EISENSTEIN PRIMES, CRITICAL VALUES AND GLOBAL TORSION.

NEIL DUMMIGAN

Abstract. We consider congruences between Eisenstein series and cusp forms (of weight k, level N and character χ of conductor N), modulo large prime divisors of $L(1-k,\chi^{-1})$. We show that such primes occur in the order of a "global torsion" group attached to the cusp form f, and (under a certain hypothesis) also in the denominator of the algebraic part of the rightmost critical value $L_f(k-1)$. These occurrences are linked by the Bloch-Kato conjecture.

1. Introduction

Let f be a normalised newform in $S_k(\Gamma_1(N),\chi)$. Here $k\geq 2$ and $N\geq 1$ are integers, $\chi:(\mathbb{Z}/N\mathbb{Z})^\times\to\mathbb{C}^\times$ is a character, and

$$\Gamma_1(N)=\left\{\begin{bmatrix}a & b\\ c & d\end{bmatrix}\in \mathrm{SL}_2(\mathbb{Z})|c\equiv 0 \pmod{N},\, d\equiv 1 \pmod{N}\right\}.$$

If $\Gamma_0(N)=\left\{\begin{bmatrix}a & b\\ c & d\end{bmatrix}\in \mathrm{SL}_2(\mathbb{Z})|c\equiv 0 \pmod{N}\right\}$ then the holomorphic function f on the complex upper half plane satisfies $f\left(\frac{a\tau+b}{c\tau+d}\right)=\chi(d)(c\tau+d)^kf(\tau)$ for all $\begin{bmatrix}a & b\\ c & d\end{bmatrix}\in\Gamma_0(N)$. Since $\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\in\Gamma_0(N)$ and $f\neq 0$, necessarily $\chi(-1)=(-1)^k$. The Fourier expansion of f is of the form $f(\tau)=\sum_{n=1}^\infty a_nq^n$ with $a_1=1$, where $q=e^{2\pi i\tau}$. The a_n lie in the ring of integers of some finite extension of $\mathbb{Q}$, and for each Hecke operator T_n, $T_nf=a_nf$.

The L-series $L_f(s)=\sum_{n=1}^\infty a_nn^{-s}$ converges for $\Re(s)>(k+1)/2$, and has an Euler product. It defines a function with an analytic continuation to the whole complex plane, with

$$(2\pi)^{-s}\Gamma(s)L_f(s)=\int_0^\infty f(iy)y^s\frac{dy}{y}.$$

In fact $L_f(s)$ is the L-function attached to a premotivic structure (see 1.1.1 of [DFG] for precise definitions). At precisely the points $s=1,\dots,k-1$ it is critical in Deligne's sense [De]. As in §7 of [De], the above integral expression for $L_f(s)$ enables one to verify the relevant case of Deligne's conjecture, which interprets the critical values, up to algebraic multiples, as certain periods. The relevant case of the

Date: May 21st, 2007.
1991 *Mathematics Subject Classification.* 11F67, 14G10.
Key words and phrases. Modular L-function, modular symbols, Bloch-Kato conjecture.

Bloch-Kato conjecture [BK] removes the ambiguity about the algebraic multiple, up to a unit. It predicts that, for $1 \leq j \leq k-1$,

$$\frac{L_f(j)}{(2\pi i)^j\Omega^{(-1)^j}} = \frac{\prod_{p\leq\infty} c_p(j)\ \#Ш(j)}{\#H^0(\mathbb{Q}, A(j))\#H^0(\mathbb{Q}, \check{A}(1-j))}. \tag{1}$$

The various terms will be defined in §3, but this should be viewed as analogous to the rank-0 case of the Birch Swinnerton-Dyer formula.

For a character χ of conductor N, consider the Eisenstein series $E_k^{\chi,\mathbf{1}}$. (We must exclude the case $k=2, N=1$.) This non-cusp form belongs to $M_k(\Gamma_1(N),\chi)$, and $T_p(E_k^{\chi,\mathbf{1}}) = (\chi(p)+p^{k-1})E_k^{\chi,\mathbf{1}}$ for every prime p. If $\lambda \nmid 6Nk!$ divides the Dirichlet L-value $L(1-k,\chi^{-1})$, we show, in §2, that there is a newform $f \in S_k(\Gamma_1(N),\chi)$ with $f \equiv E_k^{\chi,\mathbf{1}} \pmod{\lambda}$ (as Fourier expansions). Here λ is a prime divisor for a number field K large enough to contain the values of χ and the Fourier coefficients of f. (Actually, Proposition 2.1 deals with a somewhat more general type of Eisenstein series.) Congruences of this type are well-known. The case $k=12, N=1$ is Ramanujan's congruence $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$. The case $k=2, N=p, \lambda \mid p$ (not satisfying our condition $\lambda \nmid N$) was used by Ribet in [R].

The various terms in (1) depend on a choice of "S-integral premotivic structure", though the ratio of the two sides is independent of the choice. We make a natural choice as in [DFG]. Having done this, there is a 2-dimensional $\mathbb{F}_\lambda$-vector space $A[\lambda]$ with $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action, that may be viewed as analogous to (a twist of) the group of ℓ-torsion points on an elliptic curve. It follows from the fact that (for all primes p) $a_p \equiv \chi(p)+p^{k-1} \pmod{\lambda}$, that $A[\lambda]$ is reducible, with composition factors $\mathbb{F}_\lambda(\chi^{-1})$ and $\mathbb{F}_\lambda(1-k)$. *If* the latter is a submodule, then $A[\lambda](k-1)$ has a trivial submodule. This would contribute to the λ-part of $\#H^0(\mathbb{Q}, A(k-1))$ in the denominator of (1) (in the case $j=k-1$). In [Du1] I speculated that this is the case, and, following the proof of Theorem 4.6 of [FJ], I proved it in Theorem 7.3 of [Du2], in the case $N=1, K=\mathbb{Q}$ (which probably means just $k=12, \ell=691$, $k=16, \ell=3617$, $k=18, \ell=43867$, $k=20, \ell=283$ or 617, $k=22, \ell=131$ or 593, and $k=26, \ell=657931$). In §4 below, we see that the proof carries across easily to the more general case considered here. Moreover, we show that the λ-parts of the Tamagawa factors $c_p(k-1)$, appearing in the numerator, are trivial, even for $p \mid N$. Hence, unless there is non-trivial λ-torsion in $Ш(k-1)$, we expect to see λ in the denominator of $\frac{L_f(k-1)}{(2\pi i)^{k-1}\Omega^{(-1)^{k-1}}}$.

In §8 of [Du2] we noted that this can be observed in Stein's numerical data [St], in the case $N=1, K=\mathbb{Q}$. In [Du1] we saw the related (but weaker) fact that (again in the case $N=1, K=\mathbb{Q}$) $\lambda=\ell$ appears in various "period ratios", which are essentially ratios of critical values of $L_f(s)$. This was observed numerically (using data of Manin [M]), and also proved theoretically, using a formula of Kohnen and Zagier [KZ]. In §§5,6 and 7 of this paper, we give a proper proof and explanation, in the general case, of why λ appears in the denominator of $\frac{L_f(k-1)}{(2\pi i)^{k-1}\Omega^{(-1)^{k-1}}}$. We have to impose a condition, that λ is not a congruence prime for f in $S_k(\Gamma_1(N),\chi)$ (see before Lemma 6.2).

The proof uses the well-established principle that modular symbols provide a bridge between the cohomology of modular curves (with coefficients in appropriate local systems) and critical values of modular L-functions. To relate congruences of modular forms to congruences of cohomology classes, we make essential use, largely via [DFG], of the Fontaine-Lafaille integral theory of crystalline representations, and of Faltings's comparison theorem.

It is natural to ask whether the condition, that λ should not be a congruence prime for f in $S_k(\Gamma_1(N), \chi)$, is purely a technical convenience, or whether it is natural in that, should it fail, there is a reason why λ might not occur in the denominator of $\frac{L_f(k-1)}{(2\pi i)^{k-1}\Omega^{(-1)^{k-1}}}$. In §8 we look into this and see how the failure of the condition might lead to non-trivial λ-torsion in Ш$(k-1)$.

I am grateful to the referee for several helpful remarks, including the observation that more generally, if there is a newform $f \in S_k(\Gamma_1(N), \chi)$ with $f \equiv E_k^{\chi,\mathbf{1}} \pmod{\lambda^n}$, then λ^n divides both $\#H^0(\mathbb{Q}, A(k-1))$ and the denominator of $\frac{L_f(k-1)}{(2\pi i)^{k-1}\Omega^{(-1)^{k-1}}}$ (if λ is not a congruence prime for f in $S_k(\Gamma_1(N), \chi)$). The existence of such an f may be deduced from $\lambda^n \mid L(1-k, \chi^{-1})$ if λ is not a congruence prime for $S_k(\Gamma_1(N))$. For simplicity we only consider the case $n = 1$.

