VALUES OF A HILBERT MODULAR SYMMETRIC SQUARE L-FUNCTION AND THE BLOCH-KATO CONJECTURE

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Abstract. Let F be a totally real field, of narrow class number one and odd degree over $\mathbb{Q}$, and let f be a Hilbert modular eigenform over F, cuspidal and of level one and scalar weight k such that $k/2$ is odd. Analogy with earlier calculations in the case $F = \mathbb{Q}$ leads us to expect large primes dividing $\zeta_F(1-k)$ to appear in a certain critical value of the symmetric square L-function. This is verified by direct computation in what appears to be the only tractable example such that $F \neq \mathbb{Q}$, namely F is the totally real cubic field of discriminant 49, and f is of weight six. We attempt an explanation via the Bloch-Kato conjecture on special values of L-functions, and a construction of elements in a generalised Shafarevich-Tate group.

1. Introduction

In [Du1] and [Du2] we looked at the prime factorisations of certain rational numbers arising from ratios of critical values for the L-functions of classical modular forms of level one, and their tensor products and symmetric squares. They possess some unusual features (the prevalence of small primes, and the presence of large irregular primes) which may be explained using the Bloch-Kato conjecture on the special values of L-functions attached to motives. In particular, for symmetric squares and tensor products we described a very natural construction of elements of irregular prime order (which had been predicted on the basis of Bloch-Kato) in certain generalised Shafarevich-Tate groups, using Heegner cycles and Ramanujan congruences.

In this paper we look at a Hilbert modular cuspidal eigenform f of level one and scalar weight $k > 2$, over a totally real field F, with coefficients in a number field

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$E \subset \mathbb{C}$. If $k/2$ is odd, and if the degree $g = [F : \mathbb{Q}]$ is also odd, then the sign in the functional equation of the standard L-function $L_f(s)$ is negative, so $L_f(s)$ vanishes at the central point $s = k/2$.

Suppose now that the narrow class number of F is one, so that there is a unique normalised Eisenstein series $E_{k,F}$ of weight k. Under mild assumptions, one may prove the existence of a congruence relating the coefficients of f to the coefficients of $E_{k,F}$, modulo a prime ideal (λ say, of norm ℓ) dividing the numerator of the rational number $\zeta_F(1-k)$. This is analogous to Ramanujan's congruence $\tau(n) \equiv \sigma_{11}(n) (\mathrm{mod}\, 691)$.

Let $D_f(s)$ be the symmetric-square L-function. We expect, by analogy with the case $F = \mathbb{Q}$, $k = 18, 22$ or 26, that the ratio of critical values, $D_f((k/2)+k-1)/(\pi^{(k-2)g} D_f(k))$, which is known to be an element of the number field E, has λ dividing the numerator. For this analogy, it is important that the sign in the functional equation of $L_f(s)$ is negative.

Since $[F : \mathbb{Q}]$ must be odd, the simplest examples to look at (other than $F = \mathbb{Q}$, which we examined in [Du2]), are the totally real cubic fields. The totally real cubic field of smallest discriminant is $F = \mathbb{Q}(\zeta_7)^+$, of discriminant 49. The critical points to the right of the central point are the even integers from k to $2k-2$, so $(k/2)+k-1$ will be critical as long as $k/2$ is odd, and it will be different from k as long as $(k/2) > 1$. So the first weight to try is $k = 6$.

Using a generalisation of the Eichler-Selberg trace formula which may be found in [Mi] or [Tak], one finds that for $F = \mathbb{Q}(\zeta_7)^+$ and $k = 6$, the dimension of the space of cusp forms is two. We count ourselves lucky that there exists at least this one example where the dimension is so manageably small. (Already for the next example, $F = \mathbb{Q}(\zeta_9)^+$ and $k = 6$, the dimension is four, which would be very much more difficult to deal with. For the field $F = \mathbb{Q}(\zeta_7)^+$, with the next weight $k = 10$ for which $k/2$ is odd, we find that $\dim(S_k) = 8$.) We find a Galois-conjugate pair of normalised Hecke eigenforms. Their coefficients satisfy Ramanujan-style congruences modulo prime ideal divisors of $\ell = 7393$.

The critical values of $D_f(s)$ may be calculated using the formulas in [Mi] or [Tak] (though a factor of m^{1-k} in the formulas of [Tak] seemed to prevent me from getting the same answers as I did using [Mi]). The computation involves taking a certain linear combination of values of L-functions of quadratic characters over F. Some of these values are quite difficult to get at, and we have to use a formula of Hida [H]. But when we reach the final answer, the factor of 7393 is there right where we expect it to be. There is nothing in the calculation to suggest an elementary explanation for its appearance. (A similar remark applies to the calculations in the case $F = \mathbb{Q}$, referred to above. Even in this case, it would be interesting to have a proof, independent of computation, of the occurrence of the Eisenstein prime factor in the ratio of values of $D_f(s)$. Theorem 14.2 of [Du2] does this for certain tensor product L-functions attached to pairs of forms of different weights.)

We attempt an explanation via the Bloch-Kato conjecture on special values of the L-functions attached to motives, as in [Du2] for $F = \mathbb{Q}$, though hopefully in a less clumsy manner here. We assume the existence of a motive M_f over F, with coefficients in E, whose L-function is $L_f(s)$. This accords with general conjectures on the correspondence between motives and automorphic forms, and is known in the case $F = \mathbb{Q}$ [Sc]. For $\mathrm{Sym}^2 M_f((k/2) + k - 1)$ and $\mathrm{Sym}^2 M_f(k)$ we examine the λ-part of the conjecture and see that, combined with the computation, it leads us to expect the existence of non-trivial λ-torsion in a certain generalised Shafarevich-Tate group.

We give an independent construction of such non-trivial λ-torsion, assuming (more-or-less an instance of the Beilinson-Bloch conjecture [Be],[Blo]) that, since $L_f(k/2) = 0$, a certain λ-adic Selmer group (for $M_f(k/2)$) is non-trivial. The mod λ congruence is then used to produce from this a non-trivial λ-torsion element in a Selmer group for $\mathrm{Sym}^2 M_f((k/2)+k-1)$. The analogous assumption in the case $F = \mathbb{Q}$ has been proven by Skinner and Urban [SU]. Note that a consequence of their result is that, in [Du2], the application of Nekovar's p-adic Gross-Zagier formula, which in some cases was not computationally feasible, is no longer necessary.

All the computations referred to in this paper were performed using the computer algebra package Maple. I thank the referee for helpful comments on the exposition.

2. Hilbert modular forms for narrow class number one

Let F be a finite extension of the field $\mathbb{Q}$ of rational numbers. Let $g = [F : \mathbb{Q}]$ be the degree of the extension. Let $\sigma_1, \ldots, \sigma_g$ be the distinct embeddings of F into $\mathbb{C}$. Suppose that F is *totally real,* i.e., that $\sigma_i(F) \subset \mathbb{R}$ for all $1 \leq i \leq g$. For $\alpha \in F$, let $\alpha^{(i)} = \sigma_i(\alpha)$. We say that α is *totally positive,* denoted $\alpha \gg 0$, if $\alpha^{(i)} > 0$ for all $1 \leq i \leq g$. Let O_F be the ring of integers of F. Let $\mathfrak{d}$ be the different of F, the integral ideal such that $\mathfrak{d}^{-1}$ is the module dual to O_F with respect to the trace pairing.

