CONGRUENCES OF LOCAL ORIGIN AND AUTOMORPHIC INDUCTION

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ABSTRACT. We explore the possibilities for the Galois representation ρ_g attached to a weight-one newform g to be residually reducible, i.e. for the Hecke eigenvalues to be congruent to those of a weight-one Eisenstein series. A special role is played by Eisenstein series E_1^{1,η_K} of level d_K, where η_K is the quadratic character associated with a quadratic field K, of discriminant d_K, with respect to which ρ_g is of dihedral type. We prove congruences, where the modulus divides either the class number h_K or $(p - \eta_K(p))$, and g is of level d_K in the first case, level $d_K p$ or $d_K p^2$ (according as $\eta_K(p) = 1$ or -1 respectively) in the second. We also prove analogous congruences where 1 and η_K are replaced by a newform f and its twist by η_K, and g is replaced by a Siegel cusp form of genus 2 and paramodular level, induced in some sense from a Hilbert modular form.

1. INTRODUCTION

Ramanujan's famous congruence $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$ is a congruence between the Hecke eigenvalues of a cusp form and an Eisenstein series of weight 12, both of level 1, modulo a prime divisor of $\zeta(12)/\pi^{12}$. It is easily generalised to other weights k. Moreover, keeping the Eisenstein series at level 1, we may replace the cusp form of level 1 by one of level p, and prove a congruence modulo ℓ, where $\ell > 3$ divides $p^k - 1$. Such congruences "of local origin", anticipated by Harder [Ha], were proved by Billerey and Menares [BM1], and independently in [DF]. Ramanujan's congruence is also easily generalised to the case that the Eisenstein series and the cusp form are both of level N, and the modulus divides a certain Dirichlet L-value, as in [Du1, Proposition 2.1]. The present paper is based on the 2018 Sheffield Ph.D. thesis of the second-named author [Sp], supervised by the first-named author. The original problem was to prove congruences of local origin where the Eisenstein series is of level N and the cusp form is of level Np. The technique involves finding a linear combination of old Eisenstein series at level Np whose constant term at each cusp is divisible by ℓ, then using a lemma of Deligne and Serre. Billerey and Menares independently obtained a somewhat more complete result [BM2]; see Theorem 2.9 below.

This covers weights $k \geq 2$, so, following an idea of the second-named author, we decided to see what happens for weight 1. The main result on this is Theorem 2.7 in §2, where we use a cusp form of dihedral type with respect to a quadratic field. Preceding this we show, in Propositions 2.2 and 2.5, that that is essentially the only case where we can expect to see a congruence. In Theorem 2.7 the quadratic field is imaginary, and we restrict ourselves to a brief remark on the real quadratic case.

In §3 we turn to an analogous situation where the cusp form of dihedral type is replaced by a genus 2 Siegel modular form whose associated automorphic representation of $\mathrm{GSp}_2(\mathbb{A})$ is automorphically induced from a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_K)$, where now K is a real quadratic field. The main result is Theorem 3.1. This is a short paper, and the theorems are set in context by extended remarks, which we shall not repeat here.

2. Eisenstein congruences for weight one cusp forms

The following is a theorem of Deligne and Serre [DeSe, Théorème 4.1].

Theorem 2.1. *Let $f \in S_1(\Gamma_0(N), \epsilon)$ be a normalised newform, $f = \sum_{n=1}^{\infty} a_n(f)q^n$, where $\epsilon : (\mathbb{Z}/N\mathbb{Z})^{\times} \to \mathbb{C}^{\times}$ is a Dirichlet character such that $\epsilon(-1) = -1$ (i.e. ϵ is odd). Then there exists an irreducible, continuous linear representation $\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$ such that for all primes $q \nmid N$, ρ_f is unramified at q, with $\mathrm{tr}(\rho_f(\mathrm{Frob}_q^{-1})) = a_q(f)$ and $\det(\rho_f(\mathrm{Frob}_q^{-1})) = \epsilon(q)$.*

We have an associated projective representation $\rho'_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{PGL}_2(\mathbb{C})$, whose image is necessarily either dihedral of order $2n$ for some $n \geq 2$, or isomorphic to one of the groups A_4, S_4 or S_5. Since ρ_f factors through a finite group, it may be defined over a number field, in fact over the field K_f generated over $\mathbb{Q}$ by the $a_n(f)$. For any prime divisor λ in K_f, dividing a rational prime ℓ, we may choose an invariant $\mathcal{O}_\lambda$-lattice then reduce modulo λ to obtain a residual representation $\overline{\rho}_{f,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_\lambda)$, whose composition factors are independent of the choice of lattice.

Proposition 2.2.   (1) *For any ℓ, if $\rho'_f(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) \simeq S_4$ or A_5 then $\overline{\rho}_{f,\lambda}$ is irreducible.*
       (2) *Suppose that $\ell > 2$. If $\rho'_f(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) \simeq A_4$ then $\overline{\rho}_{f,\lambda}$ is irreducible.*