2. Eisenstein series and congruences with cusp forms

Choose a weight $k \geq 3$, a level $N \geq 1$ and a Dirichlet character χ of conductor dividing N. Let ψ and ϕ be primitive Dirichlet characters of conductors u, v respectively, with $\psi\phi = \chi$, $uv \mid N$, and $\chi(-1) = (-1)^k$. Then there is an Eisenstein series $E_k^{\psi,\phi}$ belonging to $M_k(\Gamma_1(N), \chi)$. In fact, the $E_k^{\psi,\phi}(t\tau)$, for all ψ, ϕ as above, and positive t such that $t \mid N/(uv)$, form a basis for $M_k(\Gamma_1(N), \chi)/S_k(\Gamma_1(N), \chi)$. If $uv = N$ then $E_k^{\psi,\phi}$ is said to be *new* at N. If $k = 2$, a slight modification is needed: for $N = 1$ the (only) triple $\psi = \phi = \mathbf{1}$, $t = 1$ must be excluded, and for $N > 1$, $t \mid N$, one uses $E_2^{\mathbf{1},\mathbf{1}}(\tau) - tE_2^{\mathbf{1},\mathbf{1}}(t\tau)$ in place of $E_2^{\mathbf{1},\mathbf{1}}(t\tau)$.

At infinity, the q-expansion is

$$E_k^{\psi,\phi}(\tau) = \delta(\psi)L(1-k, \psi^{-1}\phi) + 2\sum_{n=1}^{\infty} \sigma_{k-1}^{\psi,\phi}(n)q^n,$$

where

$$\delta(\psi) := \begin{cases} 1 & \text{if } \psi = \mathbf{1} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\sigma_{k-1}^{\psi,\phi}(n) := \sum_{m \mid n, m>0} \psi(n/m)\phi(m)m^{k-1}.$$

For all this, see Theorems 4.5.1 and 4.6.2 of [DiSh]. Recall that $L(1-k, \psi^{-1}\phi) \in \mathbb{Q}(\psi^{-1}\phi)$ (the extension of $\mathbb{Q}$ generated by the values of that character).

Proposition 2.1. *Suppose that $E_k^{\psi,\phi}$ is new at level $N \geq 1$, with $k \geq 2$. Let $\lambda' \nmid 6N$ be a prime of $\mathbb{Z}[\psi, \phi]$ such that $\mathrm{ord}_{\lambda'}(L(1-k, \psi^{-1}\phi)) > 0$. Then there exists a normalised Hecke eigenform $f = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_1(N), \chi)$ such that for all $n \geq 1$,*

$$a_n \equiv \sigma_{k-1}^{\psi,\phi}(n) \pmod{\lambda},$$

where $\lambda \mid \lambda'$ is a prime of the ring of integers of the extension of $\mathbb{Q}(\psi, \phi)$ generated by the a_n.

Proof. For any $\mathbb{Z}[1/N]$-algebra R let $M_k(\Gamma_1(N), R)$ be the R-module of Katz modular forms, and $S_k(\Gamma_1(N), R)$ the submodule of cusp forms. See §1 of [E] for the definitions and basic properties. Consider $E_k^{\psi,\phi}$ as an element of $M_k(\Gamma_1(N), R)$, where

$R = \mathbb{Z}[\zeta_N, \psi, \phi]_{(\lambda'')}$, λ'' is any prime divisor of λ' and ζ_N is a primitive N^{th}-root of unity. That we may do this follows from the q-expansion principle, 1.6 of [Katz], since $X_1(N)$ is connected, and the coefficients in the q-expansion of $E_k^{\psi,\phi}$ at the cusp ∞ lie in R. According to Theorem 3.20 of [FJ], the constant term of $E_k^{\psi,\phi}$ at each (oriented) cusp is of the form $u\, L(1-k, \psi^{-1}\phi)$, with u a unit in $\mathbb{Z}[1/(2N), \zeta_N, \psi, \phi]$. Therefore, since $\mathrm{ord}_{\lambda'}(L(1-k, \psi^{-1}\phi)) > 0$, we have $\overline{E_k^{\psi,\phi}} \in S_k(\Gamma_1(N), \mathbb{F}_{\lambda''})$, where $\overline{E_k^{\psi,\phi}}$ denotes the base-change of $E_k^{\psi,\phi}$ to $M_k(\Gamma_1(N), \mathbb{F}_{\lambda''})$, and $\mathbb{F}_{\lambda''}$ is the residue field.

For each prime p let T_p be the Hecke operator for $\Gamma_1(N)$, and for $(d, N) = 1$ let $\langle d \rangle$ be the diamond operator. Then by Proposition 5.2.3 of [DiSh], $E_k^{\psi,\phi}$ is an eigenfunction for all the T_p and $\langle d \rangle$, in fact

$$T_p E_k^{\psi,\phi} = (\psi(p) + \phi(p)p^{k-1}) E_k^{\psi,\phi},$$
$$\langle d \rangle E_k^{\psi,\phi} = \chi(d) E_k^{\psi,\phi}.$$

For $p \mid N$ this uses the fact that $E_k^{\psi,\phi}$ is new. The same equations hold for $\overline{E_k^{\psi,\phi}}$ in $S_k(\Gamma_1(N), k_{\lambda''})$. By Lemma 1.9 of [E], the base-change map from $S_k(\Gamma_1(N), \overline{\mathbb{Z}_\ell})$ to $S_k(\Gamma_1(N), \overline{\mathbb{F}_\ell})$ is surjective, where $\lambda \mid \ell$. (Note that if $N \neq 1$ or $k \not\equiv 2 \pmod{12}$, we could allow $\ell = 3$.) The existence of an eigenform $f \in S_k(\Gamma_1(N))$ with eigenvalues satisfying the desired congruences now follows easily (if not quite directly) from Lemme 6.11 of [DeSe]. That we may take $f \in S_k(\Gamma_1(N), \chi)$ is a consequence of Carayol's Lemma (1.10 of [E]). □

Note that, since the character χ has maximal conductor N, f is a newform for $\Gamma_1(N)$. (Recall that $\psi\phi = \chi$, and we assumed that $E_k^{\psi,\phi}$ is new.)

3. The Bloch-Kato conjecture

For $k \geq 2$, $N \geq 1$ and χ a character of conductor dividing N, let $f = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_1(N), \chi)$ be a normalised newform. Attached to f is its L-function $L_f(s)$, defined by the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ for $\Re(s) > \frac{k+1}{2}$, but having an analytic continuation to the whole complex plane. Also attached to f is a "premotivic structure" M_f over $\mathbb{Q}$ with coefficients in K, the extension of $\mathbb{Q}(\chi)$ generated by the a_n. Thus there are 2-dimensional K-vector spaces $M_{f,B}$ and $M_{f,\mathrm{dR}}$ (the Betti and de Rham realisations) and, for each finite prime λ of O_K, a 2-dimensional K_λ-vector space $M_{f,\lambda}$, the λ-adic realisation. These come with various structures and comparison isomorphisms, such as $M_{f,B} \otimes_K K_\lambda \simeq M_{f,\lambda}$. See 1.1.1 of [DFG] for the precise definition of a premotivic structure, and 1.6.2 of [DFG] for the construction of M_f. The λ-adic realisation $M_{f,\lambda}$ comes with a continuous linear action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. For each prime number p, the restriction to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ may be used to define a local L-factor, and the Euler product is precisely $L_f(s)$ [Ca]. As the L-function attached to a premotivic structure, its orders of vanishing and leading terms at integer points may be interpreted via the Bloch-Kato conjecture.

On $M_{f,B}$ there is an action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, and the eigenspaces $M_{f,B}^{\pm}$ are 1-dimensional. On $M_{f,\mathrm{dR}}$ there is a decreasing filtration, with F^j a 1-dimensional space precisely for $1 \leq j \leq k-1$. The de Rham isomorphism $M_{f,B} \otimes_K \mathbb{C} \simeq M_{f,\mathrm{dR}} \otimes_K \mathbb{C}$ induces isomorphisms between $M_{f,B}^{\pm} \otimes \mathbb{C}$ and $(M_{f,\mathrm{dR}}/F) \otimes \mathbb{C}$, where $F := F^1 = \ldots = F^{k-1}$. Define $\Omega^{\pm}$ to be the determinants of these isomorphisms. These depend on the choice of K-bases for $M_{f,B}^{\pm}$ and $M_{f,\mathrm{dR}}/F$, so should be viewed as elements of $\mathbb{C}^\times/K^\times$. For $1 \leq j \leq k-1$, the Tate-twisted premotivic structure $M_f(j)$ is *critical* (i.e. the above map *is* an isomorphism, with $F = F^j$), and its Deligne period c^+ (see [De]) is $(2\pi i)^j\Omega^{(-1)^j}$. Deligne's conjecture for $M_f(j)$, known in this case, asserts then that $L_f(j)/(2\pi i)^j\Omega^{(-1)^j}$ is an element of K.

