

# YOSHIDA LIFTS AND SELMER GROUPS

SIEGFRIED BÖCHERER, NEIL DUMMIGAN, AND RAINER SCHULZE-PILLOT

ABSTRACT. Let  $f$  and  $g$ , of weights  $k' > k \geq 2$ , be normalised newforms for  $\Gamma_0(N)$ , for square-free  $N > 1$ , such that, for each Atkin-Lehner involution, the eigenvalues of  $f$  and  $g$  are equal. Let  $\lambda \mid \ell$  be a large prime divisor of the algebraic part of the near-central critical value  $L(f \otimes g, \frac{k+k'-2}{2})$ . Under certain hypotheses, we prove that  $\lambda$  is the modulus of a congruence between the Hecke eigenvalues of a genus-two Yoshida lift of (Jacquet-Langlands correspondents of)  $f$  and  $g$  (vector-valued in general), and a non-endoscopic genus-two cusp form. In pursuit of this we also give a precise pullback formula for a genus-four Eisenstein series, and a general formula for the Petersson norm of a Yoshida lift.

Given such a congruence, using the 4-dimensional  $\lambda$ -adic Galois representation attached to a genus-two cusp form, we produce, in an appropriate Selmer group, an element of order  $\lambda$ , as required by the Bloch-Kato conjecture on values of L-functions.

## 1. INTRODUCTION

This paper is about congruences between modular forms, modulo large prime divisors of normalised critical values of L-functions. The first instance of this might be considered to be Ramanujan's congruence modulo 691 between the Hecke eigenvalues of the cusp form  $\Delta$  and an Eisenstein series of weight 12 for  $SL_2(\mathbb{Z})$ , the prime 691 occurring in the critical value  $\zeta(12)$ . Congruences modulo  $p$  between Eisenstein series and cusp forms (now of weight 2 and level  $p$ ) were used by Ribet [R1] to prove his converse to Herbrand's theorem. Interpreting the congruence as a reducibility modulo  $p$  of the 2-dimensional Galois representation attached to the cusp form, he used the non-trivial extension of 1-dimensional factors to construct elements of order  $p$  in the class group of  $\mathbb{Q}(\zeta_p)$ . Mazur and Wiles [MW] developed this idea further in their proof of Iwasawa's main conjecture. When Bloch and Kato [BK] proved most of their conjecture in the case of the Riemann zeta function, the Mazur-Wiles theorem was the main ingredient.

Let  $f$  and  $g$ , of weights  $k' > k \geq 2$ , be normalised newforms for  $\Gamma_0(N)$ , for square-free  $N > 1$ , such that, for each Atkin-Lehner involution, the eigenvalues of  $f$  and  $g$  are equal. Let  $\lambda \mid \ell$  be a large prime divisor of the algebraic part of the near-central critical value  $L(f \otimes g, \frac{k+k'-2}{2})$  (or equivalently of its partner  $L(f \otimes g, \frac{k+k'}{2})$ ). In this paper, we seek a congruence modulo  $\lambda$  between the Hecke eigenvalues of a Yoshida lift  $F = F_{f,g}$ , and some other genus-2 Hecke eigenform  $G$ , of the same weight  $\text{Sym}^j \otimes \det^\kappa$ , where  $j = k - 2$  and  $\kappa = 2 + \frac{k'-k}{2}$ , and level  $\Gamma_0^{(2)}(N)$ . (See §1.1 and later sections for definitions and notation.) Proposition 9.1 (and Corollary 9.2) is what we are able to prove. If  $p$  is any prime  $p \nmid \ell N$  (where  $\lambda \mid \ell$ ) and  $\mu_G(p)$  is

---

*Date:* September 28th, 2011.