We now impose the condition that F has narrow class number one, i.e. that every non-zero ideal of O_F can be generated by a totally positive element. This condition is satisfied in the case we are interested in, and makes things a little simpler. To begin with, there exists a totally positive element δ such that $\mathfrak{d} = (\delta)$. Let $\mathfrak{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the complex upper half plane. Let $z = (z_1, ..., z_g)$ be a variable in $\mathfrak{H}^g$, for $\alpha \in F$ let $\mathrm{tr}(\alpha z) = \sum_{i=1}^g \alpha^{(i)} z_i$, and for $z \in \mathbb{C}$ let $\mathbf{e}(z) = \exp(2\pi i z)$. For a positive integer k, let f be a modular form of weight k for $\mathrm{SL}(2, O_F)$ (i.e. level one). Unless kg is even, $f = 0$. For all $g \in \mathrm{SL}(2, O_F)$,

$$f(gz) = f(z) \prod_{i=1}^{g} (c^{(i)} z_i + d^{(i)})^k,$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $(gz)_i = \frac{a^{(i)} z_i + b^{(i)}}{c^{(i)} z_i + d^{(i)}}$.

The function f has a Fourier expansion

$$f(z) = a(0) + \sum_{0 \ll \alpha \in O_F} a(\alpha) \mathbf{e}(\mathrm{tr}((\alpha/\delta) z)),$$

where the coefficient $a(\alpha)$ depends only on the ideal generated by α. Since we are supposing that every non-zero ideal $\mathfrak{a}$ can be generated by a totally positive element, there is always a coefficient which we may call $a(\mathfrak{a})$. We say that f is a cusp form if $a(0) = 0$. A consequence of the narrow class number one condition is that $\mathrm{SL}(2, O_F) \backslash \mathfrak{H}^g$ has just a single cusp. There is then also a unique normalised Eisenstein series $E_{k,F}$ of weight k. Its Fourier expansion is given by:

$$E_{k,F}(z) = 1 + \frac{2^g}{\zeta_F(1-k)} \sum_{0 \ll \alpha \in O_F} \sigma_{k-1}(\alpha) \mathbf{e}(\mathrm{tr}((\alpha/\delta) z)),$$

where $\sigma_r(\alpha) := \sum_{\mathfrak{a} | (\alpha)} \mathrm{N}(\mathfrak{a})^r$.

The space of cusp forms of weight k is denoted S_k. A cusp form is said to be normalised if $a(1) = 1$. The space S_k has a basis consisting of normalised Hecke eigenforms, for which $a(\mathfrak{a})$ is the eigenvalue of the Hecke operator $T(\mathfrak{a})$. The Hecke operators have the following effect on Fourier expansions. If $f(z) = a(0) +$

$\sum_{0\ll\alpha\in O_F} a(\alpha)\mathbf{e}(\mathrm{tr}((\alpha/\delta)z))$ and $T(\mathfrak{a})f(z) = a'(0) + \sum_{0\ll\alpha\in O_F} a'(\alpha)\mathbf{e}(\mathrm{tr}((\alpha/\delta)z))$ then

$$a'(\alpha) = \sum_{\mathfrak{b}|((\alpha)+\mathfrak{a})} \mathrm{N}(\mathfrak{b})^{k-1} a((\alpha)\mathfrak{a}/\mathfrak{b}^2). \tag{1}$$

Let f be a normalised Hecke eigencuspform of weight k for $\mathrm{SL}(2, O_F)$, where F is a totally real field of narrow class number one. The standard L-function attached to f has an Euler product

$$L_f(s) = \prod_{\mathfrak{p}} (1 - a(\mathfrak{p})\mathrm{N}(\mathfrak{p})^{-s} + \mathrm{N}(\mathfrak{p})^{k-1-2s})^{-1}.$$

Let $1 - a(\mathfrak{p})X + N(\mathfrak{p})^{k-1}X^2 = (1 - \alpha(\mathfrak{p})X)(1 - \beta(\mathfrak{p})X)$. We actually know, from the Ramanujan-Petersson bound (deduced from the Weil conjectures in [Oh]), that $\alpha(\mathfrak{p})$ and $\beta(\mathfrak{p})$ are conjugate complex numbers, of absolute value $\mathrm{N}(\mathfrak{p})^{(k-1)/2}$, at least for almost all $\mathfrak{p}$. Define the symmetric square L-function

$$D_f(s) = \prod_{\mathfrak{p}} [(1 - \alpha(\mathfrak{p})^2\mathrm{N}(\mathfrak{p})^{-s})(1 - \alpha(\mathfrak{p})\beta(\mathfrak{p})\mathrm{N}(\mathfrak{p})^{-s})(1 - \beta(\mathfrak{p})^2\mathrm{N}(\mathfrak{p})^{-s})]^{-1}.$$

According to Theorem 1 of [Mi], if $k \geq 4$ then $D_f(s)$ has an analytic continuation to $\mathbb{C}$, and satisfies a functional equation

$$\mathfrak{D}(s) = \mathfrak{D}(2k - 1 - s),$$

where

$$\mathfrak{D}(s) = (\Gamma(s)(2\pi)^{-s}\Gamma((s + 2 - k)/2)\pi^{-(s+2-k)/2})^g D_f(s).$$

The critical points for $D_f(s)$ (in the sense of [De]) are $r + k - 1$ and $k - r$ for odd r such that $1 \leq r \leq k-1$. These are the numbers such that the gamma factors in the functional equation do not force a "trivial" zero at $k - r$. Strictly speaking, Deligne defines the notion of a critical point in terms of the Hodge numbers of a motive. We shall be assuming the existence of a motive associated to f, whose properties will be described in more detail in §7.

3. Ramanujan-style congruences

Let f be a normalised Hecke eigencuspform as above, of weight k. The coefficients $a(\mathfrak{a})$ generate a number field E. Let L be a normal closure of E. By applying elements of $\mathrm{Gal}(L/\mathbb{Q})$ to the coefficients of f, we get other normalised Hecke eigencuspforms of weight k. Let us suppose that we can obtain a basis for S_k in this fashion.

Lemma 3.1. *Suppose that Galois conjugates of f form a basis for S_k. Let ℓ be a prime number dividing the numerator of the rational number $\zeta_F(1-k)$. Suppose also that there exists a Hilbert modular form g of weight k for $\mathrm{SL}(2, O_F)$, with constant term 1 and algebraic Fourier coefficients which are integral at some prime dividing ℓ. Then, for some prime λ of E, dividing ℓ,*

$$a(\mathfrak{a}) \equiv \sigma_{k-1}(\mathfrak{a}) (\mathrm{mod}\,\lambda) \text{ for all } \mathfrak{a},$$

where $a(\mathfrak{a})$ is a Fourier coefficient of f, as in §2.

The proof is identical to that of Theorem 2 of [DG]. One uses the fact that g must be expressible as the sum of $E_{k,F}$ and a linear combination of f and its Galois conjugates.

4. The work of Mizumoto and Takase

For g, h Hilbert modular forms of weight k for $\mathrm{SL}(2, O_F)$, with gh a cusp form and F of narrow class number one, define the Petersson inner product

$$(g, h) = \int_{\mathrm{SL}(2,O_F)\backslash \mathfrak{H}^g} g(z)\overline{h(z)} \prod_{j=1}^{g} y_j^{k-2}\, dx_j\, dy_j.$$

This gives a non-degenerate Hermitian inner product on the space S_k of cusp forms of weight k. Fix an odd r such that $1 < r < k-1$. The functional on S_k which maps a normalised Hecke eigencuspform f to the critical value $D_f(r+k-1)$ of the symmetric square L-function, is represented by a cusp form whose Fourier coefficients may be calculated explicitly. Essentially the same formula is proved in two different ways in [Mi] and [Tak], and leads to a way of calculating $D_f(r+k-1)$. In particular $D_f(r+k-1)/(\pi^{(2r+k-1)g}(f,f))$ belongs to the algebraic number field E generated by the Fourier coefficients of f. Both methods owe something to [Z], and the introduction of that paper may be recommended for an overview in the case $F = \mathbb{Q}$, but we say a little here, for the convenience of the reader. (That $D_f(r+k-1)/(\pi^{(2r+k-1)g}(f,f))$ belongs to E was also proved by Sturm, by a different method using the Rankin product of f with a theta series [St].)