If we choose K-bases for $M_{f,B}$ and $M_{f,\mathrm{dR}}$, to pin down $\Omega^\pm$, then the Bloch-Kato conjecture predicts the prime factorisation of the element $L_f(j)/(2\pi i)^j\Omega^{(-1)^j}$ of K. In fact, we shall choose an O_K-submodule $\mathfrak{M}_{f,B}$, generating $M_{f,B}$ over K, but not necessarily free, and likewise an $O_K[1/S]$-submodule $\mathfrak{M}_{f,\mathrm{dR}}$, generating $M_{f,\mathrm{dR}}$ over K, where S is the set of primes dividing Nk! We take these as in 1.6.2 of [DFG]. They are part of the "S-integral premotivic structure" associated to f. With these choices it is still natural to talk of an element "$L_f(j)/(2\pi i)^j\Omega^{(-1)^j}$" of the group of fractional ideals of $O_K[1/S]$, and the Bloch-Kato conjecture predicts its prime factorisation.

In order to define the various terms appearing in the conjecture, we shall need the elements $\mathfrak{M}_{f,\lambda}$ of the S-integral premotivic structure, for each prime λ of O_K, and also the crystalline realisation $\mathfrak{M}_{f,\lambda\text{-crys}}$ for each $\lambda \notin S$. We choose these as in 1.6.2 of [DFG]. For each λ, $\mathfrak{M}_{f,\lambda}$ is a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice in $M_{f,\lambda}$. Let $A_\lambda := M_{f,\lambda}/\mathfrak{M}_{f,\lambda}$. Let $\check{A}_\lambda := \check{M}_{f,\lambda}/\check{\mathfrak{M}}_{f,\lambda}$, where $\check{M}_{f,\lambda}$ and $\check{\mathfrak{M}}_{f,\lambda}$ are the vector space and O_λ-lattice dual to $M_{f,\lambda}$ and $\mathfrak{M}_{f,\lambda}$ respectively, with the natural $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action. Let $A := \oplus_\lambda A_\lambda$, etc.

Following [BK] (Section 3), for $p \neq \ell$ (including $p = \infty$) let

$$H^1_f(\mathbb{Q}_p, M_{f,\lambda}(j)) = \ker\big(H^1(D_p, M_{f,\lambda}(j)) \to H^1(I_p, M_{f,\lambda}(j))\big).$$

Here D_p is a decomposition subgroup at a prime above p, I_p is the inertia subgroup, and $M_{f,\lambda}(j)$ is a Tate twist of $M_{f,\lambda}$, etc. The cohomology is for continuous cocycles and coboundaries. For $p = \ell$ let

$$H^1_f(\mathbb{Q}_\ell, M_{f,\lambda}(j)) = \ker\big(H^1(D_\ell, M_{f,\lambda}(j)) \to H^1(D_\ell, M_{f,\lambda}(j) \otimes_{\mathbb{Q}_\ell} B_{\mathrm{crys}})\big).$$

(See Section 1 of [BK] for the definition of Fontaine's ring B_{crys}.) Let $H^1_f(\mathbb{Q}, M_{f,\lambda}(j))$ be the subspace of those elements of $H^1(\mathbb{Q}, M_{f,\lambda}(j))$ that, for all primes p, have local restriction lying in $H^1_f(\mathbb{Q}_p, M_{f,\lambda}(j))$. There is a natural exact sequence

$$0 \longrightarrow \mathfrak{M}_{f,\lambda}(j) \longrightarrow M_{f,\lambda}(j) \xrightarrow{\ \pi\ } A_\lambda(j) \longrightarrow 0.$$

Let $H^1_f(\mathbb{Q}_p, A_\lambda(j)) = \pi_* H^1_f(\mathbb{Q}_p, M_{f,\lambda}(j))$. Define the λ-Selmer group $H^1_f(\mathbb{Q}, A_\lambda(j))$ to be the subgroup of elements of $H^1(\mathbb{Q}, A_\lambda(j))$ whose local restrictions lie in $H^1_f(\mathbb{Q}_p, A_\lambda(j))$ for all primes p. Note that the condition at $p = \infty$ is superfluous unless $\ell = 2$. Define the Shafarevich-Tate group

$$Ш(j) = \bigoplus_\lambda \frac{H^1_f(\mathbb{Q}, A_\lambda(j))}{\pi_* H^1_f(\mathbb{Q}, M_{f,\lambda}(j))}.$$

For a finite prime p, let $H^1_f(\mathbb{Q}_p, \mathfrak{M}_{f,\lambda}(j))$ be the inverse image of $H^1_f(\mathbb{Q}_p, M_{f,\lambda}(j))$ under the natural map. Suppose now that $p \neq \ell$. If k is odd, suppose that $j \neq (k-1)/2$. Then $H^0(\mathbb{Q}_p, M_{f,\lambda}(j))$ is trivial (because the eigenvalues of Frob_p^{-1} acting on $M_{f,\lambda}$ are algebraic integers of absolute value $p^{(k-1)/2}$). By inflation-restriction, $H^1_f(\mathbb{Q}_p, M_{f,\lambda}(j)) \simeq (M_{f,\lambda}(j)^{I_p})/(1 - \mathrm{Frob}_p)(M_{f,\lambda}(j)^{I_p})$. This is trivial, since $H^0(\mathbb{Q}_p, M_{f,\lambda}(j))$ is. Hence (using the exact sequence above) $H^1_f(\mathbb{Q}_p, \mathfrak{M}_{f,\lambda}(j))$

is the torsion part of $H^1(\mathbb{Q}_p, \mathfrak{M}_{f,\lambda}(j))$. Using again the triviality of $H^0(\mathbb{Q}_p, M_{f,\lambda}(j))$, we may then identify $H^1_f(\mathbb{Q}_p, \mathfrak{M}_{f,\lambda}(j))$ with $H^0(\mathbb{Q}_p, A_\lambda(j))$.

This has an O_λ-submodule that is given by $(M_{f,\lambda}(j)^{I_p}/\mathfrak{M}_{f,\lambda}(j)^{I_p})^{\mathrm{Frob}_p=\mathrm{id}}$, whose "order" (i.e. λ raised to its length) is the λ-part of $P_p(p^{-j})$, where $P_p(p^{-s}) = \det(1-\mathrm{Frob}_p^{-1}p^{-s}|M_{f,\lambda}^{I_p})$ is the Euler factor at p in $L_f(s)$ (strictly speaking, its reciprocal). When $p \nmid N$, so that $M_{f,\lambda}(j)^{I_p} = M_{f,\lambda}(j)$ maps surjectively to $A_\lambda(j)$, the submodule is the whole of $H^0(\mathbb{Q}_p, A_\lambda(j))$, but in general we define the λ-part of the Tamagawa factor $c_p(j)$ to be the index of the submodule.

It is also possible to define a λ-part of $c_p(j)$ for $\lambda \mid p$, using a measure of $H^1_f(\mathbb{Q}_p, \mathfrak{M}_{f,\lambda}(j))$ arising from the Bloch-Kato exponential map. (See [BK] for details.)

Conjecture 3.1. *Suppose that $1 \leq j \leq k-1$. Then the Bloch-Kato conjecture predicts the following equality of fractional ideals of $O_K[1/S]$:*

$$\frac{L_f(j)}{(2\pi i)^j\Omega^{(-1)^j}} = \frac{\prod_{p\leq\infty} c_p(j)\ \#Ш(j)}{\#H^0(\mathbb{Q}, A(j))\#H^0(\mathbb{Q}, \check{A}(1-j))}.$$

The Tamagawa factor $c_\infty(j)$ is at worst a power of 2. We shall ignore $\#Ш(j)$, except to note that it is integral. By Lemmas 4.3 and 4.6 of [DSW], for $\lambda \nmid S$ the λ-part of $c_p(j)$ can only possibly be non-trivial if $p \mid N$. (The proof of Lemma 4.6, the case $\lambda \mid p$, uses $\mathbb{V}(\mathfrak{M}_{f,\lambda\text{-crys}}) = \mathfrak{M}_{f,\lambda}$, where $\mathbb{V}$ is the version of the Fontaine-Lafaille functor used in [DFG].)

In §7 we shall show, under a certain condition, that for f and λ as in Proposition 2.1, with $\psi = \chi$ and $\phi = \mathbf{1}$ (and also $\lambda \nmid k!$),

$$\mathrm{ord}_\lambda\left(\frac{L_f(k-1)}{(2\pi i)^{k-1}\Omega^{(-1)^{k-1}}}\right) < 0.$$

(The condition is that λ is not a "congruence prime" for f in $S_k(\Gamma_1, \chi)$.) Given that $\mathrm{ord}_\lambda(\#Ш(k-1)) \geq 0$ and $\mathrm{ord}_\lambda(c_p(k-1)) \geq 0$ (for $p \nmid N$ it is even 0), the Bloch-Kato conjecture predicts that $H^0(\mathbb{Q}, A_\lambda(k-1))$ or $H^0(\mathbb{Q}, \check{A}_\lambda(2-k))$ must be non-zero. Now $\check{A}[\lambda](2-k)$ has composition factors $\mathbb{F}_\lambda(\chi)(2-k)$ and $\mathbb{F}_\lambda(1)$ (see the first paragraph of §4), neither of which is trivial. Note that $\mathbb{F}_\lambda(\chi)$ denotes a 1-dimensional $\mathbb{F}_\lambda$-vector space on which $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts via the reduction of the character χ. The non-triviality of $\mathbb{F}_\lambda(\chi)(2-k)$ follows from $\ell > k$, and the fact that χ is unramified, but non-trivial if $k=2$. Hence $H^0(\mathbb{Q}, \check{A}_\lambda(2-k))$ is zero. We shall confirm in the next section that $H^0(\mathbb{Q}, A_\lambda(k-1))$ is non-zero.