*1991 Mathematics Subject Classification.* 11F46, 11F67, 11F80, 11F33, 11G40.

the eigenvalue of the Hecke operator  $T(\mathfrak{p})$  acting on  $G$ , then the congruence is

$$\mu_G(\mathfrak{p}) \equiv \mathfrak{a}_{\mathfrak{p}}(\mathfrak{f}) + \mathfrak{p}^{(k'-k)/2} \mathfrak{a}_{\mathfrak{p}}(\mathfrak{g}) \pmod{\lambda}.$$

Our proof is modelled on Katsurada's approach to proving congruences between Saito-Kurokawa lifts and non-lifts [Ka], modulo divisors of the near-central critical values of Hecke L-functions of genus-1 cuspidal eigenforms of level 1. Thus we consider a "pullback formula" for the restriction to  $\mathfrak{H}_2 \times \mathfrak{H}_2$  of a genus-4, Eisenstein series (of weight 4) to which a certain differential operator has been applied. The coefficient of  $F \otimes F$  is some constant times a value of the standard L-function of  $F$ , divided by the Petersson norm of  $F$ .

Section 6 contains a proof of the required pullback formula (17) (derived, using also (15), from the more general (9)), using differential operators from [B1] and [BSY], and taking care to determine the precise constants occurring. Section 8 contains the proof of a formula for the Petersson norm of the Yoshida lift  $F$ , generalising [BS1], which dealt with the analogous case where  $k' = k = 2$  and  $F$  is scalar-valued of weight  $\kappa = 2$ . This proof uses another, more subtle pullback formula (16), involving an Eisenstein series of genus 4 and weight 2, also provided by Section 6. The value  $L(\mathfrak{f} \otimes \mathfrak{g}, \frac{k+k'}{2})$  thereby appears as a factor in the formula for the Petersson norm of the Yoshida lift, thus introducing  $\lambda$  into a denominator in the pullback formula referred to in the previous paragraph. The congruence is then proved by some application of Hecke operators to both sides.

For this we need to know the integrality at  $\lambda$  of the left-hand-side (dealt with in Section 7), and, more problematically, that some Fourier coefficient of a canonical scaling of the Yoshida lift  $F$  is not divisible by  $\lambda$ . (At this point Katsurada was able to use an explicit formula for the Fourier coefficients of a Saito-Kurokawa lift.) What we need on Fourier coefficients of Yoshida lifts can be reduced to a weak condition on non-divisibility by  $\lambda$  of certain normalised L-values, in the case that  $N$  is prime, Atkin-Lehner eigenvalue  $\epsilon_N = -1$  and  $k/2, k'/2$  are odd, using an averaging formula from [BS5]. This condition may be checked explicitly using a formula of Gross and Zagier. In his thesis [Ji], Johnson Jia has worked out a different approach to the problem of Fourier coefficients of Yoshida lifts mod  $\lambda$ , in the scalar-valued case.

Brown [Br] used the Galois interpretation of congruences (of Hecke eigenvalues) between Saito-Kurokawa lifts and non-lifts, to confirm a prediction of the Bloch-Kato conjecture. Likewise, in the earlier sections of this paper we use congruences between Yoshida lifts and non-lifts to produce non-zero elements of  $\lambda$ -torsion in the appropriate Bloch-Kato Selmer group. (See Proposition 5.1.) The required cohomology classes come from non-trivial extensions inside the mod  $\lambda$  reduction of Weissauer's 4-dimensional Galois representation attached to  $G$ . This mod  $\lambda$  representation is reducible thanks to the congruence.

The work of Brown is easily extended to other (not necessarily near-central) critical values of  $L_f(s)$  if one assumes a conjecture of Harder [Ha, vdG] on the existence of congruences involving vector-valued genus-2 cusp forms. It is not possible likewise to extend the present work to other critical values of the tensor-product L-function using genus-2 Siegel modular forms. The problem is that we have two fixed parameters  $k'$  and  $k$ , not allowing any freedom to vary  $j$  and  $\kappa$ . This is explained in more detail at the end of [Du2].

M. Agarwal and K. Klosin, independently of us, at the suggestion of C. Skinner, worked on using congruences between Yoshida lifts and non-lifts to construct elements in Selmer groups, to support the Bloch-Kato conjecture for tensor product L-functions at the near central point [AK]. Their approach to proving such congruences is different, resulting in different conditions, and covers the scalar-valued case ( $k = 2$ ). They use a Siegel-Eisenstein series with a character, as in [Br], and take pains to avoid our assumption (in Lemma 4.1 and Proposition 5.1) that  $\lambda$  is not a congruence prime for  $f$  or  $g$ , at the cost of restricting  $k'$  to be 10 or 14.

