g+g+g} = i^{g+2} = (-1)^{(g/2)+1}.$$

On the other hand $s_f = \mathrm{diag}(1, \underbrace{-1, \ldots, -1}_{g-2 \text{ times}}, 1, 1, 1 \underbrace{-1, \ldots, -1}_{g-2 \text{ times}}, 1)$. In the left half, $\frac{g}{2} - 1$ of the -1s are in odd position, so $\chi_{\mathrm{hol}}(s_f) = (-1)^{(g/2)+1}$. Since this is the same as $\epsilon_\psi(s_f)$, π exists. □

As already noted, $c_\infty(\pi) = \mathrm{diag}(k-1, k-2, \ldots, k-g, 0, g-k, \ldots, 2-k, 1-k)$, so as in the previous section $\pi = \pi_G$ for some cuspidal Hecke eigenform $G \in S_k(\mathrm{Sp}_g(\mathbb{Z}))$. This time $L(s, G, \mathrm{St}) = L(s, F, \mathrm{St}) \prod_{i=1}^{g-2} L(f, s + (k-g+i))$. The existence of such a G is precisely [Ib, Conjecture 3.2].

5.3. **Spinor lifts.** Now $k, g \geq 2$ even, $f \in S_{2k-g}(\mathrm{SL}(2,\mathbb{Z}))$, and F is a cuspidal Hecke eigenform for $\mathrm{Sp}_2(\mathbb{Z})$, of weight $\det^\kappa \otimes \mathrm{Sym}^j(\mathbb{C}^2)$, with $(\kappa, j) = (r+1, 2k-g-1-r)$ (so we impose $k > \frac{g}{2} + r + 1$), for some fixed odd r with $\frac{g}{2} + 1 \leq r < g$. To F we associate an automorphic representation π_F^{spin} of $\mathrm{PGSp}_2(\mathbb{A}) \simeq \mathrm{SO}(3,2)(\mathbb{A})$, with

$$c_\infty(\pi_F^{\mathrm{spin}}) = \mathrm{diag}\left(\frac{j+2\kappa-3}{2}, \frac{j+1}{2}, -\frac{j+1}{2}, -\frac{j+2\kappa-3}{2}\right)$$

$$= \mathrm{diag}\left(\frac{2k-g+r-2}{2}, \frac{2k-g-r}{2}, -\frac{2k-g-r}{2}, -\frac{2k-g+r-2}{2}\right).$$

Then

$$c_\infty(\pi_F^{\mathrm{spin}}[g+1-r])$$

$$= \mathrm{diag}(k-1,\dots,k+r-g-1,k-r,\dots,k-g,g-k,\dots,r-k,1+g-r-k,\dots,1-k),$$

where the dots denote unbroken sequences of consecutive integers. To fill in the gaps we use $\pi_f[2r-g-2]$, then to put 0 in the middle we use [1]. Thus

$$c_\infty(\pi_F^{\mathrm{spin}}[g+1-r] \oplus \pi_f[2r-g-2] \oplus [1])$$

$$= \mathrm{diag}\left(k-1,k-2,\dots,k-g,0,g-k,\dots,2-k,1-k\right).$$

Note that since $r > 2$ and $j > 0$, there are no entries in $c_\infty(\pi_F^{\mathrm{spin}})$ differing by 1, so in the Arthur parameter of π_F^{spin}, all $d_i = 1$. The possibility that π_F^{spin} is endoscopic is ruled out, since there are no holomorphic Yoshida lifts at level 1. Hence the lift of π_F^{spin} to $\mathrm{PGL}(4)(\mathbb{A})$, which is what is really meant above by π_F^{spin}, must be cuspidal, as desired.

Proposition 5.3. *If $4 \mid g$, there exists $\pi \in \Pi_{\mathrm{disc}}(\mathrm{Sp}_g)$ such that $\psi(\pi) = \pi_F^{\mathrm{spin}}[g+1-r] \oplus \pi_f[2r-g-2] \oplus [1]$.*

Proof. This time $n_1 = 4(g+1-r)$ and $n_2 = 2(2r-g-2)$ are even, while $n_3 = 1$ is odd, so we must consider $s_1 =: s_F$ and $s_2 =: s_f$. Since $\widehat{G^{\pi_f}}$ and $\widehat{G^{\pi_F^{\mathrm{spin}}}}$ are both symplectic, it follows from a theorem of Arthur (see [CR, §3.20]) that $\epsilon(\pi_f \times \pi_F^{\mathrm{spin}}) = 1$. Hence $\epsilon_\psi(s_f) = \epsilon_\infty(\pi_f \times 1)^1 = \epsilon_\infty(\pi_f) = (-1)^{g/2}$ as before, and likewise $\epsilon_\psi(s_F) = \epsilon_\infty(\pi_F^{\mathrm{spin}}) = i^{(2k-g-r+1)+(2k-g+r-1)} = (-1)^{g/2}$.

$$s_f = \mathrm{diag}(\underbrace{1,\dots,1}_{g+1-r},\underbrace{-1,\dots,-1}_{2r-g-2},\underbrace{1,\dots,1}_{2g+3-2r},\underbrace{-1,\dots,-1}_{2r-g-2},\underbrace{1,\dots,1}_{g+1-r}),$$

and on the left side the number of -1s in odd position is $r - \frac{g}{2} - 1$, so $\chi_{\mathrm{hol}}(s_f) = (-1)^{r-(g/2)-1} = (-1)^{g/2}$, since r is odd.