In fact, we shall show also that $\mathrm{ord}_\lambda(c_p(k-1)) = 0$ even for $p \mid N$, so that the contribution of $H^0(\mathbb{Q}, A_\lambda(k-1))$ to the denominator is not cancelled by Tamagawa factors. This depends on χ having exact conductor N, as will the proof that $\mathrm{ord}_\lambda\left(\frac{L_f(k-1)}{(2\pi i)^{k-1}\Omega^{(-1)^{k-1}}}\right) < 0$.

It seems worth mentioning that the functional equation relates $L_f(s)$ to $L_{f_{\chi^{-1}}}(k-s)$. The form $f_{\chi^{-1}}$, the twist of f by the character χ^{-1}, lives in $S_k(\Gamma_1(N), \chi^{-1})$, and is congruent to $E_k^{\mathbf{1},\chi^{-1}} \pmod{\lambda}$.

4. Global torsion and Tamagawa factors

Recall that we have chosen a weight $k \geq 2$, a level $N \geq 1$, a character χ of conductor precisely N, with $\chi(-1) = (-1)^k$, and a cusp form $f \equiv E_k^{\chi,\mathbf{1}} \pmod{\lambda}$, where $\lambda \nmid 6Nk!$ is a prime of O_K (K being the extension of $\mathbb{Q}(\chi)$ generated by the Fourier coefficients a_n of f at ∞) such that $\mathrm{ord}_\lambda(L(1-k,\chi^{-1})) > 0$. For all primes $p \nmid N$, $a_p \equiv \chi(p) + p^{k-1} \pmod{\lambda}$. In fact this holds even for $p \mid N$, with $\chi(p) = 0$ for such p. Since a_p is the trace of Frob_p^{-1} on $M_{f,\lambda}$, it follows that $A[\lambda]$ (i.e. the λ-torsion in A_λ) is reducible as an $\mathbb{F}_\lambda[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-module, with composition factors $\mathbb{F}_\lambda(\chi^{-1})$ and $\mathbb{F}_\lambda(1-k)$. Here we identify χ with a character of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ via an Artin map that sends p to Frob_p.

Theorem 4.1. *In the situation of the preceding paragraph, $A[\lambda]$ has $\mathbb{F}_\lambda(1-k)$ as a submodule.*

Corollary 4.2. *$H^0(\mathbb{Q}, A_\lambda(k-1))$ is non-trivial.*

Before proving this we need a few preliminaries. Diamond, Flach and Guo, in 1.4.2 of [DFG], construct "premotivic structures" $M(N,\chi), M(N,\chi)_c$ and $M(N,\chi)_!$ for the space of modular forms of level N and character χ. Fixing choices of N and χ, we call these M, M_c and $M_!$. (There is a map from M_c to M with image $M_!$.) Each has Betti, de Rham, and (for each prime λ of O_K) λ-adic realisations, denoted $M_B, M_{\mathrm{dR}}, M_\lambda$, etc. The Betti and de Rham realisations are K-vector spaces and the λ-adic realisations are K_λ-vector spaces. Here we choose K as above, though the construction works for any number field containing $\mathbb{Q}(\chi)$. Temporarily λ denotes any prime of O_K. There are various additional structures and comparison maps, discussed in detail in [DFG]. For example, M_λ supports a continuous representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. There are also S-integral premotivic structures $\mathfrak{M}, \mathfrak{M}_c$ and $\mathfrak{M}_!$, where S is the set of primes dividing $Nk!$. These have realisations $\mathfrak{M}_B$ (an O_K-lattice in M_B), $\mathfrak{M}_{\mathrm{dR}}$ (an $O_K[1/S]$-lattice in M_{dR}) and $\mathfrak{M}_\lambda$ (a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice in M_λ) for all primes λ, etc. There are canonical isomorphisms $\mathfrak{M}_B \otimes_{O_K} O_\lambda \simeq \mathfrak{M}_\lambda$, etc. For $\lambda \notin S$ there is also a crystalline realisation $\mathfrak{M}_{\lambda\text{-crys}}$.

Let $\mathbb{T}'$ be the ring generated over O_K by all the Hecke operators T_n acting on $M_k(\Gamma_1(N),\chi)$. There are compatible actions of $\mathbb{T}'$ on all of the above, by Proposition 1.3 of [DFG]. Let $\mathcal{I}$ be the ideal of $\mathbb{T}'$ generated by $T_p - (\chi(p) + p^{k-1})$ for all primes p, and let $\mathfrak{m}$ be the maximal ideal generated by $\mathcal{I}$ and λ.

For $\lambda \notin S$, $\mathfrak{M}_{\lambda\text{-crys}}$ is a filtered O_λ module with graded pieces of degrees 0 and $k-1$. There is a Hecke-equivariant isomorphism $\mathrm{Fil}^{k-1}\mathfrak{M}_{\lambda\text{-crys}} \simeq M_k(\Gamma_1(N),\chi,O_\lambda)$. It has an injective Frobenius endomorphism ϕ, and is strongly divisible in the sense that $\mathfrak{M}_{\lambda\text{-crys}} = \phi\mathfrak{M}_{\lambda\text{-crys}} + \phi_{k-1}(\mathrm{Fil}^{k-1}\mathfrak{M}_{\lambda\text{-crys}})$, where $\ell^{k-1}\phi_{k-1} : \mathrm{Fil}^{k-1}\mathfrak{M}_{\lambda\text{-crys}} \to \mathfrak{M}_{\lambda\text{-crys}}$ is the restriction of ϕ. (See the end of 1.4.2 of [DFG].) Similar statements apply to $\mathfrak{M}_c$ and $\mathfrak{M}_!$, with $S_k(\Gamma_1(N),\chi,O_\lambda)$ replacing $M_k(\Gamma_1(N),\chi,O_\lambda)$. Viewing $\mathfrak{M}_\lambda, \mathfrak{M}_{c,\lambda}$ and $\mathfrak{M}_{!,\lambda}$ as $\mathbb{Z}_\ell$-modules with $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$-action, they may be identified with, respectively, $\mathbb{V}(\mathfrak{M}_{\lambda\text{-crys}}), \mathbb{V}(\mathfrak{M}_{c,\lambda\text{-crys}})$ and $\mathbb{V}(\mathfrak{M}_{!,\lambda\text{-crys}})$, where $\mathbb{V}$ is the covariant version of Fontaine and Lafaille's functor used in [DFG].

Lemma 4.3. *Suppose that $\lambda \notin S$, $\lambda \nmid 6$ and $\mathrm{ord}_\lambda(L(1-k,\chi^{-1})) > 0$. The $\mathbb{F}_\lambda[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-module $(\mathfrak{M}_{!,\lambda}/\lambda\mathfrak{M}_{!,\lambda})[\mathfrak{m}]$ has a submodule that becomes isomorphic to $\mathbb{F}_\lambda(1-k)$ upon restriction to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_N))$, and it is the unique subquotient with this property.*

Proof. This is based on the proof of Proposition 4.6 of [FJ]. The rank-one O_λ-submodule $\mathcal{E}$ of $M_k(\Gamma_1(N),\chi,O_\lambda)$ generated by the Eisenstein series $E_k^{\chi,\mathbf{1}}$ is the kernel of $\mathcal{I}$ on $\mathfrak{M}_{\lambda\text{-crys}}$, so is stable under ϕ_{k-1}, since ϕ commutes with the Hecke operators. Since ϕ is injective and $\mathfrak{M}_{\lambda\text{-crys}}$ is strongly divisible, we must have

$\phi_{k-1}(\mathcal{E}) = \mathcal{E}$, so $\mathcal{E}$ is a strongly divisible filtered ϕ-module. The functor $\mathbb{V}$ takes $\mathcal{E}$ to a rank-one O_λ-submodule E of $\mathfrak{M}_\lambda$, stable under $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$. In fact, since $\mathbb{V}$ respects Hecke operators, $E = \mathfrak{M}_\lambda[\mathcal{I}]$ and so is stable under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. In fact, $E \simeq O_\lambda(1-k)$ as $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_N))$-module, since $M_{!,\lambda}[\mathcal{I}] = 0$, and by 1.2.0 of [Sc] the cokernel of the inclusion of $M_{!,\lambda}$ in M_λ becomes, upon restriction to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_N))$, isomorphic to a direct sum of copies of $K_\lambda(1-k)$.