Proof. For background on Schur covers and projective representations, see [HH]. Schur covers of A_4, S_4 and A_5 may be taken as $\mathrm{SL}_2(\mathbb{F}_3)$, $\mathrm{GL}_2(\mathbb{F}_3)$ and $\mathrm{SL}_2(\mathbb{F}_5)$ respectively. In each case the Schur index is 2, so these are all double covers. The projective representation ρ'_f of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, viewed as a projective representation of the finite quotient isomorphic to $G = A_4, S_4$ or A_5, lifts to a linear representation $\tilde{\rho}_f$ of the Schur cover C, and it is not difficult to show that if $\overline{\rho}_{f,\lambda}$ were reducible then so would be a reduction modulo λ of this irreducible 2-dimensional $\tilde{\rho}_f$, and we would see a congruence between the character of $\tilde{\rho}_f$ and a sum of two 1-dimensional characters of C. (Here, as elsewhere, where necessary we replace K_f by a sufficiently large finite extension, and λ by a divisor of the original λ.) A careful inspection of the character tables of $\mathrm{SL}_2(\mathbb{F}_3)$, $\mathrm{GL}_2(\mathbb{F}_3)$ and $\mathrm{SL}_2(\mathbb{F}_5)$ establishes that for $C = \mathrm{SL}_2(\mathbb{F}_5)$, for which the only 1-dimensional character is the trivial character, there are no such congruences whatsoever, and for $C = \mathrm{SL}_2(\mathbb{F}_3)$ the only such congruence is for $\ell = 2$. For $C = \mathrm{GL}_2(\mathbb{F}_3)$ the only such congruence is for $\ell = 3$, but the corresponding 2-dimensional representation of C factors not only through S_4, but through the quotient of S_4 by its Klein-4 subgroup, in which case $\rho'_f(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) \not\simeq S_4$. $\qquad\square$

Remark 2.3.   (1) It is possible a priori for $\tilde{\rho}_f$ to become reducible modulo λ only when ℓ divides the order of C, thus excluding right away all ℓ except $\{2, 3\}$ (for $\mathrm{SL}_2(\mathbb{F}_3), \mathrm{GL}_2(\mathbb{F}_3)$) and $\{2, 3, 5\}$ (for $\mathrm{SL}_2(\mathbb{F}_5)$).

(2) For $\ell > 2$, the composition factors of a 2-dimensional representation, in characteristic ℓ, of a finite group, are determined by the Brauer character. Hence reducibility of $\overline{\rho}_{f,\lambda}$ would not only imply, but be implied by, the existence of Dirichlet characters ψ, ϕ, of conductors whose product divides N, such that

$$a_q(f) \equiv \psi(q) + \phi(q) \pmod{\lambda} \quad \text{for all primes } q \nmid \ell N,$$

i.e. a congruence of Hecke eigenvalues between f and the Eisenstein series denoted $E_1^{\psi,\phi}$ in [DiSh, §4.8], which belongs to $M_1(\Gamma_0(N), \psi\phi)$.

We turn now to the remaining case, where $\rho'_f(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$ is dihedral.

Proposition 2.4. *Let $K/\mathbb{Q}$ be a quadratic field, with associated quadratic character η. Let $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathbb{C}^\times$ be a character of order $n \geq 2$, such that*

(1) *$\chi \neq \chi_\sigma$, where $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ represents the non-trivial element of $\mathrm{Gal}(K/\mathbb{Q})$ and $\chi_\sigma(g) := \chi(\sigma g \sigma^{-1}) \, \forall g \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$;*
(2) *if K is real then χ has signature $(+,-)$ at the infinite places.*

Let $\mathfrak{f}$ be the conductor of χ, and d_K the discriminant of $K/\mathbb{Q}$. Using class field theory to view χ as a Dirichlet character $\chi : (\mathcal{O}_K/\mathfrak{f})^\times \to \mathbb{C}^\times$, let $\chi_\mathbb{Q}$ be its restriction to $(\mathbb{Z}/(\mathfrak{f} \cap \mathbb{Z}))^\times$. Then

(1) *$f_\chi(z) := \sum_{\substack{\mathfrak{a} \in \mathcal{O}_K \\ \mathfrak{a} \text{ integral}}} \chi(\mathfrak{a}) q^{\mathrm{N}(\mathfrak{a})}$ is a newform in $S_1(\Gamma_0(|d_K| \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{f})), \eta\chi_\mathbb{Q})$.*
(2) *$\rho_{f_\chi} \simeq \mathrm{Ind}_{K/\mathbb{Q}}(\chi)$ is irreducible, with $\rho_{f_\chi}|_{\mathrm{Gal}(\overline{\mathbb{Q}}/K)} \simeq \chi \oplus \chi_\sigma$ and $\rho_{f_\chi}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) \simeq D_n$, the dihedral group of order $2n$, while $\rho'_{f_\chi}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) \simeq D_m$, where $m \mid n$ is the order of $\chi^{-1}\chi_\sigma$.*

Conversely, any weight 1, cuspidal normalised newform f such that $\rho'_f(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) \simeq D_m$ (with $m \geq 2$) is f_χ for some K and χ as above.

For a discussion of this, see [Se, §7]. Note that $\chi \neq \chi_\sigma$ ensures that $\mathrm{Ind}_{K/\mathbb{Q}}(\chi)$ is irreducible, while the condition on χ when K is real ensures that $\eta\chi_\mathbb{Q}$ is odd.

Proposition 2.5. *Suppose that $\rho'_f(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$ is dihedral, so we have $K/\mathbb{Q}$, η and χ as in Proposition 2.4, and $f = f_\chi \in S_1(\Gamma_0(|d_K| \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{f})), \eta\chi_\mathbb{Q})$. Suppose also that there exist Dirichlet characters ψ, ϕ, of conductors whose product divides $N := |d_K| \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{f})$, such that*

$$a_q(f) \equiv \psi(q) + \phi(q) \pmod{\lambda} \quad \text{for all primes } q \nmid \ell N.$$

Then

$$\psi \equiv \eta\phi \pmod{\lambda}.$$

Proof. For any prime q inert in K, $a_q(f_\chi) = 0$, so $\psi(q) + \phi(q) \equiv 0 \pmod{\lambda}$. Define a Dirichlet character $\alpha : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ by $\alpha = \psi/\phi$ (where each of ψ and ϕ is pulled back from its natural domain to $(\mathbb{Z}/N\mathbb{Z})^\times$). Then $\alpha(q) \equiv -1 \pmod{\lambda}$ for any prime q inert in K.