$$s_F = \mathrm{diag}(\underbrace{-1,\dots,-1}_{g+1-r},\underbrace{1,\dots,1}_{2r-g-2},\underbrace{-1,\dots,-1}_{g+1-r},1,\underbrace{-1,\dots,-1}_{g+1-r},\underbrace{1,\dots,1}_{2r-g-2},\underbrace{-1,\dots,-1}_{g+1-r}),$$

and on the left side the number of -1s in odd position is $g+1-r$, which is even, so $\chi_{\mathrm{hol}}(s_F) = 1$. Thus, though $\chi_{\mathrm{hol}}(s_f) = \epsilon_\psi(s_f)$, for $\chi_{\mathrm{hol}}(s_F) = \epsilon_\psi(s_F)$ we need the condition $4 \mid g$. □

As already noted, $c_\infty(\pi) = \mathrm{diag}(k-1,k-2,\dots,k-g,0,g-k,\dots,2-k,1-k)$, so as before, $\pi = \pi_G$ for some cuspidal Hecke eigenform $G \in S_k(\mathrm{Sp}_g(\mathbb{Z}))$. This time $L(s,G,\mathrm{St}) = \zeta(s)\prod_{i=1}^{g+1-r} L(s-i+(g-r+2)/2, F, \mathrm{spin}) \prod_{i=1}^{2r-g-2} L(f, s+(k-r+i))$, where the spinor L-function is in its automorphic normalisation, centred at $s = 1/2$. In the special case $r = \frac{g}{2}+1$ (in which case f does not actually appear), the existence of such a G is precisely [Ib, Conjecture 3.1].

5.4. **Ikeda-Miyawaki lifts.** Consider Hecke eigenforms $f \in S_{2k}(\mathrm{SL}(2,\mathbb{Z}))$, $h \in S_{k+n+1}(\mathrm{SL}(2,\mathbb{Z}))$, where $k+n+1$ is even. Let π_f be the associated cuspidal, automorphic representation of $\mathrm{PGL}(2)(\mathbb{A})$, and π_h^{st} the cuspidal automorphic representation of $\mathrm{Sp}_1(\mathbb{A}) = \mathrm{SL}_2(\mathbb{A})$ associated with h. Recall that $c_p(\pi_h^{\mathrm{st}}) = \mathrm{diag}(\alpha_p^2, 1, \alpha_p^{-2}) \in \mathrm{SO}(2,1)(\mathbb{C})$ (where $a_p(h) = p^{(k+n)/2}(\alpha_p + \alpha_p^{-1})$), and $c_\infty(\pi_h^{\mathrm{st}}) = \mathrm{diag}(k+n, 0, -k-n)$. Since $c_\infty(\pi_f) = \mathrm{diag}(\frac{2k-1}{2}, \frac{1-2k}{2})$, we see that $c_\infty(\pi_h^{\mathrm{st}} \oplus \pi_f[2n]) = \mathrm{diag}(k+n, \ldots, k-n, 0, n-k, \ldots, -n-k)$, where the dots denote unbroken sequences of consecutive integers. This is of the form $\mathrm{diag}\,(\kappa-1, \kappa-2, \ldots, \kappa-g, 0, g-\kappa, \ldots, 2-\kappa, 1-\kappa)$, where $\kappa = k+n+1$ and $g = 2n+1$.

Proposition 5.4. *There exists* $\pi \in \Pi_{\mathrm{disc}}(\mathrm{Sp}_{2n+1})$ *such that* $\psi(\pi) = \pi_h^{\mathrm{st}} \oplus \pi_f[2n]$.

Proof. Since $n_1 = 3$ is odd, while $n_2 = 4n$ is even, we consider $s_2 =: s_f$. First, $\epsilon_\psi(s_f) = \epsilon_\infty(\pi_h^{\mathrm{st}} \times \pi_f)$. The associated motive (twisted to have weight 0) would have Hodge type a union of $\{(-q,q),(q,-q)\}$, where $2q$ runs through $\{2k-1+2(k+n) = 4k+2n-1, 2k-1, 2n+1\}$. Putting this in the formula $i^{q-p+1} = i^{2q+1}$, we find that

$$\epsilon_\infty(\pi_f) = i^{4k+2n+2k+2n+2} = i^{2k+2} = (-1)^{k+1}.$$

Now $s_f = \mathrm{diag}(1, \underbrace{-1, \ldots, -1}_{2n}, 1, \underbrace{-1, \ldots, -1}_{2n}, 1)$, and in the left half, n of the -1s are in odd position, so $\chi_{\mathrm{hol}}(s_f) = (-1)^n$, which is the same as $(-1)^{k+1}$, since $n+k+1$ is even. □

As already noted, $c_\infty(\pi) = \mathrm{diag}(k+n, \ldots, k-n, 0, n-k, \ldots, -n-k)$, so $\pi = \pi_G$ for some cuspidal Hecke eigenform $G \in S_{k+n+1}(\mathrm{Sp}_{2n+1}(\mathbb{Z}))$. Also $L(s, G, \mathrm{St}) = L(s, h, \mathrm{St}) \prod_{i=1}^{2n} L(f, s+(k-n-1+i))$, and we recognise G as a lift whose existence was conjectured by Miyawaki and proved by Ikeda [Miy, Ik2].

5.5. **Lifts from genus 3 and 1.** Let f be as in the previous section, with $k+n+1$ still even. Let F be a vector-valued cuspidal Hecke eigenform of genus 3 such that if π_F^{st} is the associated automorphic representation of $\mathrm{Sp}_3(\mathbb{A})$ then $c_\infty(\pi_F^{\mathrm{st}}) = \mathrm{diag}(k+n, k+n-1, k-n, 0, n-k, -n-k+1, -n-k)$. In the language of [BFvdG, §§4.1,7], $(a,b,c) = (k+n-3, k+n-3, k-n-1)$. To fill in the gaps of length $2n-2$, we consider $\psi = \pi_F^{\mathrm{st}} \oplus \pi_f[2n-2]$. We may as well exclude the case $n=1$, in which F is already scalar-valued and π below would be just the same as π_F^{st}.