Let $M_k = M_k(\Gamma_1(N), \chi, O_\lambda)$ and $S_k = S_k(\Gamma_1(N), \chi, O_\lambda)$. The image of $\mathcal{E}$ in $M_k/\lambda M_k$ actually lies in $S_k/\lambda S_k$, as noted in the proof of Proposition 2.1. This gives a ϕ_{k-1}-stable, one-dimensional $\mathbb{F}_\lambda$-subspace $\overline{\mathcal{E}}$ of the finite-length filtered O_λ-module $\mathfrak{M}_{!,\lambda\text{-crys}}/\lambda\mathfrak{M}_{!,\lambda\text{-crys}}$, lying inside Fil^{k-1}. Since $\mathcal{E}$ is killed by $\mathcal{I}$, $\overline{\mathcal{E}} \subset (\mathfrak{M}_{!,\lambda\text{-crys}}/\lambda\mathfrak{M}_{!,\lambda\text{-crys}})[\mathfrak{m}]$.

We may apply a finite-length version of the functor $\mathbb{V}$ (see 1.1.2 of [DFG]) to get a one-dimensional subspace W of $(\mathfrak{M}_{!,\lambda}/\lambda\mathfrak{M}_{!,\lambda})[\mathfrak{m}]$. Inside $(\mathfrak{M}_\lambda/\lambda\mathfrak{M}_\lambda)[\mathfrak{m}]$, W is just the reduction of E, so is isomorphic to $\mathbb{F}_\lambda(1-k)$ as a module for $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_N))$. Let L be the finite unramified extension of $\mathbb{Q}_\ell$ corresponding to $\mathbb{Q}(\zeta_N)$, and let ψ be any (O_λ-valued) character of $\mathrm{Gal}(L/\mathbb{Q}_\ell)$.

By the q-expansion principle, $\dim_{\mathbb{F}_\lambda}(S_k(\Gamma_1(N), \chi, \mathbb{F}_\lambda)[\mathfrak{m}]) = 1$, hence

$$\mathrm{Fil}^{k-1}(((\mathfrak{M}_{!,\lambda\text{-crys}}/\lambda\mathfrak{M}_{!,\lambda\text{-crys}})[\mathfrak{m}])/\overline{\mathcal{E}}) = 0.$$

But

$$\mathbb{V}(((\mathfrak{M}_{!,\lambda\text{-crys}}/\lambda\mathfrak{M}_{!,\lambda\text{-crys}})[\mathfrak{m}])/\overline{\mathcal{E}}) = ((\mathfrak{M}_{!,\lambda}/\lambda\mathfrak{M}_{!,\lambda})[\mathfrak{m}])/W,$$

and the filtered module $\mathbb{F}_\lambda\{\psi\}\{1-k\}$ such that $\mathbb{V}(\mathbb{F}_\lambda\{\psi\}\{1-k\}) = \mathbb{F}_\lambda(\psi)(1-k)$ has $\mathrm{Fil}^{k-1}\mathbb{F}_\lambda\{\psi\}\{1-k\} = \mathbb{F}_\lambda\{\psi\}\{1-k\}$, so $(\mathfrak{M}_{!,\lambda}/\lambda\mathfrak{M}_{!,\lambda})[\mathfrak{m}]$ cannot have any more composition factors isomorphic to $\mathbb{F}_\lambda(1-k)$ upon restriction to $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/L)$. $\square$

Proof of Theorem 4.1. By construction, $\mathfrak{M}_{f,\lambda}$ is a submodule of $\mathfrak{M}_{!,\lambda}$, hence $A[\lambda]$ is a submodule of $(\mathfrak{M}_{!,\lambda}/\lambda\mathfrak{M}_{!,\lambda})[\mathfrak{m}]$. In this latter, as above, $\mathbb{F}_\lambda(1-k)$ has multiplicity at most one, and appears as a submodule, if at all. It remains to observe that the subquotients of $A[\lambda]$ are $\mathbb{F}_\lambda(\chi^{-1})$ and $\mathbb{F}_\lambda(1-k)$, so the latter must be a submodule. Note that since χ is unramified at ℓ and $k < \ell$, these factors remain distinct upon restriction to $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/L)$, the latter being ramified.
$\square$ Write $\chi = \prod_{p \mid N} \chi_p$, where the conductor of χ_p is the power of p in N.

Proposition 4.4. *In the same situation as above,* $\mathrm{ord}_\lambda(c_p(j)) = 0$*, for any integer* j *and any prime* $p \mid N$ *such that the order of* χ_p *is not a power of* ℓ *(e.g if* $\ell \nmid p-1$*).*

Proof. Since $a_p \equiv \chi(p) + p^{k-1} = p^{k-1} \not\equiv 0 \pmod{\lambda}$, the Euler factor $(1 - a_p p^{-s})$ must have degree 1. Recalling that this Euler factor is the reciprocal of $\det(1 - \mathrm{Frob}_p^{-1} p^{-s} | M_{f,\lambda}^{I_p})$, we see that $\dim M_{f,\lambda}^{I_p} = 1$. Now $\mathrm{ord}_\lambda(c_p(j))$ could be non-zero only if the map from $M_{f,\lambda}^{I_p}$ to $A_\lambda^{I_p}$ is not surjective. This would force $\dim A[\lambda]^{I_p} > 1$, so $A[\lambda]^{I_p} = A[\lambda]$. This cannot be the case, since the composition factors of $A[\lambda]$ are $\mathbb{F}_\lambda(1-k)$ and $\mathbb{F}_\lambda(\chi^{-1})$, and χ, having exact conductor N, is ramified at p. The condition on the order of χ_p ensures that the reduction $\pmod{\lambda}$ of χ (which we also call χ, by abuse of notation) is still ramified at p. $\square$

5. The Hecke action on boundary symbols

Let R be a commutative ring in which 6 is invertible. Let A be a right $R[\Sigma]$-module, where $\Sigma = M_2(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Q})$. Let $\mathcal{D}$ be the group of divisors supported on $\mathbb{P}^1(\mathbb{Q})$, with $\mathcal{D}_0$ the subgroup of divisors of degree zero. There is a natural left action of $\mathrm{GL}_2(\mathbb{Q})$ on $\mathcal{D}$. If Γ is any congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, let $\mathrm{Symb}_\Gamma(A)$ (the group of A-valued modular symbols for Γ) be the set of homomorphisms $\Phi : \mathcal{D}_0 \to A$ such that $\Phi|g = \Phi$ for all $g \in \Gamma$, where $(\Phi|g)(D) := (\Phi(gD))|g$. Replacing $\mathcal{D}_0$ by $\mathcal{D}$, we likewise define $\mathrm{Bound}_\Gamma(A)$, the group of A-valued boundary symbols for Γ. Restriction from $\mathcal{D}$ to $\mathcal{D}_0$ provides a natural homomorphism from $\mathrm{Bound}_\Gamma(A)$ to $\mathrm{Symb}_\Gamma(A)$. A useful reference for modular symbols is §4 of [GS].

For $g \in \Sigma$, if $\Gamma g \Gamma = \cup_i \Gamma g_i$ then, for $\Phi \in \mathrm{Bound}_\Gamma(A)$ or $\mathrm{Symb}_\Gamma(A)$, let $\Phi|T(g) := \sum_i \Phi|g_i$. When $g = \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}$, we abbreviate $T(g)$ to $T(p)$.

From now on, we fix a choice of weight $k \geq 2$ and level N, let $\Gamma = \Gamma_1(N)$ and $A = \mathrm{Sym}^{k-2} R^2$, the module of polynomials of degree $k-2$ over R in variables X and Y. The right Σ-action is defined by $(F|g)(X,Y) = F((X,Y)g^*)$, where for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we define $g^* = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Sometimes we may write $\mathrm{Symb}_k(\Gamma_1(N), R)$ instead of $\mathrm{Symb}_{\Gamma_1(N)}(\mathrm{Sym}^{k-2} R^2)$. If $\psi : (\mathbb{Z}/N\mathbb{Z})^\times \to R^\times$ is a character, we may view ψ as a character of $\Gamma_0(N)$ in the usual way: $\psi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \psi(d)$, then let $\mathrm{Symb}_k(\Gamma_1(N), \psi, R) := \{\Phi \in \mathrm{Symb}_k(\Gamma_1(N), R) : \Phi|g = \psi(g)\Phi \ \forall g \in \Gamma_0(N)\}$. Likewise for Bound. By Theorem 4.3 of [GS], $\mathrm{Bound}_k(\Gamma_1(N), \psi, R)$ may be viewed as a subgroup of $\mathrm{Symb}_k(\Gamma_1(N), \psi, R)$. In fact, using a theorem of Ash and Stevens (see §6 below) to identify $\mathrm{Symb}_k(\Gamma_1(N), \psi, R)$ with a certain compactly supported cohomology group, $\mathrm{Bound}_k(\Gamma_1(N), \psi, R)$ is the kernel of the projection onto parabolic cohomology.