*Acknowledgements.* We thank M. Agarwal, T. Berger, J. Bergström, J. Jia, H. Katsurada, K. Klosin, C. Poor and D. Yuen for helpful communications. We thank also M. Chida for pointing out that [We4] allows us to eliminate an unnecessary hypothesis.

**1.1. Definitions and notation.** Let  $\mathfrak{H}_n$  be the Siegel upper half plane of  $n$  by  $n$  complex symmetric matrices with positive-definite imaginary part. Let  $\Gamma^{(n)} := \mathrm{Sp}(n, \mathbb{Z}) = \mathrm{Sp}_{2n}(\mathbb{Z}) = \{M \in \mathrm{GL}_{2n}(\mathbb{Z}) : {}^t M J M = J\}$ , where  $J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$ . For  $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma^{(n)}$  and  $Z \in \mathfrak{H}_n$ , let  $M(Z) := (AZ + B)(CZ + D)^{-1}$  and  $J(M, Z) := CZ + D$ . Let  $\Gamma_0^{(n)}(N)$  be the subgroup of  $\Gamma^{(n)}$  defined by the condition  $N \mid C$ . Let  $V$  be the space of a finite-dimensional representation  $\rho$  of  $\mathrm{GL}(n, \mathbb{C})$ . A holomorphic function  $f : \mathfrak{H}_n \rightarrow V$  is said to belong to the space  $M_\rho(\Gamma_0^{(n)}(N))$  of Siegel modular forms of genus  $n$  and weight  $\rho$ , for  $\Gamma_0^{(n)}(N)$ , if

$$f(M(Z)) = \rho(J(M, Z))f(Z) \quad \forall M \in \Gamma_0^{(n)}(N), Z \in \mathfrak{H}_n.$$

In other words,  $f|M = f$  for all  $M \in \Gamma_0^{(n)}(N)$ , where  $(f|M)(Z) := \rho(J(M, Z))^{-1}f(M(Z))$  for  $M \in \mathrm{Sp}_{2n}(\mathbb{Z})$ . Such an  $f$  has a Fourier expansion

$$f(Z) = \sum_{S \geq 0} a(S)e(\mathrm{Tr}(SZ)) = \sum_{S \geq 0} a(S, f)e(\mathrm{Tr}(SZ)),$$

where the sum is over all positive semi-definite half-integral matrices, and  $e(z) := e^{2\pi iz}$ .

Denote by  $S_\rho(\Gamma_0^{(n)}(N))$ , the subspace of cusp forms, those that vanish at the boundary. They are also characterised by the condition that, for all  $M \in \mathrm{Sp}_{2n}(\mathbb{Z})$ ,  $a(S, f|M) = 0$  unless  $S$  is positive-definite. When  $\rho$  is of the special form  $\det^k \otimes \mathrm{Sym}^j(\mathbb{C}^n)$  (where  $\mathbb{C}^n$  is the standard representation of  $\mathrm{GL}_n(\mathbb{C})$ ), the Petersson inner product will be as in §2 of [Koz], and when also  $n = 2$ , the Hecke operators  $T(\mathfrak{m})$ , for  $(\mathfrak{m}, N) = 1$ , will be defined as in §2 of [Ar], replacing  $\mathrm{Sp}_4(\mathbb{Z})$  by  $\Gamma_0^{(2)}(N)$ . When  $j = 0$ , we are dealing with the usual scalar-valued Siegel cusp forms of weight  $k$ . For a Hecke eigenform  $F$ , the incomplete spinor and standard L-functions  $L^{(N)}(F, s, \mathrm{spin})$  and  $L^{(N)}(F, s, \mathrm{St})$  may be defined in terms of Satake parameters as in [An], see also §20 of [vdG].