Proposition 5.5. *There exists* $\pi \in \Pi_{\mathrm{disc}}(\mathrm{Sp}_{2n+1})$ *such that* $\psi(\pi) = \pi_F^{\mathrm{st}} \oplus \pi_f[2n-2]$.

Proof. Since $n_1 = 7$ is odd, while $n_2 = 4n-4$ is even, we consider $s_2 =: s_f$.

$$\epsilon_\psi(s_f) = \epsilon_\infty(\pi_F^{\mathrm{st}} \times \pi_f) = i^{(4k+2n)+(4k+2n-2)+(4k-2n)+2k+2n+(2n+2)+2n} = i^{2k} = (-1)^k.$$

$$s_f = \mathrm{diag}(1, 1, \underbrace{-1, \ldots, -1}_{2n-2}, 1, 0, 1, \underbrace{-1, \ldots, -1}_{2n-2}, 1, 1),$$

with $n-1$ of -1s in the left half in odd position, so $\chi_{\mathrm{hol}}(s_F) = (-1)^{n-1}$, which is the same as $(-1)^k$, since $k+n+1$ is even. □

As before, $\pi = \pi_G$ for some cuspidal Hecke eigenform $G \in S_{k+n+1}(\mathrm{Sp}_{2n+1}(\mathbb{Z}))$. We read off $L(s, G, \mathrm{St}) = L(s, F, \mathrm{St}) \prod_{i=1}^{2n-2} L(f, s+k-n+i)$.

5.6. **Lifts from genus 1, 2 and 1.** As in §5.4, consider Hecke eigenforms $f \in S_{2k}(\mathrm{SL}(2,\mathbb{Z}))$, $h \in S_{k+n+1}(\mathrm{SL}(2,\mathbb{Z}))$, where $k+n+1$ is even. Let π_f be the associated cuspidal, automorphic representation of $\mathrm{PGL}(2)(\mathbb{A})$, and π_h^{st} the cuspidal automorphic representation of $\mathrm{Sp}_1(\mathbb{A}) = \mathrm{SL}_2(\mathbb{A})$ associated with h. Let F be a cuspidal Hecke eigenform for $\mathrm{Sp}_2(\mathbb{Z})$, of weight $\det^\kappa \otimes \mathrm{Sym}^j(\mathbb{C}^2)$, with $(\kappa, j) = (r+1, 2k-1-r)$, for some fixed odd r with $n+1 \le r \le 2n-1$. To F we associate an automorphic representation π_F^{spin} of $\mathrm{PGSp}_2(\mathbb{A}) \simeq \mathrm{SO}(3,2)(\mathbb{A})$, with

$$c_\infty(\pi_F^{\mathrm{spin}}) = \mathrm{diag}\left(\frac{j+2\kappa-3}{2}, \frac{j+1}{2}, -\frac{j+1}{2}, -\frac{j+2\kappa-3}{2}\right)$$
$$= \mathrm{diag}\left(\frac{2k+r-2}{2}, \frac{2k-r}{2}, -\frac{2k-r}{2}, -\frac{2k+r-2}{2}\right).$$

Then

$$c_\infty(\pi_F^{\mathrm{spin}}[2n+1-r])$$
$$= \mathrm{diag}(k+n-1, \ldots, k+r-n-1, k+n-r, \ldots, k-n, n-k, \ldots, r-n-k, 1+n-r-k, \ldots, 1-k-n),$$

where the dots denote unbroken sequences of consecutive integers. To fill in the gaps we use $\pi_f[2r-2n-2]$, and we also add $c_\infty(\pi_h^{\mathrm{st}}) = \mathrm{diag}(k+n, 0, -n-k)$. Thus

$$c_\infty(\pi_h^{\mathrm{st}} \oplus \pi_F^{\mathrm{spin}}[2n+1-r] \oplus \pi_f[2r-2n-2])$$
$$= \mathrm{diag}\left(k+n, k+n-1, \ldots, k-n, 0, n-k, \ldots, 1-n-k, -n-k\right).$$

Proposition 5.6. *There exists $\pi \in \Pi_{\mathrm{disc}}(\mathrm{Sp}_{2n+1})$ such that $\psi(\pi) = \pi_h^{\mathrm{st}} \oplus \pi_F^{\mathrm{spin}}[2n+1-r] \oplus \pi_f[2r-2n-2]$.*

Proof. This time $n_2 = 4(2n+1-r)$ and $n_3 = 2(2r-2n-2)$ are even, while $n_1 = 3$ is odd, so we must consider $s_2 =: s_F$ and $s_3 =: s_f$. Since $\widehat{G^{\pi_f}}$ and $\widehat{G^{\pi_F^{\mathrm{spin}}}}$ are both symplectic, it follows from a theorem of Arthur (see [CR, §3.20]) that $\epsilon(\pi_f \times \pi_F^{\mathrm{spin}}) = 1$. Hence

$$\epsilon_\psi(s_f) = \epsilon_\infty(\pi_f \times \pi_h^{\mathrm{st}})^1 = i^{2k+(2n+2)+(4k+2n)} = (-1)^{k+1},$$

and likewise

$$\epsilon_\psi(s_F) = \epsilon_\infty(\pi_F^{\mathrm{spin}} \times \pi_h^{\mathrm{st}})$$
$$= i^{(2k+r-1)+(2k-r+1)+(2n+r+1)+(2n-r+3)+(4k+2n+r-1)+(4k+2n-r+1)} = 1.$$

$$s_f = \mathrm{diag}(\underbrace{1,\ldots,1}_{2n+1-r}, \underbrace{-1,\ldots,-1}_{2r-2n-2}, \underbrace{1,\ldots,1}_{4n+3-2r}, \underbrace{-1,\ldots,-1}_{2r-2n-2}, \underbrace{1,\ldots,1}_{2n+1-r}),$$

and on the left side the number of -1s in odd position is $r-n-1$, so $\chi_{\mathrm{hol}}(s_f) = (-1)^{r-n-1} = (-1)^n$, since r is odd. This is the same as $(-1)^{k+1}$, since $n+k+1$ is even.