Proposition 5.1. *Let χ be an R-valued character of conductor N. There is an element Φ_χ of $\mathrm{Bound}_k(\Gamma_1(N), \chi, R)$, supported on the $\Gamma_0(N)$-orbit of ∞, such that $\Phi_\chi(\infty) = X^{k-2}$. It satisfies, for all primes p,*

$$\Phi_\chi|T(p) = (p^{k-1} + \chi(p))\Phi_\chi.$$

It spans the submodule of $\mathrm{Bound}_k(\Gamma_1(N), \chi, R)$ comprising all Φ on which the $T(p)$ act in this manner.

Proof. The stabiliser of ∞ in $\Gamma_1(N)$ is the subgroup generated by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. The submodule of $\mathrm{Sym}^{k-2} R^2$ fixed by this subgroup is spanned by X^{k-2}. Therefore, up to a scalar, we are forced to choose $\Phi_\chi(\infty) = X^{k-2}$. Now the values of Φ_χ on the $\Gamma_0(N)$-orbit of ∞ are determined by $\Phi_\chi|g = \chi(g)\Phi_\chi$, so that $\Phi_\chi(g\infty) = \chi(g)\Phi_\chi(\infty)|g^{-1}$ for all $g \in \Gamma_0(N)$. We complete the definition of Φ_χ by decreeing that it take the value 0 outside the $\Gamma_0(N)$-orbit of ∞.

Next we calculate the action of $T(p)$. According to Proposition 5.2.1 of [DiSh], if $p \nmid N$ then

$$\Phi_\chi|T(p) = \sum_{j=0}^{p-1} \Phi_\chi \Big| \begin{bmatrix} 1 & j \\ 0 & p \end{bmatrix} + \Phi_\chi \Big| \begin{bmatrix} m & n \\ N & p \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix},$$

where $mp - nN = 1$. In the sum from 0 to $p-1$, each g_i fixes ∞, so each of the p terms, evaluated at ∞, is $(pX)^{k-2}$. If $g = \begin{bmatrix} m & n \\ N & p \end{bmatrix}$ then $g \in \Gamma_0(N)$, so $\Phi_\chi|g = \chi(p)\Phi_\chi$, so $(\Phi_\chi|g)(\infty) = \chi(p)X^{k-2}$. Now $\begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}$ fixes both ∞ and X^{k-2}.

So we find that

$$(\Phi_\chi|T(p))(\infty) = (p^{k-1} + \chi(p))\Phi_\chi(\infty).$$

The fact that $T(p)$ commutes with the diamond operators $\langle d\rangle$ (see the calculation on p.169 of [DiSh]) allows us to extend this from ∞ to the whole $\Gamma_0(N)$-orbit of ∞.

It remains to sketch a proof of the uniqueness property of Φ_χ. If ψ and ϕ are primitive Dirichlet characters of conductors u, v respectively, with $\psi\phi = \chi$, $uv = N$, then we may apply the above construction to $\psi\phi^{-1}$, then apply a twisting operator: $\Phi|R_\phi = \sum_{a=0}^{v-1}\phi(a)\Phi\Big|\begin{bmatrix}1 & a\\ 0 & v\end{bmatrix}$ (see 4.10 of [GS]). Thus we get some $\Phi \in \mathrm{Bound}_k(\Gamma_1(N), \chi, R)$ such that, for all primes p, $\Phi|T(p) = (\phi(p)p^{k-1} + \psi(p))\Phi$. If there was a $\Phi \in \mathrm{Bound}_k(\Gamma_1(N), \chi, R)$ such that, for all primes p, $\Phi|T(p) = (p^{k-1} + \chi(p))\Phi$, but Φ was not a multiple of Φ_χ, then, together with what we have constructed, it would, over $\mathbb{C}$, span a space of dimension greater than that of $M_k(\Gamma_1(N), \chi)/S_k(\Gamma_1(N), \chi)$ (recall the first paragraph of §2). This would contradict the fact that $\mathrm{Bound}_k(\Gamma_1(N), \chi, \mathbb{C})$ has the same dimension as $M_k(\Gamma_1(N), \chi)/S_k(\Gamma_1(N), \chi)$. (This fact follows from a comparison of the dimensions of the graded pieces of $M_{c,\mathrm{dR}}$ and $M_{!,\mathrm{dR}}$.) □

The involution $\iota = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ breaks $\mathrm{Symb}_k(\Gamma_1(N), \chi, R)$ and $\mathrm{Bound}_k(\Gamma_1(N), \chi, R)$ into $\pm$ eigenspaces. It is easy to check that Φ_χ is in the $(-1)^{k-2} = (-1)^k$-eigenspace, i.e. $\Phi_\chi|\iota = (-1)^k\Phi_\chi$.

6. A congruence of modular symbols

Let R be a commutative ring in which 6 is invertible, A a right $R[\Sigma]$-module, and Γ any congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Then by a theorem of Ash and Stevens [AS](Theorem 4.2 in [GS]), there is a natural isomorphism

$$\mathrm{Symb}_\Gamma(A) \simeq H^1_c(\Gamma\backslash\mathfrak{H}, A).$$

Letting $\Gamma = \Gamma(N), A = \mathrm{Sym}^{k-2}R^2$, with $R = O_\lambda$, then taking the part on which $\Gamma_0(N)$ acts via χ, we get $\mathrm{Symb}_k(\Gamma_1(N), \chi, O_\lambda)$ from the left, while from the right, following the construction in 1.2–1.4 of [DFG], we get $\mathfrak{M}_{c,\lambda}$. In other words:

Lemma 6.1. $\mathrm{Symb}_k(\Gamma_1(N), \chi, O_\lambda) \simeq \mathfrak{M}_{c,\lambda}$.

The action of ι on the left matches the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on the right.

Though it is not necessary for the truth of the lemma, we now put ourselves in the situation described in the first paragraph of §4. Recall that $\mathbb{T}'$ is the ring generated over O_K by all the Hecke operators T_n acting on $M_k(\Gamma_1(N), \chi)$, and $\mathcal{I}$ is the ideal of $\mathbb{T}'$ generated by $T_p - (\chi(p) + p^{k-1})$ for all primes p. Let $\mathcal{I}_f$ be the ideal of $\mathbb{T}'$ generated by $T_p - a_p$ for all primes p. Define Φ_f to be any generator for the free rank-1 O_λ-module $\mathrm{Symb}_k(\Gamma_1(N), \chi, O_\lambda)^{(-1)^k}[\mathcal{I}_f]$. Let Φ_χ be as in Proposition 5.1. We say that λ is a *congruence prime* for f in $S_k(\Gamma_1(N), \chi)$ if there exists $g \in S_k(\Gamma_1(N), \chi, O_{K'})$, orthogonal to f (with K' some sufficiently large finite extension of K), such that $f \equiv g \pmod{\lambda'}$ (where $\lambda' \mid \lambda$). By applying the eigenspace-killing procedure described in the proof below, one may assume that g is an eigenvector for $\mathbb{T}'$.

Lemma 6.2. *Suppose that λ is not a congruence prime for f in $S_k(\Gamma_1(N), \chi)$. We may choose Φ_f in such a way that $\Phi_f - \Phi_\chi \in \lambda\,\mathrm{Symb}_k(\Gamma_1(N), \chi, O_\lambda)$.*

Proof. Let $M_{f,E} := M_\lambda[\mathcal{I}] \oplus M_\lambda[\mathcal{I}_f]$, $M_{c,f,E} := M_{c,\lambda}[\mathcal{I}] \oplus M_{c,\lambda}[\mathcal{I}_f]$, $\mathfrak{M}_{f,E} := M_{f,E} \cap \mathfrak{M}_\lambda$ and $\mathfrak{M}_{c,f,E} := M_{c,f,E} \cap \mathfrak{M}_{c,\lambda}$. Since $\ell > k-2$, the duality morphisms of 1.5 of [DFG] induce perfect O_λ-valued pairings between $\mathfrak{M}_{c,\lambda}$ and $\mathfrak{M}_\lambda$, and between $\mathfrak{M}_{c,\lambda}^{(-1)^k}$ and $\mathfrak{M}_\lambda^{(-1)^{k-1}}$. By restriction, we get a pairing between $\mathfrak{M}_{c,f,E}^{(-1)^k}$ and $\mathfrak{M}_{f,E}^{(-1)^{k-1}}$. Let v_f be a generator for $M_\lambda^{(-1)^{k-1}}[\mathcal{I}_f] \cap \mathfrak{M}_{f,E}^{(-1)^{k-1}}$ and let v_E be a generator for $M_\lambda^{(-1)^{k-1}}[\mathcal{I}] \cap \mathfrak{M}_{f,E}^{(-1)^{k-1}}$. Let w_f be a generator for $M_{c,\lambda}^{(-1)^k}[\mathcal{I}_f] \cap \mathfrak{M}_{f,E}^{(-1)^k}$ and let w_E be a generator for $M_{c,\lambda}^{(-1)^k}[\mathcal{I}] \cap \mathfrak{M}_{f,E}^{(-1)^k}$.