## 2. CRITICAL VALUES OF THE TENSOR PRODUCT L-FUNCTION

Let  $f \in S_{k'}(\Gamma_0(N))$ ,  $g \in S_k(\Gamma_0(N))$  be normalised newforms (with  $k' > k \geq 2$ ),  $K$  some number field containing all the Hecke eigenvalues of  $f$  and  $g$ . Attached to  $f$  is a “premotivic structure”  $M_f$  over  $\mathbb{Q}$  with coefficients in  $K$ . Thus there are 2-dimensional  $K$ -vector spaces  $M_{f,B}$  and  $M_{f,dR}$  (the Betti and de Rham realisations)

and, for each finite prime  $\lambda$  of  $\mathcal{O}_K$ , a 2-dimensional  $K_\lambda$ -vector space  $M_{f,\lambda}$ , the  $\lambda$ -adic realisation. These come with various structures and comparison isomorphisms, such as  $M_{f,B} \otimes_K K_\lambda \simeq M_{f,\lambda}$ . See 1.1.1 of [DFG] for the precise definition of a premotivic structure, and 1.6.2 of [DFG] for the construction of  $M_f$ , which uses the cohomology, with, in general, non-constant coefficients, of modular curves, and pieces cut out using Hecke correspondences.

On  $M_{f,B}$  there is an action of  $\text{Gal}(\mathbb{C}/\mathbb{R})$ , and the eigenspaces  $M_{f,B}^\pm$  are 1-dimensional. On  $M_{f,dR}$  there is a decreasing filtration, with  $F^j$  a 1-dimensional space precisely for  $1 \leq j \leq k' - 1$ . The de Rham isomorphism  $M_{f,B} \otimes_K \mathbb{C} \simeq M_{f,dR} \otimes_K \mathbb{C}$  induces isomorphisms between  $M_{f,B}^\pm \otimes \mathbb{C}$  and  $(M_{f,dR}/F) \otimes \mathbb{C}$ , where  $F := F^1 = \dots = F^{k'-1}$ . Define  $\omega^\pm$  to be the determinants of these isomorphisms. These depend on the choice of  $K$ -bases for  $M_{f,B}^\pm$  and  $M_{f,dR}/F$ , so should be viewed as elements of  $\mathbb{C}^\times/K^\times$ . In exactly the same way there is also a premotivic structure  $M_g$ , but since  $k' > k$ , it turns out that it is the periods of  $f$  that will show up in the formula for the periods of the rank-4 premotivic structure  $M_{f \otimes g} := M_f \otimes M_g$ .

From the above properties of  $M_f$  and  $M_g$ , one easily obtains the following properties of  $M_{f \otimes g}$ . The eigenspaces  $M_{f \otimes g,B}^\pm$  are 2-dimensional. On  $M_{f \otimes g,dR}$  there is a decreasing filtration, with  $F^t$  a 2-dimensional space precisely for  $k \leq t \leq k' - 1$ . The de Rham isomorphism  $M_{f \otimes g,B} \otimes_K \mathbb{C} \simeq M_{f \otimes g,dR} \otimes_K \mathbb{C}$  induces an isomorphism between  $M_{f \otimes g,B}^\pm \otimes \mathbb{C}$  and  $(M_{f \otimes g,dR}/F') \otimes \mathbb{C}$ , where  $F' := F^k = \dots = F^{k'-1}$ . Define  $\Omega^\pm \in \mathbb{C}^\times/K^\times$  to be the determinants of these isomorphisms.

For use in the next section, we shall choose an  $\mathcal{O}_K$ -submodule  $\mathfrak{M}_{f,B}$ , generating  $M_{f,B}$  over  $K$ , but not necessarily free, and likewise an  $\mathcal{O}_K[1/S]$ -submodule  $\mathfrak{M}_{f,dR}$ , generating  $M_{f,dR}$  over  $K$ , where  $S$  is the set of primes dividing  $N(k')$ . We take these as in 1.6.2 of [DFG]. They are part of the “ $S$ -integral premotivic structure” associated to  $f$ , and are defined using integral models and integral coefficients. Actually, it will be convenient to enlarge  $S$  so that  $\mathcal{O}_K[1/S]$  is a principal ideal domain, then replace  $\mathfrak{M}_{f,B}$  and  $\mathfrak{M}_{f,dR}$  by their tensor products with the new  $\mathcal{O}_K[1/S]$ . These will now be free, as will be any submodules, and the quotients we consider. Choosing bases, and using these to calculate the above determinants, we pin down the values of  $\omega^\pm$  (up to  $S$ -units). Setting  $\mathfrak{M}_{f \otimes g,B} := \mathfrak{M}_{f,B} \otimes \mathfrak{M}_{g,B}$  and  $\mathfrak{M}_{f \otimes g,dR} := \mathfrak{M}_{f,dR} \otimes \mathfrak{M}_{g,dR}$ , similarly we pin down  $\Omega^\pm$  (up to  $S$ -units). We just have to imagine not including in  $S$  any prime we care about.