$$s_F = \mathrm{diag}(\underbrace{-1,\ldots,-1}_{2n+1-r}, \underbrace{1,\ldots,1}_{2r-2n-2}, \underbrace{-1,\ldots,-1}_{2n+1-r}, 1, \underbrace{-1,\ldots,-1}_{2n+1-r}, \underbrace{1,\ldots,1}_{2r-2n-2}, \underbrace{-1,\ldots,-1}_{2n+1-r}),$$

and on the left side the number of -1s in odd position is $2n+1-r$, which is even, so $\chi_{\mathrm{hol}}(s_F) = 1$. □

We have $\pi = \pi_G$ for some cuspidal Hecke eigenform $G \in S_{k+n+1}(\mathrm{Sp}_{2n+1}(\mathbb{Z}))$, and we get $L(s, G, \mathrm{St})$

$$= L(s, h, \mathrm{St}) \prod_{i=1}^{2n+1-r} L(s + \frac{2n-r}{2} + 1 - i, F, \mathrm{spin}) \prod_{j=1}^{2r-2n-2} L(f, s + k + n - r + j).$$

Note that in the case $r = n+1$, f does not appear.

6. Congruences between lifts and "non-lifts"

6.1. Congruences between Ikeda lifts and non-Ikeda lifts. The following is Theorem 4.7 of [Ka1]. The proof makes use of the proof by Katsurada and Kawamura [KK] of a conjecture of Ikeda on the Petersson norm of his lift. The normalised L-values $L_{\mathrm{alg}}(f, k)$ and $L_{\mathrm{alg}}(2i+1, f, \mathrm{St})$ are obtained from $L(f, k)$ and $L(2i+1, f, \mathrm{St})$ by dividing by suitably normalised Deligne periods, as explained in [BD, §4]. For $L_{\mathrm{alg}}(f, k)$, the Deligne period is as constructed in [Ka1, §4], using parabolic cohomology with integral coefficients. (Since $q > 2k$, we may ignore various factorials of small numbers.) For $L_{\mathrm{alg}}(2i+1, f, \mathrm{St})$ it is essentially a product $\Omega^+\Omega^-$ of normalised Deligne periods for $L(f, s)$ [Du, Lemma 5.1], but given the condition (2) below, this is as good as the $\langle f, f\rangle$ used by Katsurada (see condition (3) in [Ka1, Theorem 4.7]).

Theorem 6.1. *For $k, g \geq 2$ even, and $f \in S_{2k-g}(\mathrm{SL}(2, \mathbb{Z}))$ a Hecke eigenform, let $F \in S_k(\mathrm{Sp}_g(\mathbb{Z}))$ be the Ikeda lift, as in §5.1 above. Suppose that $k \geq 2g + 4$ and that $q > 2k$ is a prime number such that, for some divisor $\mathfrak{q} \mid q$ in a sufficiently large number field,*

$$\mathrm{ord}_{\mathfrak{q}}(L_{\mathrm{alg}}(f, k) \prod_{i=1}^{(g/2)-1} L_{\mathrm{alg}}(2i+1, f, \mathrm{St})) > 0.$$

Suppose further that

(1) *for some even integer t with $k+2 \leq t \leq 2k-2g-2$, and some fundamental discriminant D with $(-1)^{g/2}D > 0$,*

$$\mathrm{ord}_{\mathfrak{q}}\left(\frac{\zeta(t+g-k)}{\pi^{t+g-k}}\left(\prod_{i=1}^{g} L_{\mathrm{alg}}(f, t+i-1)\right) L_{\mathrm{alg}}(f, (k-2g)/2, \chi_D)D\right) = 0,$$

where χ_D is the associated quadratic character, and the Dirichlet L-value is normalised as in [Ka1]*;*

(2) *there is not a congruence mod $\mathfrak{q}$ of Hecke eigenvalues between f and another Hecke eigenform in $S_{2k-g}(\mathrm{SL}(2, \mathbb{Z}))$;*

(3) *if $g > 2$, $q \nmid \prod_{p \leq \frac{2k-g}{12}, p\ prime}(1 + p + p^2 + \ldots + p^{g-1})$.*

Then there exists a Hecke eigenform $G \in S_k(\mathrm{Sp}_g(\mathbb{Z}))$, not the Ikeda lift of any Hecke eigenform $h \in S_{2k-g}(\mathrm{SL}(2, \mathbb{Z}))$, such that for any prime p, corresponding Hecke eigenvalues for F and G, for all the Hecke operators $T(p)$ and $T_i(p^2)$ ($1 \leq i \leq g$), are congruent mod $\mathfrak{q}$.

Ikeda proved only that F is a Hecke eigenform for the $T_i(p^2)$ (defined in [Ka1, §2]), which generate a Hecke algebra associated with the pair $(\mathrm{Sp}_g(\mathbb{Q}_p), \mathrm{Sp}_g(\mathbb{Z}_p))$, but Katsurada [Ka1, Proposition 4.1] extended this to $T(p)$, which with the $T_i(p^2)$ generates a Hecke algebra associated with $(\mathrm{GSp}_g(\mathbb{Q}_p), \mathrm{GSp}_g(\mathbb{Z}_p))$. (See also the final paragraph of §3 above.) If we ignore the $T(p)$ then the congruence in the theorem is equivalent to a congruence (for all p) of Satake parameters

$$c_p(\pi_F) \equiv c_p(\pi_G) \pmod{\mathfrak{q}},$$

(or strictly speaking $p^{kg-g(g+1)/2}c_p(\pi_F) \equiv p^{kg-g(g+1)/2}c_p(\pi_G) \pmod{\mathfrak{q}}$), with