In fact we choose w_f and w_E to be the images of Φ_f and Φ_χ under the isomorphism of Lemma 6.1. We wish to show that $w_f - w_E \in \lambda\,\mathfrak{M}_{c,f,E}^{(-1)^k}$. Complex conjugation in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\mathbb{F}_\lambda(1-k)$ as $(-1)^{k-1}$ but on $\mathbb{F}_\lambda(\chi^{-1})$ as $(-1)^k$. Therefore it must be the former that is spanned by the image of v_f in $\mathfrak{M}_\lambda/\lambda\mathfrak{M}_{f,\lambda}$. Applying Lemma 4.3, we find that, choosing scalar multiples appropriately, without loss of generality, $v_f - v_E \in \lambda\,\mathfrak{M}_{f,E}^{(-1)^{k-1}}$. Let r be the largest integer such that $v_f - v_E \in \lambda^r\,\mathfrak{M}_{f,E}^{(-1)^{k-1}}$. Since the Hecke operators T_p (for $p \nmid N$) are self-adjoint for the pairing, $\langle w_f, v_E\rangle = \langle w_E, v_f\rangle = 0$. Letting λ denote also some uniformiser in O_λ, if $v_f - v_E = \lambda^r v$ then we have

$$\langle w_f, v_f\rangle = \lambda^r\langle w_f, v\rangle,\ \langle w_E, v_E\rangle = -\lambda^r\langle w_E, v\rangle.$$

If w_f and w_E do not span $\mathfrak{M}_{c,f,E}^{(-1)^k}$ then we may choose them in such a way that $w_f - w_E \in \lambda\,\mathfrak{M}_{c,f,E}^{(-1)^k}$, as required. So suppose that w_f and w_E do span $\mathfrak{M}_{c,f,E}^{(-1)^k}$.

By perfectness of the pairing between $\mathfrak{M}_{c,\lambda}^{(-1)^k}$ and $\mathfrak{M}_\lambda^{(-1)^{k-1}}$, there exists some $g \in \mathfrak{M}_\lambda^{(-1)^{k-1}}$ such that

$$\langle w_f, g\rangle = 1,\ \langle w_E, g\rangle = 0.$$

Then $g = (v_f - h)/\lambda^r$ for some $h \in \mathfrak{M}_\lambda^{(-1)^{k-1}}$ (i.e. $h = v_f - \lambda^r g$). There is a decomposition of $M_\lambda^{(-1)^{k-1}} \otimes K'_{\lambda'}$, for $K'_{\lambda'}$ some sufficiently large finite extension of K_λ, into one-dimensional $\mathbb{T}'$-eigenspaces, all "new" because χ has conductor N. This parallels the decomposition of $M_k(\Gamma_1(N),\chi)$. The element h is a linear combination of eigenvectors, with v_E and v_f excluded, since by design $\langle h, w_f\rangle = \langle h, w_E\rangle = 0$. If there is a system of eigenvalues $\{b_p\}$ such that, for some p, $a_p \not\equiv b_p \pmod{\lambda'}$, then by applying $(T_p - b_p)/(a_p - b_p)$, we can kill the eigenspace corresponding to $\{b_p\}$, in the expression $g = \frac{v_f - h}{\lambda^r}$. Since h is necessarily non-trivial, there must be a system of eigenvalues $\{b_p\}$ (corresponding to some cusp form in $S_k(\Gamma_1(N),\chi)$ different from f), such that $a_p \equiv b_p \pmod{\lambda'}$ for every p. In other words, λ is a congruence prime for f in $S_k(\Gamma_1(N),\chi)$. □

7. The denominator of the L-value

Throughout this section, we are in the situation described in the first paragraph of §4. We need to consider the period $\Omega^{(-1)^{k-1}}$. Since we are looking just at the λ-part of the Bloch-Kato conjecture, this period matters only up to a unit in $O_{(\lambda)}$, the localisation at λ of O_K. Recall that in §3, $\Omega^\pm$ were defined as determinants of isomorphisms from $M_{f,B}^\pm \otimes \mathbb{C}$ to $(M_{f,\mathrm{dR}}/F) \otimes \mathbb{C}$, calculated with respect to bases arising from $\mathfrak{M}_{f,B}$ and $\mathfrak{M}_{f,\mathrm{dR}}$. (We can choose bases for $\mathfrak{M}_{f,B} \otimes O_{(\lambda)}$ and $\mathfrak{M}_{f,\mathrm{dR}} \otimes O_{(\lambda)}$.) Let $\omega^\pm$ be the determinants of isomorphisms going the other way, from $F \otimes \mathbb{C}$ to $M^{\pm}_{f,B} \otimes \mathbb{C}$, arising from the inverse of the comparison isomorphism $I : M_{f,B} \otimes \mathbb{C} \to M_{f,\mathrm{dR}} \otimes \mathbb{C}$, also calculated with respect to bases coming from $\mathfrak{M}_{f,B}$ and $\mathfrak{M}_{f,\mathrm{dR}}$.

Lemma 7.1. *Up to a unit in $O_{(\lambda)}$,*

$$\omega^{\pm} = (2\pi i)^{k-1} \Omega^{\mp}.$$

Proof. Applying Lemma 5.1.6 of [De], $\omega^{\pm} = \Omega^{\mp}/\det I$, where the determinant is calculated using bases coming from $\mathfrak{M}_{f,B}$ and $\mathfrak{M}_{f,\mathrm{dR}}$. Now $\det(\mathfrak{M}_{f,B}) = \eta_f \mathfrak{M}_{\chi}(1-k)_B$ and $\det(\mathfrak{M}_{f,\mathrm{dR}}) = \eta_f \mathfrak{M}_{\chi}(1-k)_{\mathrm{dR}}$, in the notation of 1.7.3 of [DFG] (η_f is a certain fractional ideal of K and $M_{\chi}(1-k)$ is a Tate-twist of a Dirichlet motive). We may realise $\det I$ as the period of $M_{\chi}(1-k)$ with respect to the integral structure $\mathfrak{M}_{\chi}(1-k)$, namely $G(\chi)(2\pi i)^{1-k}$ (see 1.1.3 of [DFG]). The Gauss sum $G(\chi)$ is coprime to λ. □

Recall from §4 the premotivic structures M_c, M and $M_!$ associated with $M_k(\Gamma_1(N), \chi)$, and that $M_!$ is the image of M_c in M. The premotivic structure M_f attached to f is $M_![\mathcal{I}_f]$. Likewise $\mathfrak{M}_f = \mathfrak{M}_![\mathcal{I}_f]$. There are natural identifications of $\mathrm{Fil}^{k-1} M_{c,\mathrm{dR}}$ and $\mathrm{Fil}^{k-1} M_{!,\mathrm{dR}}$ with $S_k(\Gamma_1(N), \chi, K)$, such that $\mathrm{Fil}^{k-1}\mathfrak{M}_{f,\mathrm{dR}} = O_K[1/S]f$ (1.4.2 and 1.6.2 of [DFG]).

To f we associate a modular symbol $\psi_f \in \mathrm{Symb}_k(\Gamma_1(N), \chi, \mathbb{C})$, defined by

$$\psi_f(\{b\} - \{a\}) = (2\pi i)^{k-1} \int_a^b f(\tau)(\tau X + Y)^{k-2}\, d\tau.$$

Via the Ash-Stevens isomorphism we may view it as an element of $M_{c,B} \otimes \mathbb{C}$, which then maps to an element Ψ_f of $M_{!,B} \otimes \mathbb{C}$. In fact ψ_f and Ψ_f are killed by $\mathcal{I}_f$. The following lemma was used implicitly in §5 of [DSW].

Lemma 7.2. *Under the comparison (de Rham) isomorphism $M_{!,\mathrm{dR}} \otimes \mathbb{C} \to M_{!,B} \otimes \mathbb{C}$, the image of $f \in \mathrm{Fil}^{k-1} M_{f,\mathrm{dR}}$ is Ψ_f.*

We sketch the reason. As on the line preceding (4) of [DFG], f corresponds to a differential $(2\pi i)^{k-1} f(\tau)\, dz^{\otimes k-2}\, d\tau$ on $\Gamma_1(N)\backslash\mathfrak{H}$, with coefficients in a certain local system with fibres $\mathrm{Sym}^{k-2} H^1_{\mathrm{dR}}(E_{\tau})$, where $E_{\tau} = \mathbb{C}/\langle 1, \tau \rangle_{\mathbb{Z}}$. In the cohomology of E_{τ}, the class of dz may be identified with $\tau X + Y$, where X and Y are certain generators for the integral cohomology of E_{τ}. Standard arguments show that the comparison map defined in 1.2.4 of [DFG] (in terms of resolving a locally constant sheaf) is effected via integration along chains in $\Gamma_1(N)\backslash\mathfrak{H}$.