Now view the reduction $\overline{\alpha}$ as a character from $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to $\overline{\mathbb{F}}_\ell^\times$. We have just seen that if $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \backslash \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ then $\overline{\alpha}(\sigma) = -1$. For any $g \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$,

$$\overline{\alpha}(g) = \overline{\alpha}(g\sigma^{-1})\overline{\alpha}(\sigma) = (-1)(-1) = 1.$$

We see then that $\overline{\alpha} = \overline{\eta}$, so $\psi \equiv \eta\phi \pmod{\lambda}$. (Though the various characters have different conductors, we can view this as a congruence of pullbacks to $(\mathbb{Z}/N\mathbb{Z})^\times$, or of associated characters of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.) $\square$

Remark 2.6. If we assume that $\ell > 2$ then the congruence implies reducibility of $\overline{\rho}_{f,\lambda}$, and ± 1 are distinct in $\overline{\mathbb{F}}_\ell$. But the proof works even for $\ell = 2$, in which case $\overline{\eta}$ is trivial.

Theorem 2.7. *Let K be an imaginary quadratic field, with associated quadratic character η, discriminant $N = d_K$. Let h_K be the class number of $\mathcal{O}_K$, and $w = 2, 4$ or 6 the number of units in $\mathcal{O}_K$. Let ℓ be a rational prime such that $\ell \nmid w$.*

(1) *If $\ell \mid h_K$ then (for $\lambda \mid \ell$ in a sufficiently large number field) there exists a newform $f \in S_1(\Gamma_0(N), \eta')$, for some character $\eta' \equiv \eta \pmod{\lambda}$, such that*

$$a_q(f) \equiv 1 + \eta(q) \pmod{\lambda} \text{ for all primes } q \neq \ell.$$

(2) *If $\ell \nmid h_K$ but $\ell \mid (p - 1)$, for some prime p split in K, then there exists a newform $f \in S_1(\Gamma_0(Np), \eta')$, for some character $\eta' \equiv \eta \pmod{\lambda}$, such that*

$$a_q(f) \equiv 1 + \eta(q) \pmod{\lambda} \text{ for all primes } q \nmid \ell p.$$

(3) *If $\ell \nmid h_K$ but $\ell \mid (p + 1)$, for some prime p inert in K, then there exists a newform $f \in S_1(\Gamma_0(Np^2), \eta')$, for some character $\eta' \equiv \eta \pmod{\lambda}$, such that*

$$a_q(f) \equiv 1 + \eta(q) \pmod{\lambda} \text{ for all primes } q \neq \ell.$$

Proof. (1) Since $\ell \mid h_K$, there exists a character $\chi : \text{Gal}(\overline{\mathbb{Q}}/K) \to \mathbb{C}^\times$ of order ℓ, factoring through $\text{Gal}(H/K)$, where H is the Hilbert class field of K. As noted by Serre [Se, §7.2], the fact that $\mathfrak{a}\sigma(\mathfrak{a})$ is always principal implies that $\chi_\sigma = \chi^{-1}$. Since ℓ is odd, it follows that $\chi_\sigma \neq \chi$. Let $f = f_\chi$, which is a newform in $S_1(\Gamma_0(N), \eta\chi_{\mathbb{Q}})$. The values of χ are ℓ^{th} roots of unity, so $\chi(\mathfrak{a}) \equiv 1 \pmod{\lambda}$ for any ideal $\mathfrak{a}$ of $\mathcal{O}_K$. In particular, $\eta\chi_{\mathbb{Q}} \equiv \eta \pmod{\lambda}$. If q is inert in K (so $\eta(q) = -1$) then

$$a_q(f) = 0 = 1 + \eta(q),$$

while if $q = \mathfrak{q}\overline{\mathfrak{q}}$ is split in K (so $\eta(q) = 1$) then

$$a_q(f) = \chi(\mathfrak{q}) + \chi(\overline{\mathfrak{q}}) \equiv 1 + 1 = 1 + \eta(q) \pmod{\lambda}.$$

If $q = \mathfrak{q}^2$ is ramified in K (so $\eta(q) = 0$) then

$$a_q(f) = \chi(\mathfrak{q}) \equiv 1 = 1 + \eta(q) \pmod{\lambda}.$$

(2) Say $(p) = \mathfrak{p}\overline{\mathfrak{p}}$. Since $\ell \mid (p - 1)$, there exists a character $\chi : (\mathcal{O}_K/\mathfrak{p})^\times \to \mathbb{C}^\times$ of exact order ℓ. Since $\ell \nmid w$, χ kills the image of the units in all cases, and since $\ell \nmid h_K$ it lifts to a character of the ray class group of conductor $\mathfrak{p}$, and may be considered $\chi : \text{Gal}(\overline{\mathbb{Q}}/K) \to \mathbb{C}^\times$. Since χ_σ has conductor $\overline{\mathfrak{p}}$, $\chi_\sigma \neq \chi$. Letting $f = f_\chi$, which is a newform in $S_1(\Gamma_0(Np), \eta\chi_{\mathbb{Q}})$, we proceed exactly as before, except we must exclude $q = p$, since $\chi(\mathfrak{p}) + \chi(\overline{\mathfrak{p}}) = 0 + 0 \neq 2 = 1 + \eta(p)$.