For $\Re(s) > 3/2$ there is a cusp form Ψ_s of weight k for $\mathrm{SL}(2, O_F)$ such that

$$\zeta_F(2s)(\Psi_s, f) = \left(\frac{\Gamma(k-1)}{(4\pi)^{k-1}}\right)^g d_F^{k-(1/2)} D_f(s+k-1). \tag{2}$$

If $\{f_1, \ldots, f_m\}$ is a basis of normalised eigen cusp forms for $S_k(\mathrm{SL}(2, O_F)$ then necessarily

$$\Psi_s = \sum_{j=1}^{m} \frac{(\Psi_s, f_j)}{(f_j, f_j)} f_j. \tag{3}$$

For r as above, the coefficients of Ψ_r may be calculated explicitly. Takase generalises the method used by Zagier when $F = \mathbb{Q}$, whereas Mizumoto (following a suggestion of Zagier) uses the expression

$$\Psi_s(z) = \sum_{(\beta), 0 \ll \beta \in O_F} N(\beta)^{k-1-s} G_{\beta^2}(z),$$

together with Gundlach's determination of the Fourier expansion of the Poincaré series $G_\mu(z)$. This expression (2.5 of [Mi]) comes from the easy

$$D_f(s) = \zeta_F(2s - 2k + 2) \sum_{\mathfrak{a}} a(\mathfrak{a}^2) N(\mathfrak{a})^{-s}, \tag{4}$$

where the sum is over all non-zero integral ideals of O_F.

Fixing r as above, equating Fourier coefficients on the two sides of (3), and using (2), leads to linear equations for the $D_{f_j}(r + k - 1)$. The equations (5) and (6) in §6 come from the coefficients of $\mathbf{e}(\mathrm{tr}((1/\delta)z))$ and $\mathbf{e}(\mathrm{tr}((2/\delta)z))$, respectively. The coefficients of the x_i are Fourier coefficients of f_i, and the right-hand-sides involve the Fourier coefficients of Ψ_r, with $r = k/2 = 3$.

The equation (3) may be extended to $r = 1$, but using the formula for $D_f(k)$ coming from the Rankin-Selberg method (see 5.8 of [Mi]) shows that the coefficient of f_i on the right-hand-side is independent of i. Thus, equating coefficients of $\mathbf{e}(\mathrm{tr}((\alpha/\delta)z))$ gives a formula for the trace of $T(\alpha)$ (used in the next section for $\alpha = 1$).

5. The simplest example with a cubic field

Let $F = \mathbb{Q}(\zeta_7)^+$ be the real subfield of the cyclotomic field generated by a primitive seventh root of unity. Thus, $F = \mathbb{Q}(\beta)$ where $\beta = \zeta_7 + \zeta_7^{-1}$ is a root of the irreducible cubic polynomial $f(x) = x^3 + x^2 - 2x - 1$. The real roots of f are

$\beta_1 = 2\cos(2\pi/7)$, $\beta_2 = 2\cos(4\pi/7)$ and $\beta_3 = 2\cos(6\pi/7)$. Moreover, $\{1, \beta, \beta^2\}$ is an integral basis for O_F, and we shall write (a, b, c) for $a + b\beta + c\beta^2$.

One easily checks that

$$\mathrm{N}_{F/\mathbb{Q}}((a,b,c)) = a^3 - a^2b + 5a^2c - 2ab^2 + b^3 - b^2c + 6ac^2 - 2bc^2 + c^3 - abc$$

and $\mathrm{tr}_{F/\mathbb{Q}}((a,b,c)) = 3a - b + 5c$. Also, the class number is one, and, examining the signs of the units β and $1+\beta$, in fact the narrow class number is one. The unique ramified prime is 7, with $(7) = (2-\beta)^3$, and a generator for the different $\mathfrak{d}$ is $\delta = (2-\beta)^2$. For any prime $p \neq 7$, p splits completely iff $p \equiv \pm 1 (\mathrm{mod} 7)$, and is inert otherwise.

The Dedekind zeta function $\zeta_F(s)$ is a product $\zeta_F(s) = \zeta(s)L(s,\chi)L(s,\overline{\chi})$ of the Riemann zeta function and two Dirichlet L-functions, where χ is a character of order three of $(\mathbb{Z}/7\mathbb{Z})^*$. Therefore its values at negative odd integers are easily evaluated using the formulas

$$\zeta(1-r) = -B_r/r,\ L(1-r,\chi) = -7^{r-1}B_{r,\chi}/r,$$

where $B_{r,\chi} = \sum_{a=1}^{6} \chi(a)B_r(a/7)$ and the Bernoulli numbers and polynomials are defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{r=0}^{\infty} B_r(x)t^r/r!,\ B_r = B_r(0).$$

We find that

$$\zeta_F(1-2) = \frac{-1}{21},\ \zeta_F(1-4) = \frac{79}{210},\ \zeta_F(1-6) = \frac{-7393}{63}.$$

(Note that the rationality of these values also follows from a general theorem of Siegel [Si1], made explicit in [Si2].) Hence

$$E_{2,F} = 1 - 168 \sum_{0 \ll \alpha \in O_F} \sigma_1(\alpha)\mathbf{e}(\mathrm{tr}((\alpha/\delta)z)),$$

$$E_{4,F} = 1 + \frac{1680}{79} \sum_{0 \ll \alpha \in O_F} \sigma_3(\alpha)\mathbf{e}(\mathrm{tr}((\alpha/\delta)z))$$

and

$$E_{6,F} = 1 - \frac{504}{7393} \sum_{0 \ll \alpha \in O_F} \sigma_5(\alpha)\mathbf{e}(\mathrm{tr}((\alpha/\delta)z)).$$

All the totally positive elements of O_F with trace bounded by a given constant may be found by manipulating some inequalities and doing a computer search. I calculated a pair of normalised cusp forms of weight six, $A = \frac{7393}{3725568}(E_{6,F} - E_{2,F}^3)$

and $B = \frac{584047}{85659840}(E_{6,F} - E_{2,F}E_{4,F})$. I only calculated very few coefficients, our aim being to obtain the eigenvalues of the Hecke operator $T(2)$.

$$A = \mathbf{e}(\mathrm{tr}((1/\delta)z)) - \frac{1798}{11}\mathbf{e}(\mathrm{tr}((2/\delta)z)) + \frac{855552}{11}\mathbf{e}(\mathrm{tr}((4/\delta)z)) + \dots$$
$$B = \mathbf{e}(\mathrm{tr}((1/\delta)z)) - \frac{33352}{607}\mathbf{e}(\mathrm{tr}((2/\delta)z)) + \frac{49318592}{607}\mathbf{e}(\mathrm{tr}((4/\delta)z)) + \dots$$

(The terms for $\alpha = 3$ have been omitted.) Using (1) we calculate

$$T(2)A = \tfrac{-1798}{11}\mathbf{e}(\mathrm{tr}((1/\delta)z)) + \tfrac{1216000}{11}\mathbf{e}(\mathrm{tr}((2/\delta)z)) + \dots$$

$$T(2)B = \tfrac{-33352}{607}\mathbf{e}(\mathrm{tr}((1/\delta)z)) + \tfrac{69208768}{607}\mathbf{e}(\mathrm{tr}((2/\delta)z)) + \dots$$

$$T(2)^2A = \tfrac{1216000}{11}\mathbf{e}(\mathrm{tr}((1/\delta)z)) + \dots$$

$$T(2)^2B = \tfrac{69208768}{607}\mathbf{e}(\mathrm{tr}((1/\delta)z)) + \dots.$$

Assuming for the moment that the space of cusp forms of weight 6 is two-dimensional, we can solve the linear equations resulting from $T(2)^2 + aT(2) + b = 0$ (apply to A and B and look at the first coefficient), to find $a = -32$ and $b = -115776$. It follows that the eigenvalues of $T(2)$ are

$$16 \pm 56\sqrt{37}.$$

It is remarkable that $16 + 56\sqrt{37} \approx 356.63$, which is only slightly smaller than the Ramanujan-Petersson bound $2N((2))^{(k-1)/2} \approx 362.04$. In fact, $\cos^{-1}\left(\frac{16+56\sqrt{37}}{2\mathrm{N}((2))^{(k-1)/2}}\right) \approx 0.173$. Assuming that, as $\mathfrak{p}$ varies, the corresponding argument "follows" a Sato-Tate distribution with density function $\frac{2}{\pi}\sin^2(\theta)$ (see, for instance, Section 3 of [Sha]), the "probability" of being this far from $\pi/2$ is less than 0.004.