Theorem 7.3. *For f and λ as above, suppose that λ is not a congruence prime for f in $S_k(\Gamma_1(N), \chi)$. Then*

$$\mathrm{ord}_{\lambda}\left(\frac{L_f(k-1)}{(2\pi i)^{k-1}\Omega^{(-1)^{k-1}}}\right) < 0.$$

Proof. Let Φ_f be a generator of $\mathrm{Symb}_k(\Gamma_1(N), \chi, O_{(\lambda)})^{(-1)^k}[\mathcal{I}_f]$. (Tensoring with O_{λ}, this can be viewed as the same Φ_f in Lemma 6.2.) Let θ_f be a generator for $\mathfrak{M}^{(-1)^k}_{f,B} \otimes O_{(\lambda)}$. Say that Φ_f maps to $b\theta_f$ under the natural map. Say also that $\Psi_f^{(-1)^k} = c\Phi_f$. Then (using Lemma 7.2) $f \in \mathrm{Fil}^{k-1}\mathfrak{M}_{f,\mathrm{dR}}$ maps to $bc\theta_f$, so by definition, $\omega_f^{(-1)^k} = bc$. By Lemma 7.1, $(2\pi i)^{k-1}\Omega^{(-1)^{k-1}} = bc$, up to a unit in $O_{(\lambda)}$.

The coefficient of X^{k-2} in $\Psi_f^{(-1)^k}(\{\infty\}-\{0\})$ is $(2\pi i)^{k-1}\int_0^{i\infty} f(\tau)\tau^{k-2}\,d\tau = \Gamma(k-1)L_f(k-1)$. Since $\ell > k-2$, the factor of $\Gamma(k-1)$ does not matter. By Lemma 6.2, the coefficient of X^{k-2} in $\Phi_f(\{\infty\}-\{0\})$ is congruent to $1 \pmod{\lambda}$, since $\Phi_\chi(\{\infty\}) = X^{k-2}$ and $\Phi_\chi(\{0\}) = 0$. Hence $L_f(k-1) = cu$, for some unit $u \in O_{(\lambda)}$. We find now that

$$\frac{L_f(k-1)}{(2\pi i)^{k-1}\Omega^{(-1)^{k-1}}} = \frac{cu}{bc} = \frac{u}{b},$$

so it suffices to prove that $\mathrm{ord}_\lambda(b) > 0$. But this is a direct consequence of Lemma 6.2, given that $\Phi_\chi \in \mathrm{Bound}_k(\Gamma_1(N),\chi,O_{(\lambda)})$, which is the kernel of the map from $\mathrm{Symb}_k(\Gamma_1(N),\chi,O_{(\lambda)})$ to $\mathfrak{M}_{!,B}\otimes O_{(\lambda)}$. □

8. Failure of the congruence prime condition

Suppose that λ is a congruence prime for f in $S_k(\Gamma_1(N),\chi)$. Then (if we make K big enough) there is another newform g such that $f \equiv g \pmod{\lambda}$. Let ρ_f and ρ_g be the λ-adic realisations of $\mathfrak{M}_f$ and $\mathfrak{M}_g$, considered as representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The reductions $\overline{\rho}_f$ and $\overline{\rho}_g$ are both extensions of $\mathbb{F}_\lambda(\chi^{-1})$ by $\mathbb{F}_\lambda(1-k)$. Unlike the irreducible case, we cannot be sure that they are isomorphic, but it is conceivable that it could sometimes happen (e.g. if $\dim(\mathfrak{M}_{!,\lambda}/\lambda\mathfrak{M}_{!,\lambda})[\mathfrak{m}] = 2$). Let us consider the case $\overline{\rho}_f \simeq \overline{\rho}_g$. Then ρ_f and ρ_g are both deformations of $\overline{\rho}_f$. Note that the space of $\overline{\rho}_f$ is $A[\lambda]$. Let r be minimal such that ρ_f and ρ_g are different $\pmod{\lambda^{r+1}}$. Then

$$\rho_g(\sigma) \equiv \rho_f(\sigma)(I + \lambda^r(\theta(\sigma))) \pmod{\lambda^{r+1}}$$

defines a cocycle θ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, representing a non-zero cohomology class $[\theta] \in H^1(\mathbb{Q},\mathrm{ad}^0(\overline{\rho}_f))$.

Bearing in mind the composition series for $A[\lambda]$, we have an exact sequence of $\mathbb{F}_\lambda[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-modules:

$$0 \longrightarrow \mathbb{F}_\lambda(\chi, 1-k) \longrightarrow \mathrm{ad}^0(\overline{\rho}_f) \xrightarrow{\pi} A[\lambda](k-1) \longrightarrow 0.$$

The projection π is evaluation on a generator of the submodule $\mathbb{F}_\lambda(1-k)$. This gives us $\pi_*[\theta] \in H^1(\mathbb{Q}, A[\lambda](k-1))$. Without going into laborious detail, it is plausible that sometimes this might give us a non-zero element of the Selmer group $H^1_f(\mathbb{Q}, A_\lambda(k-1))$. By finiteness of this Selmer group [Kato], this would give a non-zero element of λ-torsion in Ш$(k-1)$. In (1) (for $j = k-1$), this could cancel the contribution from $\#H^0(\mathbb{Q}, A(k-1))$, making it unnecessary for λ to occur in the denominator of $\frac{L_f(k-1)}{(2\pi i)^{k-1}\Omega^{(-1)^{k-1}}}$.

References

[AS] A. Ash, G. Stevens, Modular forms in characteristic ℓ and special values of their L-functions, *Duke Math. J.* **53** (1986), 849–868.

[BK] S. Bloch, K. Kato, L-functions and Tamagawa numbers of motives, The Grothendieck Festschrift Volume I, 333–400, Progress in Mathematics, 86, Birkhäuser, Boston, 1990.

[Ca] H. Carayol, Sur les représentations ℓ-adiques associées aux formes modulaires de Hilbert, *Ann. Sci. École Norm. Sup. (4)* **19** (1986), 409–468.

[De] P. Deligne, Valeurs de Fonctions L et Périodes d'Intégrales, *AMS Proc. Symp. Pure Math.,* Vol. 33 (1979), part 2, 313–346.

[DeSe] P. Deligne, J.-P. Serre, Formes modulaires de poids 1, *Ann. Sci. Ec. Norm. Sup.* **7** (1974), 507–530.

[DFG] F. Diamond, M. Flach, L. Guo, The Tamagawa number conjecture of adjoint motives of modular forms, *Ann. Sci. École Norm. Sup. (4)* **37** (2004), 663–727.

[DiSh] F. Diamond, J. Shurman, *A First Course in Modular Forms,* Graduate Texts in Mathematics 228, Springer, New York, 2005.

[Du1] N. Dummigan, Period ratios of modular forms, *Math. Ann.* **318** (2000), 621–636.

[Du2] N. Dummigan, Rational torsion on optimal curves, *Int. J. Number Theory* **1** (2005), 513–531.

[DSW] N. Dummigan, W. Stein, M. Watkins, Constructing elements in Shafarevich-Tate groups of modular motives, in *Number Theory and Algebraic Geometry,* M. Reid, A. Skorobogatov, eds., London Math. Soc. Lecture Note Series 303, 91–118, Cambridge University Press, 2003.

[E] B. Edixhoven, Serre's Conjecture, *in* Modular Forms and Fermat's Last Theorem, (G. Cornell, J. H. Silverman, G. Stevens, eds.), 209–242, Springer-Verlag, New York, 1997.

[FJ] G. Faltings, B. Jordan, Crystalline cohomology and $\mathrm{GL}(2,\mathbb{Q})$, *Israel J. Math.* **90** (1995), 1–66.

[GS] R. Greenberg, G. Stevens, p-adic L-functions and p-adic periods of modular forms, *Invent. Math.* **111** (1993), 407–447.

[Kato] K. Kato, p-adic Hodge theory and values of zeta functions of modular forms, *in* Cohomologies p-adiques et applications arithmétiques III, *Astérisque* **295** (2004), 117–290.

[Katz] N. M. Katz, p-adic properties of modular schemes and modular forms, in *Modular Functions of One Variable III,* Lect. Notes Math. **350,** 69–190, Springer-Verlag, 1973.

[KZ] W. Kohnen, D. Zagier, Modular Forms with Rational Periods, *in* Modular Forms (R. A. Rankin, ed.), Ellis Horwood Ltd., (1984), 197–249.

[M] Ju. I. Manin, Periods of parabolic forms and p-adic Hecke series, *Math. Sbornik* **92** (1973), 371–93.

[R] K. Ribet, A modular construction of unramified p-extensions of $\mathbb{Q}(\mu_p)$, *Invent. Math.* **34** (1976), 151–162.

[Sc] A. J. Scholl, Motives for modular forms, *Invent. Math.* **100** (1990), 419–430.

[St] W. Stein, Rationals part of the special values of the L-functions of level 1, http://modular.fas.harvard.edu/Tables/lratios_level1.html

University of Sheffield, Department of Pure Mathematics, Hicks Building, Hounsfield Road, Sheffield, S3 7RH, U.K.

E-mail address: n.p.dummigan@shef.ac.uk