(3) Since $\ell \mid (p^2 - 1) = \#(\mathcal{O}_K/(p))^\times$, we get a character $\chi : \text{Gal}(\overline{\mathbb{Q}}/K) \to \mathbb{C}^\times$ as above, of exact order ℓ but this time of conductor (p). The automorphism $\sigma \in \text{Gal}(K/\mathbb{Q})$ maps to the Frobenius element in $\text{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$, so $\chi_\sigma = \chi^p = \chi^{-1}$ since $\ell \mid (p + 1)$, so again $\chi_\sigma \neq \chi$ because ℓ is odd. Letting $f = f_\chi$, which is a newform in $S_1(\Gamma_0(Np^2), \eta\chi_{\mathbb{Q}})$, we prove the congruence as above. This time $q = p$ is OK, since $1 + \eta(p) = 0$. $\qquad\square$

Remark 2.8. (1) The right hand side of the congruence is the Hecke eigenvalue at q of the Eisenstein series $E_1^{1,\eta}$, which lives in $M_1(\Gamma_0(N), \eta)$.

(2)

$$L(1, \eta) = \frac{2\pi h_K}{w\sqrt{|d_K|}}.$$

If $L_{\{p\}}(s, \eta)$ is the Dirichlet series with the Euler factor at p omitted, then

$$L_{\{p\}}(1, \eta) = \begin{cases} \frac{2\pi h_K(1-p)}{pw\sqrt{|d_K|}} & \text{if } p \text{ is split in } K; \\ \frac{2\pi h_K(1+p)}{pw\sqrt{|d_K|}} & \text{if } p \text{ is inert in } K. \end{cases}$$

So, given that $\ell \nmid w$, the condition on ℓ is

$$\begin{cases} \mathrm{ord}_\ell\left(\frac{L(1,\eta)}{2\pi/\sqrt{|d_K|}}\right) > 0 & \text{in case (1);} \\ \mathrm{ord}_\ell\left(\frac{L_{\{p\}}(1,\eta)}{2\pi/\sqrt{|d_K|}}\right) > 0 & \text{in cases (2),(3).} \end{cases}$$

(3) For a congruence with $\psi(q) + \phi(q)$ (the Hecke eigenvalue at q of $E_1^{\psi,\phi}$) on the right hand side, where $\psi = \eta\phi$, we may just twist f_χ by ϕ, or equivalently multiply the original χ by the restriction to $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ of the Galois character associated to ϕ.

(4) In the case that K is real quadratic, $L(1, \eta)$ is not a critical value, but nonetheless we can give a sufficient condition for the existence of a congruence like that in the theorem. The order of the ray class group of conductor $\mathfrak{p}$ is $\frac{2h_K(\mathrm{Nm}(\mathfrak{p})-1)}{[U:U_1(\mathfrak{p})]}$, where U is the group of units and $U_1(\mathfrak{p})$ is the subgroup of those congruent to 1 modulo $\mathfrak{p}$. So the condition is that ℓ should divide $\frac{h_K(1-\eta(p))}{[U:U_1(\mathfrak{p})]}$. For example if $K = \mathbb{Q}(\sqrt{3})$, for which $h_K = 1$, a fundamental unit is $u = 2 + \sqrt{3}$. We find that $(u^7 - 1)$ is divisible by a prime ideal $\mathfrak{p}$ of norm 71, so $[U : U_1(\mathfrak{p})] = 7$, and since $\frac{71-1}{7}$ is still divisible by 5, we may take $\ell = 5$.

For comparison, we look briefly at the situation for higher weights.

Theorem 2.9. *Let ψ, ϕ be Dirichlet characters of conductors u, v respectively, and let $N = uv$. Let $k \geq 2$ be an integer. Suppose that $p \nmid N$ is a prime, and that $\lambda \mid \ell$ with $\ell > k + 1$ and $\ell \nmid N$. Let Σ_N be the set of primes dividing N.*

(1) *If*

$$\mathrm{ord}_\lambda\left(\frac{L_{\Sigma_N}(k, \psi\phi^{-1})}{(2\pi)^k}\right) > 0,$$

and $(N, k) \neq (1, 2)$, there exists a cuspidal eigenform $f \in S_k(\Gamma_0(N), \chi)$, with $\chi \equiv \psi\phi \pmod{\lambda}$, such that

$$a_q(f) \equiv \psi(q) + q^{k-1}\phi(q) \pmod{\lambda} \quad \text{for all primes } q \nmid \ell N.$$

(2) *If*

$$\mathrm{ord}_\lambda\left(\frac{L_{\Sigma_{Np}}(k, \psi\phi^{-1})}{(2\pi)^k}\right) > 0,$$

and $(N, k) \neq (1, 2)$, there exists a cuspidal eigenform $f \in S_k(\Gamma_0(Np), \chi)$, with $\chi \equiv \psi\phi \pmod{\lambda}$, such that

$$a_q(f) \equiv \psi(q) + q^{k-1}\phi(q) \pmod{\lambda} \quad \text{for all primes } q \nmid \ell p N.$$

This follows from work of Billerey and Menares [BM2, Theorems 1,2]. The right hand side is the Hecke eigenvalue at q of the Eisenstein series $E_k^{\psi,\phi} \in M_k(\Gamma_0(N), \psi\phi)$. Their proof involves finding an element of $M_k(\Gamma_0(Np), \psi\phi)$, in the old space for $E_k^{\psi,\phi}$, with constant terms at all cusps divisible by ℓ, lifting its reduction modulo ℓ to a cusp form, then applying the Deligne-Serre lemma [DeSe, Lemme 6.1]. In the case $(N, k) = (1, 2)$, the corresponding condition, which would be $\ell \mid (p^2 - 1)$, is strengthened to $\ell \mid (p - 1)$, following a theorem of Mazur [Maz].