We have a basis of Galois conjugate Hecke eigenforms, $f = f_1$ and f_2, with

$$f(z) = \sum_{0 \ll \alpha \in O_F} a(\alpha)\mathbf{e}(\mathrm{tr}((\alpha/\delta)z))$$

$$= \mathbf{e}(\mathrm{tr}((1/\delta)z)) + (16+56\sqrt{37})\mathbf{e}(\mathrm{tr}((2/\delta)z)) + (83520+1792\sqrt{37})\mathbf{e}(\mathrm{tr}((4/\delta)z)) + \dots.$$

One checks that the other condition of Lemma 3.1 is satisfied (with $g = E_{2,F}^3$), so there is some prime ideal λ dividing 7393 such that $a(\mathfrak{a}) \equiv \sigma_5(\mathfrak{a}) (\mathrm{mod} \lambda)$ for all $\mathfrak{a}$.

It remains to show that $\dim(S_6) = 2$. For this we use the generalisation, due independently to Mizumoto and Takase (using an idea of Zagier), of the Eichler-Selberg trace formula for Hecke operators. We apply Theorem 3 of [Mi] in the case $\nu = 1$ to get the trace of $T(1)$, which is the same as the dimension of S_6. We need to find all the elements $\eta \in O_F$ such that $\eta^2 \ll 4$ (see the formula below). Since the narrow class number is one, every totally positive unit is a square. Consequently, any totally positive element whose ideal is a square is a square. By searching through elements of square norm with appropriately bounded trace, we find that the five values of η^2 such that $\eta^2 \ll 4$ are $0, 1, (3, -1, -1), (2, 1, 0), (0, 0, 1)$. The values of $\mathrm{N}(4-\eta^2)$ are, respectively, $64, 27, 7, 7, 7$. The quadratic fields $F(\sqrt{\eta^2 - 4})$ are all abelian extensions of $\mathbb{Q}$, all of class number one, and the sum in Mizumoto's formula reduces to

$$(-1)^g \sum_{\eta^2 \ll 4} N_{F/\mathbb{Q}}(p_{k,1}(\eta, 1)) \frac{h(F(\sqrt{\eta^2 - 4}))}{w(F(\sqrt{\eta^2 - 4}))} - 2\zeta_F(-1)(k-1)^g/2^g$$

$$= (-1)((1/4) - 2((1/6) + (1/14) + (1/14) + (1/14))) + (125/84) = 2.$$

Here, $p_{k,r}(\ell, \nu)$ is the coefficient of x^{k-r-1} in $(1 - \ell x + \nu x^2)^{-r}$ (it is always ± 1 in the above), and h and w are the class number and the number of roots of unity.

6. The results of the calculation

Recall that F is the totally real cubic field $\mathbb{Q}(\zeta_7)^+$ of discriminant $d_F = 49$. Let $k = 6$ and let $f = f_1$ and f_2 be the Galois conjugate Hecke eigenforms considered above. Let λ be the divisor of $\ell = 7393$ such that $a(\mathfrak{a}) \equiv \sigma_5(\mathfrak{a}) (\mathrm{mod} \lambda)$ for all $\mathfrak{a}$, where $f(z) = \sum_{0 \ll \alpha \in O_F} a(\alpha) \mathbf{e}(\mathrm{tr}((\alpha/\delta)z))$. In the proposition below, $E = \mathbb{Q}(\sqrt{37}), g = 3$ and $k = 6$.

Proposition 6.1. *$D_f((k/2) + k - 1)/(\pi^{(k-2)g} D_f(k))$ is a non-zero element of E, and is divisible by λ.*

The computations that prove this proposition are summarised in the remainder of this section.

Formula 5.8 of [Mi] (proved by the Rankin-Selberg method) shows that

$$D_f(k)/(\pi^{(k+1)g}(f, f)) = \frac{2^{26}}{3^3 5^3 7^{14}},$$

so we need to show that

$$D_f((k/2) + k - 1)/(\pi^{(2k-1)g}(f, f))$$

is a non-zero element of E, divisible by λ.

For $i = 1, 2$ let $x_i = D_{f_i}((k/2) + k - 1)/(\pi^{(2k-1)g}(f_i, f_i))$. Then it follows from 5.9 and 5.2 of [Mi] that

$$x_1 + x_2 = A_1 B \tag{5}$$

$$(16 + 56\sqrt{37})x_1 + (16 - 56\sqrt{37})x_2 = A_2 B \tag{6}$$

where

$$B = \left[\frac{-2!4!2^{15}}{7!}\right]^3 \Big/ 2d_F^{11} = \frac{-2^{44}}{3^3 5^3 7^{25}},$$

$$A_1 = \sum_{\eta^2 \ll 4} \mathrm{N}(p_{6,3}(\eta, 1)) L_F(1-3, \eta^2 - 4) - \frac{\zeta_F(6) 7^{11} (7!)^3}{2^{17}\pi^{18}},$$

$$A_2 = \sum_{\eta^2 \ll 8} \mathrm{N}(p_{6,3}(\eta, 2)) L_F(1-3, \eta^2 - 8).$$

The abelian L-function $L_F(s, D)$ is such that

$$\zeta_{F(\sqrt{D})}(s) = \zeta_F(s) L_F(s, D),$$

and $p_{k,r}(\eta, \nu)$ is the coefficient of x^{k-r-1} in $(1 - \eta x + \nu x^2)^{-r}$.

The relevant L-values are in the tables below. We know already that they are rational [Si3], from which it follows that the values we calculate for the x_i must belong to E.

The following table lists the relevant data for $\eta^2 \ll 4$.

η^2	$\mathrm{N}(4-\eta^2)$	$\mathrm{N}(p_{6,3}(\eta,1))$	$L_F(1-3,\eta^2-4)$
0	64	-27	-2306
1	27	27	$-2408/9$
$(3,-1,-1)$	7	-189	$-64/7$
$(2,1,0)$	7	-189	$-64/7$
$(0,0,1)$	7	-189	$-64/7$

(Recall that (a, b, c) stands for $a + b\beta + c\beta^2$, where $\beta = 2\cos(2\pi/7)$.) Using this, one finds that $A_1 = -2115360$. The following table lists the relevant data for $\eta^2 \ll 8$.