For each prime  $\lambda$  of  $\mathcal{O}_K$  (say  $\lambda \mid \ell$ ), the  $\lambda$ -adic realisation  $M_{f,\lambda}$  comes with a continuous linear action of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ . For each prime number  $p \neq \ell$ , the restriction to  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$  may be used to define a local L-factor  $[\det(I - \text{Frob}_p^{-1} p^{-s} | M_{f,\lambda}^{I_p})]^{-1}$  (which turns out to be independent of  $\lambda$ ), and the Euler product is precisely  $L_f(s)$ . (Here  $I_p$  is an inertia subgroup at  $p$ , and  $\text{Frob}_p$  is a Frobenius element reducing to the generating  $p^{\text{th}}$ -power automorphism in  $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ .) In exactly the same way we may use the Galois representation  $M_{f \otimes g,\lambda} = M_{f,\lambda} \otimes M_{g,\lambda}$  to define the tensor product L-function  $L_{f \otimes g}(s)$ . According to Deligne’s conjecture [De], for each integer  $t$  in the critical range  $k \leq t \leq k' - 1$ ,

$$L_{f \otimes g}(t)/\Omega(t) \in K,$$

where  $\Omega(t) = (2\pi i)^{2t} \Omega^{(-1)^t}$  is the Deligne period for the Tate twist  $M_{f \otimes g}(t)$ .

It is more convenient to use  $\langle f, f \rangle$  than  $\Omega^\pm$ , so we consider the relation between the two. Calculating as in (5.18) of [Hil], using Lemma 5.1.6 of [De] and the latter

part of 1.5.1 of [DFG], one recovers the well-known fact that, up to S-units,

$$(1) \quad \langle f, f \rangle = i^{k'-1} \omega^+ \omega^- c(f),$$

where  $c(f)$ , the ‘‘cohomology congruence ideal’’, is, as the cup-product of basis elements for  $\mathfrak{M}_{f,B}$ , an integral ideal. Moreover, calculating as in Lemma 5.1 of [Dul], we find that

$$\Omega^+ = \Omega^- = 2(2\pi i)^{1-k} \omega^+ \omega^-.$$

Hence Deligne’s conjecture is equivalent to

$$\frac{L_{f \otimes g}(t)}{\pi^{2t-(k-1)} \langle f, f \rangle} \in \mathbb{K}$$

(for each integer  $k \leq t \leq k' - 1$ ). This is known to be true, using Shimura’s Rankin-Selberg integral for  $L_{f \otimes g}(s)$  [Sh4]. In the next section we consider the integral refinement of Deligne’s conjecture.

### 3. THE BLOCH-KATO CONJECTURE

We shall need the elements  $\mathfrak{M}_{f,\lambda}$  of the S-integral premotivic structure, for each prime  $\lambda$  of  $\mathcal{O}_K$ . These are as in 1.6.2 of [DFG]. For each  $\lambda$ ,  $\mathfrak{M}_{f,\lambda}$  is a  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -stable  $\mathcal{O}_\lambda$ -lattice in  $M_{f,\lambda}$ . Similarly we have  $\mathfrak{M}_{g,\lambda}$ , and  $\mathfrak{M}_{f \otimes g,\lambda} := \mathfrak{M}_{f,\lambda} \otimes \mathfrak{M}_{g,\lambda}$ .