3. Congruences between genus one and genus two cusp forms

The paramodular group of level N is given by

$$\Gamma^{\mathrm{para}}(N) = \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \frac{1}{N}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix} \cap \mathrm{Sp}_2(\mathbb{Q}),$$

where $\mathrm{Sp}_2(\mathbb{Q}) := \{g \in M_4(\mathbb{Q}) : {}^t gJg = J\}$, $J = \begin{pmatrix} 0_2 & -I_2 \\ I_2 & 0_2 \end{pmatrix}$. Let F be a cuspidal Hecke eigenform of weight $\rho = \mathrm{Sym}^j \otimes \det^\kappa$ for $\Gamma^{\mathrm{para}}(N)$. Then $F : \mathfrak{H}_2 \to V$, where $\mathfrak{H}_2 = \{Z \in M_2(\mathbb{C}) : {}^t Z = Z, \mathrm{Im}(Z) > 0\}$ is Siegel's upper half space of genus 2, V is the space of the representation $\rho = \mathrm{Sym}^j(\mathbb{C}^2) \otimes \det^\kappa$ of $\mathrm{GL}_2(\mathbb{C})$, and

$$F\left((AZ + B)(CZ + D)^{-1}\right) = \rho(CZ + D)(f(Z)) \ \forall \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{\mathrm{para}}(N).$$

Let the elements $T(q), T(q^2)$ of the genus-2 Hecke algebra be as in [vdG, §16] (with the scaling as following Definition 8). Let $\lambda_F(q), \lambda_F(q^2)$ be the respective eigenvalues for these operators acting on F. The spinor L-function of F is $L(s, F, \mathrm{Spin}) = \prod_{q \mathrm{\ prime}} L_q(s, F, \mathrm{Spin})$, where for primes $q \nmid N$, $L_q(s, F, \mathrm{Spin})^{-1}$

$$= 1 - \lambda_F(q)q^{-s} + (\lambda_F(q)^2 - \lambda_F(q^2) - q^{j+2\kappa-4})q^{-2s} - \lambda_F(q)q^{j+2\kappa-3-3s} + q^{2j+4\kappa-6-4s}.$$

Theorem 3.1. *Let $f \in S_k(\Gamma_0(N))$ be a normalised newform, with $k \geq 2$. Let K be a real quadratic field with discriminant d_K, narrow class number h'_K and associated quadratic character η_K. Suppose that $(d_K, N) = 1$, and that f is not monomial with respect to K (equivalently $a_q(f) \neq 0$ for some prime q inert in K).*

(1) *Choose a prime p that splits in K, $(p) = \mathfrak{p}\mathfrak{p}^\sigma$. Suppose that $p \nmid \ell N$, $\ell \nmid (p+1)$ and*

$$a_p(f)^2 \equiv p^{k-2}(p + 1)^2 \pmod{\lambda},$$

for some $\lambda \mid \ell$. Further assume that there exists a prime q, split in K, such that $a_q(f)^{h'_K} \not\equiv \left(p^{\frac{k}{2}} + p^{\frac{k}{2}-1}\right)^{h'_K} \pmod{\lambda}$. Let $\rho = \mathrm{Sym}^{k-2}(\mathbb{C}^2) \otimes \det^2$.

Then there exists a Siegel paramodular cusp form $F \in S_\rho(\Gamma^{\mathrm{para}}(N^2 d_K^2 p))$ satisfying

$$\lambda_F(q) \equiv a_q(f)(1 + \eta_K(q)) \pmod{\lambda} \quad \text{for all primes } q \nmid Nd_Kp.$$

(2) *Choose a prime p that is inert in K. Suppose that $p \nmid \ell N$, $\ell \nmid (p^2 + 1)$ and*

$$a_p(f)^2 \equiv -p^{k-2}(p - 1)^2 \pmod{\lambda},$$

for some $\lambda \mid \ell$. Further assume that there exists a prime q, split in K, such that $a_q(f)^{h'_K} \not\equiv \left(p^k + p^{k-2}\right)^{h'_K}$ (mod λ), and that $\overline{\rho}_{f,\lambda}|_{\mathrm{Gal}(\overline{\mathbb{Q}}/K)}$ is irreducible. Let $\rho = \mathrm{Sym}^{k-2}(\mathbb{C}^2) \otimes \det^2$.

Then there exists a Siegel paramodular cusp form $F \in S_\rho(\Gamma^{\mathrm{para}}(N^2 d_K^2 p^2))$ satisfying

$$\lambda_F(q) \equiv a_q(f)\left(1 + \eta_K(q)\right) \quad (\mathrm{mod} \ \lambda) \quad \textit{for all primes } q \nmid N d_K p.$$

Proof. (1) Let π_f be the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ associated to f. Since π_f is not monomial with respect to K, the base change $\pi' := \mathrm{BC}_{K/\mathbb{Q}}(\pi_f)$ is a *cuspidal* automorphic representation of $\mathrm{GL}_2(\mathbb{A}_K)$ [L][Ge, §6.1], of weight (k, k), level (N) and trivial central character. The associated λ-adic Galois representation is $\rho_{f,\lambda}|_{\mathrm{Gal}(\overline{\mathbb{Q}}/K)}$, of conductor (N).