η^2	$\mathrm{N}(8-\eta^2)$	$\mathrm{N}(p_{6,3}(\eta,2))/6^3$	$L_F(1-3,\eta^2-8)$
0	512	-1	-416268
1	343	0	$-64/7$
$(3,-1,-1)$	239	-1	-62208
$(2,1,0)$	239	-1	-62208
$(0,0,1)$	239	-1	-62208
$(5,-1,-2)$	167	1	-25536
$(0,-1,1)$	167	1	-25536
$(1,2,1)$	167	1	-25536
$(8,-1,-3)$	7	-7	$-64/7$
$(1,-2,1)$	7	-7	$-64/7$
$(1,3,2)$	7	-7	$-64/7$
4	64	27	-2306
$(7,-1,-1)$	7	91	$-64/7$
$(6,1,0)$	7	91	$-64/7$
$(4,0,1)$	7	91	$-64/7$

Using this, one finds that $A_2 = 109548288$. In most cases the abelian L-function is a product of Dirichlet L-functions over $\mathbb{Q}$, so its values may be calculated using Bernoulli polynomials. However, when $\mathrm{N}(8-\eta^2)$ equals 239 or 167 this is not the case. Shintani [Shi] proved a general formula for the values at negative integers of abelian L-functions over totally real fields. This seems difficult to use for computation, but since 239 and 167 are prime we are lucky enough to be able to use an alternative formula due to Hida [H]. It involves sums over totally positive elements of bounded trace, and various multiplicative functions. The formula in question is 2.21 of [H]. Some of the terms are defined in 2.6, 2.8, Remark 2.1 and the corollary to Prop. 1.1 of [H]. The formula and its proof are too complicated to describe here, but the method is somehow related to that by which Siegel explicitly determined the rational values $\zeta_F(1-r)$, using their occurrence in the constant coefficients of Eisenstein series [Si2].

In both cases Hida's formula shows that we will get a rational number with denominator bounded by some number less than 4000, so a sufficiently good approximation using a partial Euler product will enable us to deduce the exact answer. To get the values in the above table I did this using primes $p \leq 7919$. It was not too difficult to compute the necessary character values. Note that it would be infeasible to perform directly, a partial Euler product approximation to the values of the symmetric square L-function, since one would need to know a large number of coefficients for f.

Solving the linear equations (5) and (6), we find that

$$x_1/B, x_2/B = -1057680 \pm \frac{1280304}{37}\sqrt{37}.$$

The norm of these quadratic irrationals is $\frac{39752240016384}{37}$, which factorises as

$$\frac{2^{10}\,3^7\,7^4\,7393}{37}.$$

To show that x_1/B (the one with the '+') is divisible by the same divisor of 7393 modulo which we have the Ramanujan-style congruence for f_1, we just check that the norm of $(-1057680 + \frac{1280304}{37}\sqrt{37}) + (16 + 56\sqrt{37}) - (1 + 8^5)$ is divisible by 7393. But this norm is $\frac{423501445117 17}{37} = \frac{3^2.7.90927163.7393}{37}$.

7. Galois representations and motives

Our goal is to use the Bloch-Kato conjecture on special values of L-functions to explain the divisibility revealed by the computation above. This is a conjecture about the L-functions attached to motives. According to Conjecture 4.5 of [Cl], associated to f there should be a motive M_f over F, with coefficients in some finite extension E' of E. Without going into detail about precisely what that means, we shall assume that M_f exists, and, for convenience, that it has coefficients in E. We could avoid this latter assumption by considering the Bloch-Kato conjecture with coefficients in some E' bigger than E. Blasius and Rogawski [BR] prove the existence of a motive associated to f, defined over an imaginary quadratic extension of F rather than over F itself.

Actually we really need only the existence of some of the realisations of M_f, and the relations between them (for more on realisations of motives, see [Fo]):

(1) a de Rham realisation M_{dR}. This is a free $E \otimes_{\mathbb{Q}} F$-module of rank 2 with a filtration $\{\mathrm{Fil}^i\}_{i\in\mathbb{Z}}$ such that

$$\mathrm{rank}_{E\otimes F}(\mathrm{Fil}^i) = \begin{cases} 2 & i \leq 0 \\ 1 & 1 \leq i \leq k-1 \\ 0 & i \geq k. \end{cases}$$

(2) For each embedding $\iota : F \to \mathbb{R}$, a Betti realisation $M_{B,\iota}$. This is a 2-dimensional E-vector space with a natural action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. For each embedding $\phi : E \to \mathbb{C}$ there should be a Hodge structure on $M_{B,\iota} \otimes_{E,\phi} \mathbb{C}$, pure of weight $k-1$.

(3) Comparison isomorphisms

$$M_{B,\iota} \otimes_{E,\phi} \mathbb{C} \to M_{\mathrm{dR}} \otimes_{E\otimes F,\phi\otimes\iota} \mathbb{C},$$

respecting the Hodge filtrations.

(4) For each (finite) prime λ of E (say $\lambda \mid \ell$), a 2-dimensional E_λ-vector space V_λ with a representation

$$\rho_\lambda : \mathrm{Gal}(\overline{F}/F) \to \mathrm{Aut}(E_\lambda)$$

that is unramified outside primes of F dividing ℓ. If $\mathfrak{p} \nmid \ell$ is a prime of F and $\mathrm{Frob}_{\mathfrak{p}}$ is an arithmetic Frobenius element at $\mathfrak{p}$ then

$$\det(1 - \mathrm{Frob}_{\mathfrak{p}}^{-1} T | V_\lambda) = 1 - a_{\mathfrak{p}} T + \mathrm{N}(\mathfrak{p})^{k-1} T^2.$$

If $\mathfrak{p} \mid \ell$ is a prime of F then V_λ is a crystalline representation of $\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$. Assume henceforth for simplicity that $F_{\mathfrak{p}} : \mathbb{Q}_\ell$ is unramified. Then

$$\det_{E_\lambda \otimes F_{\mathfrak{p}}}(1 - \phi T | D_{\mathrm{cris},\mathfrak{p}}(V_\lambda)) = 1 - a_{\mathfrak{p}} T + \mathrm{N}(\mathfrak{p})^{k-1} T^2,$$

where $D_{\mathrm{cris},\mathfrak{p}}(V_\lambda) = (V_\lambda \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}})^{\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})}$ and ϕ is the crystalline Frobenius. (See Section 1 of [BK] for definitions of Fontaine's rings B_{cris} and B_{dR}.)

(5) Comparison isomorphisms $M_{B,\iota} \otimes_E E_\lambda \simeq V_\lambda$, compatible with the actions of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

(6) Comparison isomorphisms

$$D_{\mathrm{cris},\mathfrak{p}}(V_\lambda) \simeq M_{\mathrm{dR}} \otimes_{E\otimes F} E_\lambda \otimes F_{\mathfrak{p}},$$

and hence also

$$V_{\text{cris}}(M_{\text{dR},\mathfrak{p},\lambda}) \simeq V_\lambda,$$

where $M_{\text{dR},\mathfrak{p},\lambda} := M_{\text{dR}} \otimes_{(E\otimes F)} (E_\lambda \otimes F_\mathfrak{p})$ and $V_{\text{cris}}(M_{\text{dR},\mathfrak{p},\lambda}) := (\text{Fil}^0(B_{\text{cris}} \otimes_{F_\mathfrak{p}} M_{\text{dR},\mathfrak{p},\lambda}))^{\phi=1}$.

In our case, in which $[F:\mathbb{Q}]$ is odd, the Galois representations V_λ, with the required property at $\mathfrak{p} \nmid \ell$, may be constructed using local systems on Shimura curves [Oh] (see also [Ca], [W]). That they are crystalline at $\mathfrak{p} \mid \ell$ is a consequence of Faltings's comparison theorem [Fa] (see also [Tay]). We shall only need V_λ for the $\lambda \mid \ell = 7393$ modulo which the Ramanujan-style congruence holds, since we shall only concern ourselves with the λ-part of the Bloch-Kato conjecture. One can construct Betti and de Rham realisations with coefficients in some extension E' of E, as in §2 of [Ca].