$$c_p(\pi_F) = \mathrm{diag}(\alpha_{1,F}, \dots, \alpha_{g,F}, 1, \alpha_{g,F}^{-1}, \dots, \alpha_{1,F}^{-1}) \in \hat{T}(\mathbb{C}),$$

and likewise for G. We should interpret the congruence as being between $c_p(\pi_F)$ and some element in the orbit of $c_p(\pi_G)$ under the action of a Weyl group that can permute the indices $1, \dots, g$ and switch pairs $\alpha_{i,F}$ and $\alpha_{i,F}^{-1}$, in fact $c_p(\pi_F)$ really should be thought of as a conjugacy class in $\hat{G}(\mathbb{C})$, represented by the above element of $\hat{T}(\mathbb{C})$. To include $T(p)$ as well, we would need to consider also $\alpha_{0,F}$ with $\alpha_{0,F}^2 \prod_{i=1}^g \alpha_{i,F} = 1$, for each p.

6.2. **Congruences between Ikeda-Miyawaki lifts and non-Ikeda-Miyawaki lifts.** The following is taken from Conjecture B and Problem B' of [IKPY], which are inspired by a conjecture of Ikeda on the Petersson norm of the Ikeda-Miyawaki lift. The normalised L-values $L_{\mathrm{alg}}(2i+1, f, \mathrm{St})$ are as above. The meaning of $L_{\mathrm{alg}}(f \otimes \mathrm{Sym}^2 h, 2k+2n)$ in [IKPY] is left a little vague. In theory we take it as in [BD, §4]. Ibukiyama, Katsurada, Poor and Yuen use a practical substitute when they prove an instance of the congruence in [IKPY, §5].

Conjecture 6.2. *For Hecke eigenforms $f \in S_{2k}(\mathrm{SL}(2,\mathbb{Z}))$, $h \in S_{k+n+1}(\mathrm{SL}(2,\mathbb{Z}))$, where $k+n+1$ is even, let $F \in S_{k+n+1}(\mathrm{Sp}_{2n+1}(\mathbb{Z}))$ be the Ikeda-Miyawaki lift, as in §5.4. Suppose that $q > 2k+2n-2$ is a prime number such that, for some divisor $\mathfrak{q} \mid q$ in a sufficiently large number field,*

$$\mathrm{ord}_{\mathfrak{q}}\left(L_{\mathrm{alg}}(f \otimes \mathrm{Sym}^2 h, 2k+2n) \prod_{i=1}^{n-1} L_{\mathrm{alg}}(2i+1, f, \mathrm{St})\right) > 0.$$

Then there exists a Hecke eigenform $G \in S_{k+n+1}(\mathrm{Sp}_{2n+1}(\mathbb{Z}))$, not the Ikeda-Miyawaki lift of any Hecke eigenforms $f' \in S_{2k}(\mathrm{SL}(2,\mathbb{Z}))$, $h' \in S_{k+n+1}(\mathrm{SL}(2,\mathbb{Z}))$, such that for any prime p, corresponding Hecke eigenvalues for F and G, for all the Hecke operators $T(p)$ and $T_i(p^2)$ $(1 \leq i \leq g)$, are congruent mod $\mathfrak{q}$.

Remarks about congruences of Satake parameters, similar to the previous subsection, apply.

7. Accounting for some of the congruences

7.1. **Ikeda lifts and standard lifts: $L_{\mathrm{alg}}(f,k)$.** We have $2k-g = j+2\kappa-2, k = j+\kappa$, if $(\kappa, j) = (k+2-g, g-2)$, in agreement with §5.2 above. Harder's conjecture [H, vdG] may be formulated, given $\mathfrak{q} \mid q$ with $q > 2k-g$ and $\mathrm{ord}_{\mathfrak{q}}(L_{\mathrm{alg}}(f,k)) > 0$, as the existence of a Hecke eigenform F for $\mathrm{Sp}_2(\mathbb{Z})$, of weight $\det^{\kappa} \otimes \mathrm{Sym}^j(\mathbb{C}^2)$, such that if π_F^{st} is the associated automorphic representation of $\mathrm{Sp}_2(\mathbb{A})$ then for all primes p,

$$c_p(\pi_F^{\mathrm{st}}) \equiv \mathrm{diag}(\alpha_p p^{(g-1)/2}, \alpha_p p^{(1-g)/2}, 1, \alpha_p^{-1} p^{(g-1)/2}, \alpha_p^{-1} p^{(1-g)/2}) \pmod{\mathfrak{q}},$$

where $c_p(\pi_f) = \mathrm{diag}(\alpha_p, \alpha_p^{-1})$. The $\frac{g-1}{2} = \frac{j+1}{2}$ is what we called s in [BD]. Note that if we let $\alpha_{1,F} = \alpha_p p^s$, $\alpha_{2,F} = \alpha_p p^{-s}$ and $\alpha_{0,F} = \alpha_p^{-1}$ (so $\alpha_0^2 \alpha_1 \alpha_2 = 1$) then

$$\alpha_{0,F} + \alpha_{0,F}\alpha_{1,F} + \alpha_{0,F}\alpha_{2,F} + \alpha_{0,F}\alpha_{1,F}\alpha_{2,F} = \alpha_p + \alpha_p^{-1} + p^{-s} + p^s,$$

which when scaled by $p^{(j+2\kappa-3)/2}$ gives the familiar $a_p(f) + p^{\kappa-2} + p^{j+\kappa-1}$ on the right hand side of Harder's conjecture (as a Hecke eigenvalue for $T(p)$ on an induced representation). For simplicity we actually ignore $T(p)$, and consider only the Hecke algebra generated by $T_1(p^2)$ and $T_2(p^2)$. This is because we are looking at an automorphic representation of $\mathrm{Sp}_2(\mathbb{A})$ rather than of $\mathrm{GSp}_2(\mathbb{A})$. In [BD, §7], we looked at Harder's conjecture as a congruence of Hecke eigenvalues between a cuspidal automorphic representation of $\mathrm{GSp}_2(\mathbb{A})$ and a representation induced from the Levi subgroup $(\mathrm{GL}_1 \times \mathrm{GL}_2)(\mathbb{A})$ of the Siegel maximal parabolic (and worked it out explicitly only for $T(p)$). Here we can either restrict to $\mathrm{Sp}_2(\mathbb{A})$ or just consider directly Sp_2 with the Levi subgroup $\mathrm{GL}_1 \times \mathrm{SL}_2$ of its Siegel parabolic.