Let  $A_\lambda := M_{f \otimes g,\lambda} / \mathfrak{M}_{f \otimes g,\lambda}$ , and  $A[\lambda] := A_\lambda[\lambda]$  the  $\lambda$ -torsion subgroup. Let  $\check{A}_\lambda := \check{M}_{f \otimes g,\lambda} / \check{\mathfrak{M}}_{f \otimes g,\lambda}$ , where  $\check{M}_{f \otimes g,\lambda}$  and  $\check{\mathfrak{M}}_{f \otimes g,\lambda}$  are the vector space and  $\mathcal{O}_\lambda$ -lattice dual to  $M_{f \otimes g,\lambda}$  and  $\mathfrak{M}_{f \otimes g,\lambda}$  respectively, with the natural  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -action. Let  $A := \bigoplus_\lambda A_\lambda$ , etc.

Following [BK] (Section 3), for  $p \neq \ell$  (where  $\lambda \mid \ell$ , including  $p = \infty$ ) let

$$H_f^1(\mathbb{Q}_p, M_{f \otimes g,\lambda}(t)) = \ker(H^1(D_p, M_{f \otimes g,\lambda}(t)) \rightarrow H^1(I_p, M_{f \otimes g,\lambda}(t))).$$

Here  $D_p$  is a decomposition subgroup at a prime above  $p$ ,  $I_p$  is the inertia subgroup, and  $M_{f \otimes g,\lambda}(t)$  is a Tate twist of  $M_{f \otimes g,\lambda}$ , etc. The cohomology is for continuous cocycles and coboundaries. For  $p = \ell$  let

$$H_f^1(\mathbb{Q}_\ell, M_{f \otimes g,\lambda}(t)) = \ker(H^1(D_\ell, M_{f \otimes g,\lambda}(t)) \rightarrow H^1(D_\ell, M_{f \otimes g,\lambda}(t) \otimes_{\mathbb{Q}_\ell} B_{\text{crys}})).$$

(See Section 1 of [BK] or §2 of [Fo1] for the definition of Fontaine’s ring  $B_{\text{crys}}$ .) Let  $H_f^1(\mathbb{Q}, M_{f \otimes g,\lambda}(t))$  be the subspace of those elements of  $H^1(\mathbb{Q}, M_{f \otimes g,\lambda}(t))$  that, for all primes  $p$ , have local restriction lying in  $H_f^1(\mathbb{Q}_p, M_{f \otimes g,\lambda}(t))$ . There is a natural exact sequence

$$0 \longrightarrow \mathfrak{M}_{f \otimes g,\lambda}(t) \longrightarrow M_{f \otimes g,\lambda}(t) \xrightarrow{\pi} A_\lambda(t) \longrightarrow 0.$$

Let  $H_f^1(\mathbb{Q}_p, A_\lambda(t)) = \pi_* H_f^1(\mathbb{Q}_p, M_{f \otimes g,\lambda}(t))$ . Define the  $\lambda$ -Selmer group  $H_f^1(\mathbb{Q}, A_\lambda(t))$  to be the subgroup of elements of  $H^1(\mathbb{Q}, A_\lambda(t))$  whose local restrictions lie in  $H_f^1(\mathbb{Q}_p, A_\lambda(t))$  for all primes  $p$ . Note that the condition at  $p = \infty$  is superfluous unless  $\ell = 2$ . Define the Shafarevich-Tate group

$$\text{III}(t) = \bigoplus_\lambda \frac{H_f^1(\mathbb{Q}, A_\lambda(t))}{\pi_* H_f^1(\mathbb{Q}, M_{f \otimes g,\lambda}(t))}.$$

Tamagawa factors  $c_p(t)$  may be defined as in 11.3 of [Fo2] (where the notation is  $\text{Tam}^0 \dots$ ). The  $\lambda$  part (for  $\ell \neq p$ ) is trivial if  $A_\lambda^{1p}$  is divisible (for example if  $p \nmid N$ ). The following is equivalent to the relevant cases of the Fontaine-Perrin-Riou extension of the Bloch-Kato conjecture to arbitrary weights (i.e. not just

points right of the centre) and not-necessarily-rational coefficients. (This follows from 11.4 of [Fo2].) Note that by “#” we really mean the Fitting ideal.