Since p is split, the congruence may be read as

$$a_{\mathfrak{p}}(\pi')^2 \equiv \left(\mathrm{Nm}(\mathfrak{p})^{\frac{k}{2}} + \mathrm{Nm}(\mathfrak{p})^{\frac{k}{2}-1}\right)^2 \quad (\mathrm{mod} \ \lambda).$$

By a level-raising theorem of Taylor for Hilbert modular forms [T, Theorem 1], there exists an automorphic representation $\tilde{\pi}$ of $\mathrm{GL}_2(\mathbb{A}_K)$, of weight (k, k), level $(N)\mathfrak{p}$ and trivial central character, such that

$$a_{\mathfrak{q}}(\tilde{\pi}) \equiv a_{\mathfrak{q}}(\pi') \quad (\mathrm{mod} \ \lambda) \quad \text{for all primes } \mathfrak{q} \neq \mathfrak{p}.$$

Note that the conditions $a_q(f)^{h'_K} \not\equiv \left(p^{\frac{k}{2}} + p^{\frac{k}{2}-1}\right)^{h'_K}$ (mod λ) and $\ell \nmid (p+1)$ allow us to ignore the "error term" in Taylor's theorem. Strictly speaking, Taylor's theorem gives us $\tilde{\pi}$ of central character trivial modulo λ, but we can replace it by one of trivial character using Jarvis's generalisation of Carayol's Lemma to Hilbert modular forms [Ja, Theorem 4.1], which uses the irreducibility of $\overline{\rho}_{f,\lambda}|_{\mathrm{Gal}(\overline{\mathbb{Q}}/K)}$. In his notation, $B = \mathrm{GL}_{2/K}$, S is the set of infinite places together with divisors of $(N)\mathfrak{p}$, $U = U_0((N)\mathfrak{p})$, $U_1 = U_1((N)\mathfrak{p})$, r is essentially the central character, and χ is its inverse.

Since $\tilde{\pi}$ has level $(N)\mathfrak{p}$, $(\tilde{\pi})^\sigma$ has level $(N)\mathfrak{p}^\sigma$, so $(\tilde{\pi})^\sigma \neq \tilde{\pi}$. We now employ a mild generalisation of a special case of a theorem of Johnson-Leung and Roberts [JR, Main Theorem]. It is likewise an application of a theorem of Roberts [Ro, Theorem 8.6, Introduction]. The analysis at finite places (leading to paramodular level) is exactly as in [JR]. The only difference is at archimedean places. In [JR] they have a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_K)$ whose components at the infinite places are isomorphic to discrete series representations D_2 and D_{2n+2}, leading to a scalar-valued Siegel cusp form of weight $n + 2$. For us it is D_k and D_k, so the case $n = 0$ for them is the case $k = 2$ for us. To make the generalisation, we simply observe that the L-packet $\Pi(\phi(\pi_{0,\infty}))$ (in the notation of [JR, §3]) contains the limit of discrete series representation denoted $\pi_\lambda[c]$ in [Mo, p. 207], with $c = 0$ and Harish-Chandra parameter $\lambda = (\lambda_1, \lambda_2) = (k - 1, 0)$. To cover the fact that $\lambda_2 > 0$ does not hold, see [Mo, p.210, Remark (ii)]. The Blattner parameter is $(\Lambda_1, \Lambda_2) = (\lambda_1, \lambda_2) + (1, 2) = (k, 2)$. This is $(j + \kappa, \kappa)$, where the lowest K_∞-type is $\mathrm{Sym}^j(\mathbb{C}^2) \otimes \det^\kappa$, so we recover $j = k - 2$, $\kappa = 2$.

As in [JR, Main Theorem (iii)], $L(s, F, \mathrm{Spin})$ is the same as the standard L-function of $\tilde\pi$. Looking at the coefficient of q^{-s}, we could write

$$\lambda_F(q) = \begin{cases} a_{\mathfrak{q}}(\tilde\pi) + a_{\mathfrak{q}^\sigma}(\tilde\pi) & q \text{ split;} \\ 0 & q \text{ inert.} \end{cases}$$

Since $a_{\mathfrak{q}}(\tilde\pi) \equiv a_{\mathfrak{q}}(\pi') = a_q(f) \pmod{\lambda}$ and $a_{\mathfrak{q}^\sigma}(\tilde\pi) \equiv a_{\mathfrak{q}^\sigma}(\pi') = a_q(f)$ $\pmod{\lambda}$, the required congruence follows.

(2) This time for Taylor's level-raising theorem, since $\mathrm{Nm}(\mathfrak{p}) = p^2$ and since the Hecke eigenvalue for $\mathfrak{p}$ on π' is $a_p(f)^2 - 2p^{k_1}$, we need λ to divide