Now $c_p(\pi_f[g])$

$$= \mathrm{diag}(\alpha_p p^{(g-1)/2}, \alpha_p p^{(g-3)/2}, \ldots, \alpha_p p^{(1-g)/2}, \alpha_p^{-1} p^{(g-1)/2}, \ldots, \alpha_p^{-1} p^{(1-g)/2}),$$

and

$$c_p(\pi_f[g-2]) = \mathrm{diag}(\alpha_p p^{(g-3)/2}, \ldots, \alpha_p p^{(3-g)/2}, \alpha_p^{-1} p^{(g-3)/2}, \ldots, \alpha_p^{-1} p^{(3-g)/2}),$$

so the congruence can be read as

$$c_p(\pi_F^{\mathrm{st}} \oplus \pi_f[g-2]) \equiv c_p(\pi_f[g] \oplus [1]) \pmod{\mathfrak{q}}.$$

Comparing with §5.1 and §5.2, we see that in the case of $\mathfrak{q} \mid L_{\mathrm{alg}}(f,k)$, we can explain the congruence between the Ikeda lift and a non-Ikeda lift in Theorem 6.1 as a congruence between the Ikeda lift and a "standard lift" as constructed in §5.2. So the congruence in Theorem 6.1 is derived from that in Harder's conjecture via lifting to scalar-valued large genus forms. In the excluded case $g = 2$, Harder's conjecture is replaced by its degeneration, a congruence between a Saito-Kurokawa lift and non-lift, which does not require further lifting.

7.2. **Ikeda lifts and spinor lifts:** $L_{\mathrm{alg}}(2i+1, f, \mathrm{St})$**.** If $r = 2i+1$ then as i runs from 1 to $\frac{g}{2} - 1$, r runs through odd numbers from 3 to $g-1$. We shall only be able to account for the congruence in Conjecture 6.1 if $4 \mid g$ and $\frac{g}{2} + 1 \leq r \leq g-1$. We also require $q > 4k - 2g$. Let $(\kappa, j) = (r+1, 2k-g-1-r)$, so $\kappa + j = 2k - g$ and $r = s+1$, where $s = \kappa - 2$ as in [BD, §6]. Then a conjectural congruence of Kurokawa-Mizumoto type (instances of which were proved in [Ku, Miz, Sa, Du]) may be formulated, given $\mathrm{ord}_{\mathfrak{q}}(L_{\mathrm{alg}}(r, f, \mathrm{St})) > 0$, as the existence of a Hecke eigenform F for $\mathrm{Sp}_2(\mathbb{Z})$, of weight $\det^{\kappa} \otimes \mathrm{Sym}^j(\mathbb{C}^2)$, such that if π_F^{spin} is the associated automorphic representation of $\mathrm{SO}(3,2)(\mathbb{A})$ then for all primes p,

$$c_p(\pi_F^{\mathrm{spin}}) \equiv \mathrm{diag}(\alpha_p p^{s/2}, \alpha_p p^{-s/2}, \alpha_p^{-1} p^{s/2}, \alpha_p^{-1} p^{-s/2}) \pmod{\mathfrak{q}},$$

where $c_p(\pi_f) = \mathrm{diag}(\alpha_p, \alpha_p^{-1})$. Note that the trace of the right hand side, when scaled by $p^{(j+2\kappa-3)/2}$, becomes the familiar $a_p(f)(1+p^{\kappa-2})$. Recalling that $s = r-1$, this would imply that $c_p(\pi_F^{\mathrm{spin}}[g+1-r])$

$$\equiv \mathrm{diag}(\alpha_p p^{(g-1)/2}, \ldots, \alpha_p p^{(2r-g-1)/2}, \alpha_p p^{(1+g-2r)/2}, \ldots, \alpha_p p^{(1-g)/2},$$
$$\alpha_p^{-1} p^{(g-1)/2}, \ldots, \alpha_p^{-1} p^{(2r-g-1)/2}, \alpha_p^{-1} p^{(1+g-2r)/2}, \ldots, \alpha_p^{-1} p^{(1-g)/2}).$$

The right hand side is the "difference" between $c_p(\pi_f[g])$ and $c_p(\pi_f[2r-g-2])$. Thus we can read the congruence as

$$c_p(\pi_F^{\mathrm{spin}}[g+1-r] \oplus \pi_f[2r-g-2] \oplus [1]) \equiv c_p(\pi_f[g] \oplus [1]),$$

i.e. as a congruence between the Ikeda lift and one of the "spinor lifts" constructed in §5.3. In the case of $\mathfrak{q} \mid L_{\mathrm{alg}}(2i+1, f, \mathrm{St})$, with $4 \mid g$, $\frac{g}{4} \leq i \leq \frac{g}{2} - 1$ and $q > 4k - 2g$, we can explain the congruence between the Ikeda lift and a non-Ikeda lift in Theorem 6.1 (at least if we ignore $T(p)$) as a congruence between the Ikeda lift and a spinor lift. Thus the congruence in Theorem 6.1 is derived from that of Kurokawa-Mizumoto type via lifting to scalar-valued, large genus forms. Note that we have had to impose a stronger lower bound for q.