Actually, the motive for which we consider the λ-part of the Bloch-Kato conjecture is the symmetric square motive $\text{Sym}^2 M_f$ of rank 3, which we also denote M'_f. Each realisation is the symmetric square of the corresponding realisation of M_f. The Hodge structure on $M'_{B,\iota} \otimes_{E,\phi} \mathbb{C}$ is pure of weight $2k-2$. Let $(M'_{B,\iota})^{\pm}$ be the eigenspaces for the action of $\text{Gal}(\mathbb{C}/\mathbb{R})$. The filtration on M'_{dR} is such that

$$\text{rank}_{E\otimes F}(\text{Fil}^i) = \begin{cases} 3 & i \leq 0 \\ 2 & 1 \leq i \leq k-1 \\ 1 & k \leq i \leq 2k-2 \\ 0 & i \geq 2k-1. \end{cases}$$

Recall that the Galois representation V'_λ is actually known to exist. It is unramified at $\mathfrak{p} \nmid \ell$ and crystalline at $\mathfrak{p} \mid \ell$. It should give rise to the Euler factors of the symmetric square L-function $D_f(s)$, and, as in (4), this is known for $\mathfrak{p} \nmid \ell$.

8. The Bloch-Kato conjecture

There exists now a beautiful generalisation and reformulation of the original conjecture of Bloch and Kato [BK], applying uniformly to the leading term at $s = 0$ of the L-function attached to any motive, hence, via twisting, to the leading term at any integer of the L-function of a motive [Fo]. The statement we give below in our special case is more like the original formulation, and our handling of twists will be rather inelegant.

In §6 it was unnatural to pick out $f = f_1$ from the conjugate pair $\{f_1, f_2\}$, in fact the method of calculation produced results for f_1 and f_2 simultaneously. From now on we consider f to have Fourier coefficients in an abstract number field E. The forms f_1 and f_2 are obtained via the two embeddings σ_1 and σ_2 of E into $\mathbb{C}$. The L-functions $L_f(s)$ and $D_f(s)$ may be viewed as taking values in $E \otimes_{\mathbb{Q}} \mathbb{C}$. The congruence $a_{\mathfrak{p}} \equiv 1 + \mathrm{N}(\mathfrak{p})^{k-1} \pmod{\lambda}$ holds for an ideal $\lambda \mid 7393$ in the abstract field E. Congruences for the coefficients of f_1 and f_2 are obtained from this by applying the embeddings.

Integer points $s = j = k - 1 + r$, for odd r such that $1 \leq r \leq k - 1$ are *critical* for M'_f, in the sense that, for any given embeddings $\iota : F \to \mathbb{R}$ and $\phi : E \to \mathbb{C}$, the comparison isomorphism

$$M'_{B,\iota} \otimes_{E,\phi} \mathbb{C} \to M'_{\mathrm{dR}} \otimes_{E\otimes F, \phi\otimes\iota} \mathbb{C},$$

induces an isomorphism

$$\theta_{\iota,\phi}(j) : (M'_{B,\iota})^{(-1)^j} \otimes_{E,\phi} \mathbb{C} \to (M'_{\mathrm{dR}}/\mathrm{Fil}^j) \otimes_{E\otimes F,\phi\otimes\iota} \mathbb{C}.$$

This can be confirmed by checking the dimension of the left hand side. The points $s = k - r$, paired with the above by the functional equation, are also said to be critical, but we shall not consider them further.

With respect to some fixed choices of bases for $M'_{B,\iota}$ and M'_{dR}, we get a determinant for the map $\theta_{\iota,\phi}(j)$. This is an element of $\mathbb{C}$, but by varying over ϕ we get an element of $E \otimes_{\mathbb{Q}} \mathbb{C}$. Taking the product of these determinants for all $\iota : F \to \mathbb{R}$ gives us an element of $E \otimes \mathbb{C}$ which will be denoted vol_∞. (Since the j we consider all have the same parity, it does not depend on j.)

According to Deligne's conjecture [De], the ratio $D_f(j)/(2\pi i)^{2jg}\mathrm{vol}_\infty$ in $E \otimes \mathbb{C}$ is actually an element of $E^\times$. The Bloch-Kato conjecture predicts the prime factorisation of this element. Here we care only about the λ-part. We can state the conjecture as follows, where we view each side as a fractional ideal of E:

$$D_f(j)/(2\pi i)^{2jg}\mathrm{vol}_\infty = \frac{\prod_{\mathfrak{p}} c_{\mathfrak{p}}(j)\#\text{Ш}(j)}{\#\Gamma_F(j)\#\Gamma_F(2k-1-j)}.$$

We shall define the λ-part of each of the terms on the right hand side. Like vol_∞, they depend on choices of bases for $M'_{B,\iota}$ and (for the $c_\mathfrak{p}(j)$) M'_{dR}, though the ratio of the two sides is independent of these choices. Once we have made our choices, let $T'_{\lambda,\iota}$ be the free $O_{E,\lambda}$-submodule of $M'_{B,\iota}\otimes E_\lambda \simeq V'_\lambda$ generated by the chosen basis, and let $L'_{\mathrm{dR},\mathfrak{p},\lambda}$ be the free $O_{E,\lambda}\otimes_{\mathbb{Z}_\ell} O_{F,\mathfrak{p}}$-submodule of $M'_{\mathrm{dR},\mathfrak{p},\lambda} := M'_{\mathrm{dR}}\otimes_{(E\otimes F)}(E_\lambda\otimes F_\mathfrak{p})$ generated by the chosen basis. We choose the bases for the $M'_{B,\iota}$ in such a way that all the $T'_{\lambda,\iota}$ are equal to some fixed, $\mathrm{Gal}(\overline{F}/F)$-invariant O_λ-lattice T'_λ in V'_λ. A particular choice of T'_λ will be specified later.

First we define the λ-part of the global torsion $\Gamma_F(j)$. Define $A'_\lambda := V'_\lambda/T'_\lambda$. Adjusting the action of $\mathrm{Gal}(\overline{F}/F)$ by powers of the ℓ-adic cyclotomic character in the usual way, one can define Tate twists $T'_\lambda(j)$, $V'_\lambda(j)$ and $A'_\lambda(j)$.
Then $\mathrm{ord}_\lambda(\#\Gamma_F(j))$ is defined to be the length of the finite O_λ-module $(A'_\lambda(j))^{\mathrm{Gal}(\overline{F}/F)}$.

Next we define the λ-part of the Shafarevich-Tate group $Ш(j)$. For $\mathfrak{p}\nmid\ell$ let

$$H^1_f(F_\mathfrak{p},V'_\lambda(j)) = \ker(H^1(D_\mathfrak{p},V'_\lambda(j))\to H^1(I_\mathfrak{p},V'_\lambda(j))).$$

(The subscript f does not refer to the cusp form f.) Here, $D_\mathfrak{p}$ is a decomposition subgroup at a prime above $\mathfrak{p}$, $I_\mathfrak{p}$ is the inertia subgroup, and the cohomology is for continuous cocycles and coboundaries. For $\mathfrak{p}\mid\ell$ let

$$H^1_f(F_\mathfrak{p},V'_\lambda(j)) = \ker(H^1(D_\mathfrak{p},V'_\lambda(j))\to H^1(D_\mathfrak{p},V'_\lambda(j)\otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}})).$$

Let $H^1_f(F,V'_\lambda(j))$ be the subspace of elements of $H^1(F,V'_\lambda(j))$ whose local restrictions lie in $H^1_f(F_\mathfrak{p},V'_\lambda(j))$ for all primes $\mathfrak{p}$. (We assume that ℓ is odd, to avoid having to deal with places at infinity.)