$$(a_p(f)^2 - 2p^{k-1})^2 - (p^k + p^{k-2})^2.$$

This factorises as

$$(a_p(f)^2 + p^{k-2}(p-1)^2)(a_p(f)^2 - p^{k-2}(p+1)^2),$$

and we have assumed that λ divides the first of these factors, so we get $\tilde\pi$ as before. We need to know that $(\tilde\pi)^\sigma \neq \tilde\pi$. If $(\tilde\pi)^\sigma = \tilde\pi$ then by a theorem of Langlands [L], $\tilde\pi$ would be in the image of the base change map, say $\tilde\pi = \mathrm{BC}_{K/\mathbb{Q}}(\pi_g)$. The level raising congruence implies that $\overline{\rho}_{g,\lambda}|_{\mathrm{Gal}\overline{\mathbb{Q}}/K} \simeq \overline{\rho}_{f,\lambda}|_{\mathrm{Gal}\overline{\mathbb{Q}}/K}$. Replacing g by a quadratic twist by η if necessary (which does not alter its base change), we may assume that $\overline{\rho}_{g,\lambda} \simeq \overline{\rho}_{f,\lambda}$. (We have used the irreducibility of $\overline{\rho}_{f,\lambda}|_{\mathrm{Gal}(\overline{\mathbb{Q}}/K)}$ again.) Since $\tilde\pi = \mathrm{BC}_{K/\mathbb{Q}}(\pi_g)$, the level of g is exactly divisible by p, whereas the level of f is not divisible by p. The (easier) necessary condition for raising the level of f at p implies that λ divides the second factor $(a_p(f)^2 - p^{k-2}(p+1)^2)$. This would imply that λ divides the difference of the two factors, i.e. $2p^{k-2}(p^2+1)$, which it does not. Hence $(\tilde\pi)^\sigma \neq \tilde\pi$, and we have everything we need to proceed as before. $\square$

Remark 3.2. (1) We may view this as an analogue of (2) and (3) of Theorem 2.7. There the ray-class character χ of conductor $\mathfrak{p}$ is congruent modulo λ to the trivial character of conductor 1, which may be viewed as a "base-change" to K of the trivial character of $\mathbb{A}^\times/\mathbb{Q}^\times$, whose value at q contributes the 1 to the right-hand-side of the congruence. Here $\tilde\pi$ of level $(N)\mathfrak{p}$ is congruent modulo λ to π' of level (N), which is the base-change to K of π_f, whose Hecke eigenvalue at q contributes to the right-hand-side of the congruence. Note that it was essential to level-raise, indeed to level-raise *after* base-changing rather than before, to ensure the condition of $\mathrm{Gal}(K/\mathbb{Q})$ non-invariance. Just as in Theorem 2.7 the L-function attached to $\mathrm{Ind}_K^{\mathbb{Q}}(\chi)$ is $L(s, f_\chi)$, here the L-function attached to $\rho_{F,\lambda} := \mathrm{Ind}_K^{\mathbb{Q}}(\rho_{\tilde\pi,\lambda})$ is $L(s, F, \mathrm{Spin})$.

(2) In (2) of Remark 2.8, we had the L-values $L(1,\eta)$ and $L_{\{p\}}(1,\eta)$, which could just as well have been described as $L_{\Sigma_N}(1,\eta)$ and $L_{\Sigma_N \cup \{p\}}(1,\eta)$, since the Euler factors at primes dividing N are trivial. The congruence there could be interpreted as reducibility of $\overline{\rho}_{f,\lambda}$, with composition factors $\mathbb{F}_\lambda$, $\mathbb{F}_\lambda(\eta)$. Combining this with the irreducibility of $\rho_{f,\lambda}$, a construction of Ribet [Ri] ensures the existence of a non-trivial extension of $\mathbb{F}_\lambda$ by $\mathbb{F}_\lambda(\eta)$, hence a non-zero element of $H^1(\mathbb{Q}, \mathrm{Hom}(\mathbb{F}_\lambda, \mathbb{F}_\lambda(\eta))) = H^1(\mathbb{Q}, \mathbb{F}_\lambda(\eta))$. Since $\rho_{f,\lambda}$ is crystalline at ℓ (if $\ell > k$) and unramified at all $q \nmid \ell p N$ (say in cases

(2) and (3)), this produces a non-zero element in the λ part of a Bloch-Kato Selmer group appearing in the numerator of a conjectural formula for $L_{\Sigma_N \cup \{p\}}(1, \eta)$, cf. [DF, Propositions 3.1, 4.2]. This accounts for the involvement of that L-value. Note that the Bloch-Kato conjecture for Dirichlet characters is actually proved, in work completed by Huber and Kings [HK, Theorem 5.4.1].

Here, the congruence may be interpreted as reducibility of $\overline{\rho}_{F,\lambda}$, with composition factors $\overline{\rho}_{f,\lambda}$, $\overline{\rho}_{f,\lambda}(\eta)$ (assuming their irreducibility). Now

$$\mathrm{Hom}(\overline{\rho}_{f,\lambda}, \overline{\rho}_{f,\lambda}(\eta)) \simeq \mathrm{ad}(\overline{\rho}_{f,\lambda})(\eta),$$

but since the image of $\rho_{F,\lambda}$ acts by symplectic similitudes, in fact we get a cocycle with values in $\mathrm{ad}^0(\overline{\rho}_{f,\lambda})(\eta)$ (trace 0 endomorphisms, twisted by η), cf. [Du2, Lemma 6.5]. Since $\mathrm{ad}^0(\rho_{f,\lambda}) \simeq \mathrm{Sym}^2 \rho_{f,\lambda}(k-1)$, we are led to consider $L_{\Sigma_N \cup \{p\}}(k, \mathrm{Sym}^2 f, \eta)$. (Note that $L(k, \mathrm{Sym}^2 f, \eta)$ is paired with $L(k-1, \mathrm{Sym}^2 f, \eta^{-1}) = L(k-1, \mathrm{Sym}^2 f, \eta)$ by the functional equation.) Indeed, it is easy to check that

$$L_p(k, \mathrm{Sym}^2(f), \eta)^{-1} = \frac{1-p}{p^{k+1}} \left(a_p(f)^2 - \left(p^{\frac{k}{2}} + p^{\frac{k}{2}-1} \right)^2 \right),$$

so the level-raising condition (in the case that p is split) implies that $\mathrm{ord}_\lambda(L_p(k, \mathrm{Sym}^2(f), \eta)^{-1}) > 0$, and is analogous to the condition $\ell \mid (p-1)$ in Theorem 2.7.