7.3. **Ikeda-Miyawaki lifts:** $L_{\mathrm{alg}}(f \otimes \mathrm{Sym}^2 h, 2k+2n)$**.** Recall that we consider Hecke eigenforms $f \in S_{2k}(\mathrm{SL}(2,\mathbb{Z}))$, $h \in S_{k+n+1}(\mathrm{SL}(2,\mathbb{Z}))$, where $k+n+1$ is even. Let $a_p(f) = p^{(2k-1)/2}(\alpha_p + \alpha_p^{-1})$ and $b_p(h) = p^{(k+n)/2}(\beta_p + \beta_p^{-1})$. Let $(a,b,c) = (k+n-3, k+n-3, k-n-1)$, as in §5.5 above. Then $b+c+4 = 2k$, $a+4 = k+n+1$ (the weights of f and h), $a+b+6 = 2k+2n$ and $s := \frac{b-c+1}{2} = \frac{2n-1}{2}$. Comparing with [BD, §8, Case 2], the conjecture there (see also [BFvdG, Conjecture 10.8]), given $\mathrm{ord}_{\mathfrak{q}}(L_{\mathrm{alg}}(f \otimes \mathrm{Sym}^2 h, 2k+2n) > 0$ with $q > a+b+2c+8 = 4k$, can be formulated (ignoring $T(p)$) as the existence of a cuspidal Hecke eigenform F for $\mathrm{Sp}_3(\mathbb{Z})$, vector-valued of type (a,b,c), such that

$$c_p(\pi_F^{\mathrm{st}}) \equiv \mathrm{diag}(\alpha_p p^s, \alpha_p^{-1} p^s, \beta_p^2, 1, \beta_p^{-2}, \alpha_p p^{-s}, \alpha_p^{-1} p^{-s}) \pmod{\mathfrak{q}}.$$

To get the diagonal entries, apply the cocharacters $f_1, f_2, f_3, 0, -f_3, -f_2, -f_1$ to $\chi_p + s\tilde{\alpha} = -\log_p(\alpha_p)(e_1 - e_2) - \log_p(\beta_p) + s(e_1 + e_2)$ in [BD, §8], omitting e_0 since we are really dealing with $G = \mathrm{Sp}_3$, $M \simeq \mathrm{GL}_2 \times \mathrm{SL}_2$.

Since $c_p(\pi_h^{\mathrm{st}}) = \mathrm{diag}(\beta_p^2, 1, \beta_p^{-2})$, and since $s = \frac{2n-1}{2}$, we can read this as

$$c_p(\pi_F^{\mathrm{st}} \oplus \pi_f[2n-2]) \equiv c_p(\pi_h^{\mathrm{st}} \oplus \pi_f[2n]) \pmod{\mathfrak{q}},$$

i.e. as a congruence between the Ikeda-Miyawaki lift and one of the lifts constructed in §5.5. Thus the congruence in Conjecture 6.2, between the Ikeda-Miyawaki lift and a non-Ikeda-Miyawaki lift, can be derived from the conjectured genus 3 Eisenstein congruence, via lifting to scalar-valued, large genus forms. Again, we have had to impose a stronger lower bound for q. In the excluded case $n = 1$, the Eisenstein congruence degenerates to a congruence between an Ikeda-Miyawaki lift and a non-Ikeda-Miyawaki lift, without any further lifting.

7.4. **Ikeda-Miyawaki lifts:** $L_{\mathrm{alg}}(2i+1, f, \mathrm{St})$**.** If $r = 2i+1$ then as i runs from 1 to $n-1$, r runs through odd numbers from 3 to $2n-1$. We shall only be able to account for the congruence in Theorem 6.2 if $n+1 \leq r \leq 2n-1$. We also require $q > 4k$. Let $(\kappa, j) = (r+1, 2k-1-r)$, so $\kappa + j = 2k$ and $r = s+1$, where $s = \kappa - 2$ as in [BD, §6]. Then a conjecture of Kurokawa-Mizumoto type, given $\mathrm{ord}_{\mathfrak{q}}(L_{\mathrm{alg}}(r, f, \mathrm{St})) > 0$, predicts the existence of a Hecke eigenform F for $\mathrm{Sp}_2(\mathbb{Z})$, of weight $\det^\kappa \otimes \mathrm{Sym}^j(\mathbb{C}^2)$, such that if π_F^{spin} is the associated automorphic representation of $\mathrm{SO}(3,2)(\mathbb{A})$ then for all primes p,

$$c_p(\pi_F^{\mathrm{spin}}) \equiv \mathrm{diag}(\alpha_p p^{s/2}, \alpha_p p^{-s/2}, \alpha_p^{-1} p^{s/2}, \alpha_p^{-1} p^{-s/2}) \pmod{\mathfrak{q}},$$

where $c_p(\pi_f) = \mathrm{diag}(\alpha_p, \alpha_p^{-1})$. Recalling that $s = r-1$, this would imply that $c_p(\pi_F^{\mathrm{spin}}[2n+1-r])$

$$\equiv \mathrm{diag}(\alpha_p p^{(2n-1)/2}, \ldots, \alpha_p p^{(2r-2n-1)/2}, \alpha_p p^{(1+2n-2r)/2}, \ldots, \alpha_p p^{(1-2n)/2},$$
$$\alpha_p^{-1} p^{(2n-1)/2}, \ldots, \alpha_p^{-1} p^{(2r-2n-1)/2}, \alpha_p^{-1} p^{(1+2n-2r)/2}, \ldots, \alpha_p^{-1} p^{(1-2n)/2}).$$

The right hand side is the "difference" between $c_p(\pi_f[2n])$ and $c_p(\pi_f[2r-2n-2])$. Thus we can read the congruence as

$$c_p(\pi_h^{\mathrm{st}} \oplus \pi_F^{\mathrm{spin}}[2n+1-r] \oplus \pi_f[2r-2n-2]) \equiv c_p(\pi_h^{\mathrm{st}} \oplus \pi_f[2n]),$$

i.e. as a congruence between the Ikeda-Miyawaki lift and one of the lifts constructed in §5.6. In the case of $\mathfrak{q} \mid L_{\mathrm{alg}}(2i+1, f, \mathrm{St})$, with $\lceil \frac{n}{2} \rceil \leq i \leq n-1$ and $q > 4k$, we can explain the congruence between the Ikeda-Miyawaki lift and a non-Ikeda-Miyawaki lift in Conjecture 6.2 (at least if we ignore $T(p)$) as a congruence between the Ikeda-Miyawaki lift and a lift from §5.6. Thus the congruence in Conjecture 6.2 is derived from that of Kurokawa-Mizumoto type via lifting to scalar-valued, large genus forms. Again, we have had to impose a stronger lower bound for q.