**Conjecture 3.1.** *Suppose that  $k \leq t \leq k' - 1$ . Then we have the following equality of fractional ideals of  $\mathcal{O}_K[1/S]$ :*

$$(2) \quad \frac{L_{f \otimes g}(t)}{\Omega(t)} = \frac{\prod_{p \leq \infty} c_p(t) \# \text{III}(t)}{\#H^0(\mathbb{Q}, A(t)) \# H^0(\mathbb{Q}, \check{A}(1-t))}.$$

In other words,

$$(3) \quad \frac{L_{f \otimes g}(t)}{\pi^{2t-(k-1)} \langle f, f \rangle} = \frac{\prod_{p \leq \infty} c_p(t) \# \text{III}(t)}{\#H^0(\mathbb{Q}, A(t)) \# H^0(\mathbb{Q}, \check{A}(1-t)) c(f)}.$$

Let  $f = \sum a_n(f) q^n$ . Let  $\rho_f : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Aut}(M_{f,\lambda})$  be the 2-dimensional  $\lambda$ -adic Galois representation attached to  $f$ . Let  $\bar{\rho}_f$  be its reduction  $(\text{mod } \lambda)$ , which is unambiguously defined if it is irreducible. Likewise for  $\rho_g$  and  $\bar{\rho}_g$ .

**Lemma 3.2.** (1) *Suppose that  $\bar{\rho}_f$  and  $\bar{\rho}_g$  are irreducible, that  $\ell > k'$  and  $\ell \nmid N$ . Suppose (for some  $p \parallel N$ ) that there is no normalised newform  $h$  of level dividing  $N/p$  and trivial character, of weight  $k'$  with  $a_q(h) \equiv a_q(f) \pmod{\lambda}$  for all primes  $q \nmid \ell N$ , or of weight  $k$  with  $a_q(h) \equiv a_q(g) \pmod{\lambda}$  for all primes  $q \nmid \ell N$ . Then the  $\lambda$  part of  $c_p(t)$  is trivial (for any  $t$ ).*

(2) *If  $\lambda \mid \ell$  with  $\ell \nmid N$  and  $\ell > k' + k - 1$  then the  $\lambda$  part of  $c_\ell(t)$  is trivial (for any  $t$ ).*

*Proof.* (1) Applying a level-lowering theorem (Theorem 1.1 of [Di], see also [R2, R3]),  $\bar{\rho}_f$  and  $\bar{\rho}_g$  are both ramified at  $p$ . However, since  $p \parallel N$ , the action of  $I_p$  on each of  $M_{f,\lambda}$  and  $M_{g,\lambda}$  is unipotent, by Theorem 7.5 of [La], as recalled in Theorem 4.2.7(3)(b) of [Hi2], for a convenient reference. It follows that both  $\bar{\rho}_f \otimes \bar{\rho}_g$  and  $\rho_f \otimes \rho_g$  have  $I_p$ -fixed subspace of dimension precisely 2, hence that  $A_\lambda^{I_p}$  is divisible. As noted above, this implies that the  $\lambda$ -part of  $c_p(t)$  is trivial.

(2) It follows from Lemma 5.7 of [DFG] (whose proof relies on an application, at the end of Section 2.2, of the results of [Fa]) that  $\mathfrak{M}_{f \otimes g, \lambda}$  is the  $\mathcal{O}_\lambda[\text{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)]$ -module associated to the filtered  $\phi$ -module  $\mathfrak{M}_{f \otimes g, \text{dR}} \otimes \mathcal{O}_\lambda$  (identified with the crystalline realisation) by the functor they call  $\mathbb{V}$ . (This property is part of the definition of an  $S$ -integral premotivic structure given in Section 1.2 of [DFG].) Given this, the lemma follows from Theorem 4.1(iii) of [BK]. (That  $\mathbb{V}$  is the same as the functor used in Theorem 4.1 of [BK] follows from the first paragraph of 2(h) of [Fa].)