(3) When p is inert, the level-raising condition is that λ divides $L_{(p)}(1, \mathrm{ad}^0(\pi'))^{-1}$, which naturally arises from Galois deformation theory. This factors as $L_p(1, \mathrm{ad}^0(f), \eta)^{-1} L_p(1, \mathrm{ad}^0(f))^{-1}$, and to get $(\tilde{\pi})^\sigma \neq \tilde{\pi}$ we have imposed the stronger condition that λ divides the first factor. This is analogous to what happened with $p^2 - 1 = (p+1)(p-1)$ (i.e. $(1-p^{-2}) = (1-\eta(p)p^{-1})(1-p^{-1})$) in (3) of Theorem 2.7.

(4) One might ask, what is the analogue of (1) of Theorem 2.7? Given the above discussion, clearly we should find λ in the numerator of a suitably normalised $L_{\Sigma_N}(k, \mathrm{Sym}^2 f, \eta) = L_{\Sigma_N}(1, \mathrm{ad}^0(f), \eta)$, a "twisted adjoint L-value". At least in the case $N = 1$, such λ are the subject of a conjecture of Doi, Hida, and Ishii [DHI],[Gh1, §7, Conjecture 2], which says that there should be a congruence modulo λ of Hecke eigenvalues between the base change $\pi' := \mathrm{BC}_{K/\mathbb{Q}}(\pi_f)$ and some non-base change cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_K)$. We could substitute this for $\tilde{\pi}$ in the above proof, obtaining a congruence involving Hecke eigenvalues of an F of paramodular level $N^2 d_K^2$. Note again that it is crucial for it to be non-base change, to get the required $\mathrm{Gal}(K/\mathbb{Q})$ non-invariance. There is numerical support for the conjecture of Doi, Hida and Ishii, in their paper, and subject to a conjectured compatibility of periods, it is supported by a theorem of Ghate [Gh2, Theorem 5, Corollary 2] (which does not restrict to level 1).

(5) In [BD] we give a general conjecture on congruences of Hecke eigenvalues between cuspidal automorphic representations of split connected reductive groups G and representations induced from cuspidal automorphic representations of Levi subgroups M of maximal parabolic subgroups P. In §6 we work out the example where $G = \mathrm{GSp}_2$, P is the Klingen parabolic and $M \simeq \mathrm{GL}_2 \times \mathrm{GL}_1$. There we start with the representation $\pi_f \times 1$ of

$M(\mathbb{A})$. If instead we start with $\pi_f \times \eta$, we arrive at a conjecture that if $j + \kappa = k$, with $j \geq 0$ even and $\kappa \geq 4$, and if λ (for $\ell > 2k - 1$) divides $L_{\Sigma_{Nd_K}}(k - 2 + \kappa, \mathrm{Sym}^2 f, \eta)/\Omega$ (for a suitable Deligne period Ω) then there exists a genus 2 cuspidal Hecke eigenform F of weight $\mathrm{Sym}^j(\mathbb{C}^2) \otimes \det^\kappa$, associated automorphic representation π_F of $\mathrm{GSp}_2(\mathbb{A})$ unramified away from Σ_{Nd_K}, and

$$\lambda_F(q) \equiv a_q(f)\left(q^{\kappa - 2} + \eta_K(q)\right) \pmod{\lambda} \quad \text{for all primes } q \nmid Nd_K.$$

(Likewise with all three occurrences of Nd_K replaced by $Nd_K p$.) Theorem 3.1 proves a degenerate case of such a congruence, with $\kappa = 2$, which is smaller than what is allowed by the conjecture in [BD]. (For $\kappa = 2$ we have $s = 0$ in the notation of [BD], rather than the required $s > 1$.) Similarly Theorem 2.7 is related to the case $G = \mathrm{GL}_2, M \simeq \mathrm{GL}_1 \times \mathrm{GL}_1$ [BD, §5], with $k = 1$ rather than the required $k \geq 3$. Theorem 2.9 supports the conjecture for $G = \mathrm{GL}_2, M \simeq \mathrm{GL}_1 \times \mathrm{GL}_1$, while Propositions 2.2 and 2.5 show that it does not extend to $k = 1$ without severe restrictions.

(6) We have used a form F of weight $\mathrm{Sym}^{k-2}(\mathbb{C}^2) \otimes \det^2$ and level $\Gamma^{\mathrm{para}}(N^2 d_K^2 p)$, following Johnson-Leung and Roberts. We might have tried to use instead a form of level $\Gamma_0^{(2)}(Nd_k p) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{Z}) : C \equiv 0_2 \pmod{Nd_k p} \right\}$, with the same spinor L-function. Such a function was constructed by Yoshida [Y] for $k = 2$, and by Hsieh and Namikawa [HN] in general, as a theta lift from the multiplicative group $D^\times$ of a definite quaternion algebra. However, they require N to be square-free, and there is also a strong restriction on Atkin-Lehner eigenvalues to ensure that the theta lift does not vanish. The construction of Roberts uses a theta lift from $\mathrm{GO}(2,2)$ instead, and avoids these restrictions.

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