References

[A] J. Arthur, The endoscopic classification of representations. Orthogonal and symplectic groups. American Mathematical Society Colloquium Publications, 61. American Mathematical Society, Providence, RI, 2013. xviii+590 pp.

[AS] M. Asgari, R. Schmidt, Siegel modular forms and representations, *Manuscripta Math.* **104** (2001), 173–200.

[BD] J. Bergström, N. Dummigan, Eisenstein congruences for split reductive groups, *Selecta Mathematica,* 2015, DOI 10.1007/s00029-015-0211-0.

[BFvdG] J. Bergström, C. Faber, G. van der Geer, Siegel modular forms of degree three and the cohomology of local systems, *Selecta Math.* (N.S.) **20** (2014), 83–124.

[Br] J. Brown, Saito-Kurokawa lifts and applications to the Bloch-Kato conjecture, *Compos. Math.* **143** (2007), 290–322.

[CL] G. Chenevier, C. Lannes, Formes automorphes et voisins de Kneser des réseaux de Niemeier, 388 pages, arXiv:1409.7616, 26/09/2014.

[CR] G. Chenevier, D. Renard, Level one algebraic cuspforms of classical groups of small rank, *Mem. Amer. Math. Soc.* **1121**, vol. 237, 128 pp., 2015.

[De] P. Deligne, Valeurs de Fonctions L et Périodes d'Intégrales, *AMS Proc. Symp. Pure Math.,* Vol. 33 (1979), part 2, 313–346.

[Du] N. Dummigan, Symmetric square L-functions and Shafarevich-Tate groups, II, *Int. J. Number Theory* **5** (2009), 1321–1345.

[FvdG] C. Faber, G. van der Geer, Sur la cohomologie des systèmes locaux sur les espaces de modules des courbes de genre 2 et des surfaces abéliennes, I, II, *C. R. Math. Acad. Sci. Paris* **338** (2004), 381–384 and 467–470.

[H] G. Harder, A congruence between a Siegel and an elliptic modular form, manuscript, 2003, *reproduced in* The 1-2-3 of Modular Forms (K. Ranestad, ed.), 247–262, Springer-Verlag, Berlin Heidelberg, 2008.

[Ib] T. Ibukiyama, Lifting conjectures from vector valued Siegel modular forms of degree two, *Comment. Math. Univ. St. Pauli* **61** (2012), 87–102.

[IKPY] T. Ibukiyama, H. Katsurada, C. Poor, D. Yuen, Congruences to Ikeda-Miyawaki lifts and triple L-values of elliptic modular forms, *J. Number Theory* **134** (2014), 142–180.

[Ik1] T. Ikeda, On the lifting of elliptic cusp forms to Siegel cusp forms of degree 2n, *Ann. of Math. (2)***154** (2001), 641681.

[Ik2] T.Ikeda, Pullback of lifting of elliptic cusp forms and Miyawaki's conjecture, *Duke Math. J.* **131** (2006), 469–497.

[Ka1] H. Katsurada, Congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts, arXiv:1101.3377.

[Ka2] H. Katsurada, Congruence of Siegel modular forms and special values of their standard zeta functions, *Math. Z.* **259** (2008), 97–111.

[KK] H. Katsurada, K. Kawamura, Ikeda's conjecture on the period of the Duke-Imamoglu-Ikeda lift, *Proc. London Math. Soc. (3)* **111** (2015), 445–483.

[KM] H. Katsurada, S. Mizumoto, Congruences for Hecke eigenvalues of Siegel modular forms, *Abh. Math. Semin. Univ. Hambg.* **82** (2012), 129–152.

[Ku] N. Kurokawa, Congruences between Siegel modular forms of degree 2, *Proc. Japan Acad.* **55A** (1979), 417–422.

[Miy] I. Miyawaki, Numerical examples of Siegel cusp forms of degree 3 and their zeta functions, *Mem. Fac. Sci. Kyushu Univ.* **46** (1992), 307–339.

[Miz] S. Mizumoto, Congruences for eigenvalues of Hecke operators on Siegel modular forms of degree two, *Math. Ann.* **275** (1986), 149–161.

[PRY] C. Poor, N. Ryan, D. S. Yuen, Lifting puzzles in degree four, *Bull. Aust. Math. Soc.* **80** (2009), 65–82.

[Sa] T. Satoh, On certain vector valued Siegel modular forms of degree two, *Math. Ann.* **274** (1986), 335–352.

[T] O. Taibi, Dimensions of spaces of level one automorphic forms for split classical groups using the trace formula, preprint, 2014, `http://arxiv.org/abs/1406.4247`.

[vdG] G. van der Geer, Siegel Modular Forms and Their Applications, *in* The 1-2-3 of Modular Forms (K. Ranestad, ed.), 181–245, Springer-Verlag, Berlin Heidelberg, 2008.

University of Sheffield, School of Mathematics and Statistics, Hicks Building, Hounsfield Road, Sheffield, S3 7RH, U.K.

E-mail address: `n.p.dummigan@shef.ac.uk`