There is a natural exact sequence

$$0 \longrightarrow T'_\lambda(j) \longrightarrow V'_\lambda(j) \xrightarrow{\ \pi\ } A'_\lambda(j) \longrightarrow 0.$$

Let $H^1_f(F_\mathfrak{p},A'_\lambda(j)) = \pi_* H^1_f(F_\mathfrak{p},V'_\lambda(j))$. Define the λ-Selmer group $H^1_f(F_\mathfrak{p},A'_\lambda(j))$ to be the subgroup of elements of $H^1(F,A'_\lambda(j))$ whose local restrictions lie in $H^1_f(F_\mathfrak{p},A'_\lambda(j))$ for all primes $\mathfrak{p}$. Define the λ-part of the Shafarevich-Tate group

$$Ш_\lambda(j) = H^1_f(F,A'_\lambda(j))/\pi_* H^1_f(F,V'_\lambda(j)). \tag{7}$$

Define $\mathrm{ord}_\lambda(\#Ш)$ to be the length of this finite O_λ-module.

Next we define the λ-parts of the Tamagawa (or fudge) factors $c_{\mathfrak{p}}(j)$. Denote by $(P_{\mathfrak{p}}(\mathrm{N}(\mathfrak{p})^{-s}))^{-1}$ the Euler factor at $\mathfrak{p}$ in $D_f(s)$. For $\mathfrak{p} \nmid \ell$, $\mathrm{ord}_\lambda(c_{\mathfrak{p}}(j))$ is defined to be

$$\begin{aligned} &\text{length } H^1_f(F_{\mathfrak{p}}, T'_\lambda(j))_{\mathrm{tors}} - \mathrm{ord}_\lambda(P_{\mathfrak{p}}(\mathrm{N}(\mathfrak{p})^{-j})) \\ &\quad = \text{length } \left(H^0(F_{\mathfrak{p}}, A'_\lambda(j))/H^0\left(F_{\mathfrak{p}}, V'_\lambda(j)^{I_p}/T'_\lambda(j)^{I_p}\right)\right). \end{aligned}$$

Finally, if $\mathfrak{p} \mid \ell$ then it follows from Theorem 4.1(ii) of [BK] that, for the j in which we are interested, the Bloch-Kato exponential map

$$\exp : M'_{\mathrm{dR},\mathfrak{p},\lambda}/\mathrm{Fil}^j \to H^1_f(F_{\mathfrak{p}}, V'_\lambda)$$

is an isomorphism of E_λ-vector spaces. The image $\exp(L'_{\mathrm{dR},\mathfrak{p},\lambda}/(\mathrm{Fil}^j \cap L'_{\mathrm{dR},\mathfrak{p},\lambda}))$ may not be contained in $i_*(H^1_f(F_{\mathfrak{p}}, T'_\lambda))$, where $i : T'_\lambda \to V'_\lambda$ is the natural inclusion. However, it still makes sense to talk of a fractional ideal $I = [i_*(H^1_f(F_{\mathfrak{p}}, T'_\lambda)) : \exp(L'_{\mathrm{dR},\mathfrak{p},\lambda}/(\mathrm{Fil}^j \cap L'_{\mathrm{dR},\mathfrak{p},\lambda}))]$, then to define

$$\mathrm{ord}_\lambda(c_{\mathfrak{p}}(j)) := \text{length}\, H^1_f(F_{\mathfrak{p}}, T'_\lambda(j))_{\mathrm{tors}} + \mathrm{ord}_\lambda(I) - \mathrm{ord}_\lambda(P_{\mathfrak{p}}(\mathrm{N}(\mathfrak{p})^{-j})).$$

9. Global torsion and Tamagawa factors

If T_λ is a $\mathrm{Gal}(\overline{F}/F)$-invariant O_λ-lattice in V_λ and $A_\lambda := V_\lambda/T_\lambda$, define $A[\lambda]$ to be the λ-torsion in A_λ. This is a 2-dimensional representation of $\mathrm{Gal}(\overline{F}/F)$ over the residue field $\mathbb{F}_\lambda$. Recall the Ramanujan-style congruence

$$a_{\mathfrak{p}} \equiv 1 + \mathrm{N}(\mathfrak{p})^{k-1} \pmod{\lambda}.$$

It follows from (4) of §7 that, for $\mathfrak{p} \nmid \ell$, the trace of $\mathrm{Frob}_{\mathfrak{p}}^{-1}$ acting on $A[\lambda]$ is $1+\mathrm{N}(\mathfrak{p})^{k-1}$. Then it follows from the Chebotarev density theorem and the Brauer-Nesbitt theorem that the composition factors of $A[\lambda]$ are $\mathbb{F}_\lambda$ and $\mathbb{F}_\lambda(1-k)$. We choose T_λ in such a way that $\mathbb{F}_\lambda(1-k)$ is a submodule of $A[\lambda]$. Then we choose T'_λ to be $\mathrm{Sym}^2 T_\lambda$.

Lemma 9.1. *In the case at hand,* $\mathrm{ord}_\lambda(\#\Gamma_F(k)) = \mathrm{ord}_\lambda(\#\Gamma_F(k-1+(k/2))) = \mathrm{ord}_\lambda(\#\Gamma_F(k/2)) = 0$.

Proof. The composition factors of $A'[\lambda]$ are $\mathbb{F}_\lambda$, $\mathbb{F}_\lambda(1-k)$ and $\mathbb{F}_\lambda(2-2k)$. Those of $A'[\lambda](k)$ are therefore $\mathbb{F}_\lambda(k)$, $\mathbb{F}_\lambda(1)$ and $\mathbb{F}_\lambda(2-k)$. Similarly, those of $A'[\lambda](k-1+(k/2))$ are $\mathbb{F}_\lambda(k-1+(k/2))$, $\mathbb{F}_\lambda(k/2)$ and $\mathbb{F}_\lambda(1-(k/2))$, and those of $A'[\lambda](k/2)$

are $\mathbb{F}_\lambda(k/2)$, $\mathbb{F}_\lambda(1-(k/2))$ and $\mathbb{F}_\lambda(2-(3k/2))$. Since $k>2$ and $\ell>3k/2$, none of these is trivial. □

Lemma 9.2. *For* $\mathfrak{p}\nmid\ell, \mathrm{ord}_\lambda(c_\mathfrak{p}(j))=0$ *for all* j.

This is a trivial consequence of the fact that $V_\lambda(j)$ is unramified at such $\mathfrak{p}$.

We have already made a special choice of $\mathrm{Gal}(\overline{F}/F)$-invariant O_λ-lattice T'_λ, and chosen bases for $M'_{B,\iota}$ whose images under the comparison isomorphisms generate T'_λ. For each $\mathfrak{p}\mid\ell$, using §1.1.2 of [DFG], we see that there exists a "strongly divisible" filtered $O_{E,\lambda}\otimes O_{F,\mathfrak{p}}$-submodule $L'(\mathfrak{p})$ of $M'_{\mathrm{dR},\mathfrak{p},\lambda}$ such that $\mathbb{V}(L'(\mathfrak{p}))=T'_\lambda$, where $\mathbb{V}$, described in §1.1.2 of [DFG], is an integral version of the functor V_{cris}. (Bloch and Kato call this functor "T".) Recall that V_{cris} was described in (6) of §7, and takes us from a filtered $(E_\lambda\otimes F_\mathfrak{p})$-module to a λ-adic representation of $\mathrm{Gal}(\overline{F}_\mathfrak{p}/F_\mathfrak{p})$. Now choose a basis of M'_{dR} in such a way that, for all $\mathfrak{p}\mid\ell$, $L'_{\mathrm{dR},\mathfrak{p},\lambda}=L'(\mathfrak{p})$.

Lemma 9.3. *With the above choices of bases,* $\mathrm{ord}_\lambda(c_\mathfrak{p})=0$ *for all* $\mathfrak{p}\mid\ell$*, assuming, as in (4) of §6, that*

$$\det_{E_\lambda\otimes F_\mathfrak{p}}(1-\phi T|D_{\mathrm{cris},\mathfrak{p}}(V_\lambda))=1-a_\mathfrak{p}T+\mathrm{N}(\mathfrak{p})^{k-1}T^2.$$

This follows directly from Theorem 4.1(iii) of [BK]. Note that $F_\mathfrak{p}:\mathbb{Q}_\ell$ is unramified, in the case at hand.

Combining the above lemmas with the results of the computation, and recalling that vol_∞ is independent of j, leads to the following.

Proposition 9.4. *The Bloch-Kato conjecture predicts that*

$$\mathrm{ord}_\lambda\left(\frac{\#\Gamma_F(k-1)\#\text{Ш}(k-1+(k/2))}{\#\text{Ш}(k)}\right)=1.$$

Corollary 9.5. *If* $\mathrm{ord}_\lambda(\#\Gamma_F(k-1))=0$ *then the Bloch-Kato conjecture predicts that*

$$\mathrm{ord}_\lambda(\#\text{Ш}(k-1+(k/2)))>0.$$

In the next section we give a conditional construction of non-trivial λ-torsion in $\text{Ш}(k-1+(k/2))$.

We need to worry about $\mathrm{ord}_\lambda(\#\Gamma_F(k-1))$, but the proof of Lemma 9.1 does not apply, because the composition factors of $A'[\lambda](k-1)$ are $\mathbb{F}_\lambda(k-1)$, $\mathbb{F}_\lambda$ and $\mathbb{F}_\lambda(1-k)$.

Since $A[\lambda]$ has a trivial 1-dimensional quotient, with kernel $\mathbb{F}_\lambda(1-k)$, $A'[\lambda]$ has a quotient isomorphic to $A[\lambda]$, with non-trivial kernel. Hence if $\mathrm{ord}_\lambda(\#\Gamma_F(k-1)) \neq 0$ then $A[\lambda]$ must have a trivial 1-dimensional submodule, which forces $A[\lambda] \simeq \mathbb{F}_\lambda \oplus \mathbb{F}_\lambda(1-k)$. It seems difficult to exclude this possibility, so we shall simply guess that it is not the case, so that $\mathrm{ord}_\lambda(\#\Gamma_F(k-1)) = 0$. Though the truth of this guess would allow us to apply Corollary 9.5, it is not necessary for the construction in the next section.

When we consider the Bloch-Kato conjecture for the *ratio* $D_f(k-1+(k/2))/D_f(k)$, the cancellation of vol_∞ occurs no matter what we choose as the basis for M'_{dR}. This is why we had the freedom to arrange for $\mathbb{V}(L'_{\mathrm{dR},\mathfrak{p},\lambda})$ to be T'_λ, which then gave us (assuming (4) of §7) $\mathrm{ord}_\lambda(c_\mathfrak{p}) = 0$ (for all $\mathfrak{p} \mid \ell$).

10. From a Selmer group to the Shafarevich-Tate group

We retain the notation and choices of earlier sections.

Theorem 10.1. *Suppose that* $H^1_f(F, V_\lambda(k/2)) \neq \{0\}$. *Then* $H^1_f(F, A'_\lambda(k-1+(k/2))) \neq \{0\}$.

Proof. Since $H^1_f(F, V_\lambda(k/2)) \neq \{0\}$, we may choose some element $d \in H^1_f(F, T_\lambda(k/2))$ such that $d \notin \lambda H^1_f(F, T_\lambda(k/2))$. This element d reduces to some non-zero class $c \in H^1(F, A[\lambda](k/2))$. Considering the exact sequence

$$0 \to A[\lambda](k/2) \to A'[\lambda](k-1+(k/2)) \to \mathbb{F}_\lambda(k-1+(k/2)) \to 0,$$

and the fact that $H^0(F, \mathbb{F}_\lambda(k-1+(k/2)))$ is trivial (because $\ell > 3k/2$), we see that c maps on to a non-zero element $c' \in H^1(F, A'[\lambda](k-1+(k/2)))$.

None of the composition factors of $A'[\lambda](k-1+(k/2))$ is trivial, since $\ell > 3k/2$ and $k > 2$, so $H^0(F, A'_\lambda(k-1+(k/2)))$ is trivial. Considering the cohomology of the exact sequence

$$\begin{array}{ccccc} 0 & \longrightarrow & A'[\lambda](k-1+(k/2)) & \longrightarrow & A'_\lambda(k-1+(k/2)) \\ & \xrightarrow{\lambda} & A'_\lambda(k-1+(k/2)) & \longrightarrow & 0 \end{array},$$

we see that c' gives a non-zero element $d' \in H^1(F, A'_\lambda(k-1+(k/2)))$.

We need to show that $\mathrm{res}_\mathfrak{p}(d') \in H^1_f(F_\mathfrak{p}, A'_\lambda(k-1+(k/2)))$, for all $\mathfrak{p}$. Suppose first that $\mathfrak{p} \nmid \ell$. Since $d \in H^1_f(F, V_\lambda(k/2))$ we know that $\mathrm{res}_\mathfrak{p}(c) \in H^1(F_\mathfrak{p}^{ur}/F_\mathfrak{p}, A[\lambda](k/2))$, hence that $\mathrm{res}_\mathfrak{p}(c') \in H^1(F_\mathfrak{p}^{ur}/F_\mathfrak{p}, A'[\lambda](k-1+(k/2)))$ and then $\mathrm{res}_\mathfrak{p}(d') \in$

$H^1(F_{\mathfrak{p}}^{ur}/F_{\mathfrak{p}}, A'_{\lambda}(k-1+(k/2)))$. By [Fl], line 3 of p. 125, $H^1_f(F_{\mathfrak{p}}, A'_{\lambda}(k-1+(k/2))$ is equal to, not just contained in, $H^1(F_{\mathfrak{p}}^{ur}/F_{\mathfrak{p}}, A'_{\lambda}(k-1+(k/2)))$. Hence $\mathrm{res}_{\mathfrak{p}}(d') \in H^1_f(F_{\mathfrak{p}}, A'_{\lambda}(k-1+(k/2))$ for all $\mathfrak{p} \nmid \ell$. The case $\mathfrak{p} \mid \ell$ is more technical but may be dealt with just as in the proof of Proposition 9.2 of [Du2]. This uses again the connection between filtered modules and λ-adic representations of $\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$, already mentioned in §§7 and 9, but including the case where the modules and representations are torsion. One uses the theory of §4 of [BK], which involves Fontaine-Lafaille modules [FL]. □

There is a conjectural formula for the order of vanishing at $s = 0$ (hence, by twisting, at any integer point) of the L-function attached to any motive M, let's say over F with coefficients in E. According to the "conjecture" $C_r(M)$ in §1 of [Fo],

$$\mathrm{ord}_{s=0}(L(M,s)) = \dim_E H^1_f(F, M^*(1)) - \dim_E H^0(F, M^*(1)),$$

the difference in dimensions of certain motivic cohomology groups of the twisted dual motive. Taking λ-adic realisations (and assuming $C^i_{\lambda}(M)$ in §6.5 of [Fo]) leads to the following conjectures in our situation.

$$\dim_{E_{\lambda}} H^1_f(F, V_{\lambda}(k/2)) = \mathrm{ord}_{s=k/2} L_f(s) > 0$$

(since we arranged for the sign in the functional equation to be negative), and

$$\dim_{E_{\lambda}} H^1_f(F, V'_{\lambda}(k-1+(k/2))) = \mathrm{ord}_{s=k/2} D_f(s) = 0$$

(note that $M'_f(k-1+(k/2)) = (M'_f(k/2))^*(1))$. If these are true then Theorem 10.1 produces non-trivial λ-torsion in $\text{Ш}(k-1+(k/2))$ (recall the definition (7)). As noted in the introduction, $\dim(H^1_f(F, V_{\lambda}(k/2))) > 0$ has been proved in the analogous situation when $F = \mathbb{Q}$ [SU].

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