□

**Corollary 3.3.** *Suppose that  $N$  is square-free. Assume the conditions of Lemma 3.2(1), for all primes  $p \mid N$ , and of Lemma 3.2(2), and also that (for some  $k \leq t \leq k' - 1$ )*

$$\text{ord}_\lambda \left( \frac{L_{f \otimes g}(t)}{\pi^{2t-(k-1)} \langle f, f \rangle} \right) > 0.$$

*Then the Bloch-Kato conjecture predicts that  $\text{ord}_\lambda(\# \text{III}(t)) > 0$ , so predicts that the Selmer group  $H_f^1(\mathbb{Q}, A_\lambda(t))$  is non-trivial.*

The goal of this paper is to construct (under further hypotheses) a non-zero element of  $H_f^1(\mathbb{Q}, A_\lambda(t))$ , in the case that  $t$  is the near-central point  $t = \frac{k'+k-2}{2}$ .

**Lemma 3.4.** *If  $\ell \nmid N$ ,  $\ell > k' - 1$  and  $k < t < k' - 1$  then the  $\lambda$ -parts of  $\#H^0(\mathbb{Q}, A(t))$  and  $\#H^0(\mathbb{Q}, \check{A}(1 - t))$  are trivial.*

*Proof.* If not, then either  $A[\lambda](t)$  or  $\check{A}[\lambda](1 - t)$  would have a trivial composition factor. The composition factors of  $\bar{\rho}_f|_{I_\ell}$  are either  $\chi^0, \chi^{1-k}$  (in the ordinary case, with  $\chi$  the cyclotomic character) or  $\psi^{1-k}, \psi^{\ell(1-k)}$  (in the non-ordinary case, with  $\psi$  a fundamental character of level 2). This follows from theorems of Deligne and Fontaine, which are Theorems 2.5 and 2.6 of [Ed]. Noting that  $\psi$  has order  $\ell^2 - 1$ , with  $\psi^{\ell+1} = \chi$ , the composition factors of  $(\bar{\rho}_f \otimes \bar{\rho}_g)|_{I_\ell}$  are of the form  $\psi^a, \psi^b, \psi^c, \psi^d$ , with  $1 - \ell^2 < a, b, c, d \leq 0$  and each of  $a, b, c, d$  congruent to either  $0, 1 - k, 1 - k'$  or  $2 - k - k'$  (mod  $\ell$ ). Twisting by  $t$  is the same as multiplying by  $\psi^{(\ell+1)t}$ . This exponent is congruent to  $t$  (mod  $\ell$ ), and  $k < t < k' - 1$ . Adding to this the possible values for  $a, b, c, d$  (mod  $\ell$ ) can never produce 0 or 1. Hence neither  $A[\lambda](t)$  nor  $\check{A}[\lambda](1 - t)$  can have a trivial composition factor (even when restricted to  $I_\ell$ ).  $\square$

#### 4. A 4-DIMENSIONAL GALOIS REPRESENTATION

Let  $f, g$  be as in §§2,3, both of exact level  $N > 1$ . Let  $\lambda \mid \ell$  be a divisor of  $\frac{L_{f \otimes g}(t)}{\pi^{2t - (k-1)\langle f, f \rangle}}$ , with  $\ell \nmid N(k'!) and  $t = (k' + k - 2)/2$ . Now suppose that  $f$  and  $g$  have the same Atkin-Lehner eigenvalues for each  $p \mid N$ , and let  $F_{f, g}$  be some genus-2 Yoshida lift associated with a factorisation  $N = N_1 N_2$ , as in §8 below. (It is of type  $\text{Sym}^j \otimes \det^\kappa$ , with  $j = k - 2, \kappa = 2 + \frac{k' - k}{2}$ . Note that  $j + 2\kappa - 3 = k' - 1$ .)$

Suppose that there is a cusp form  $G$  for  $\Gamma_0^{(2)}(N)$ , an eigenvector for all the local Hecke algebras at  $p \nmid N$ , not itself a Yoshida lift of the same  $f$  and  $g$ , such that there is a congruence (mod  $\lambda$ ) of all Hecke eigenvalues (for  $p \nmid N$ ) between  $G$  and  $F_{f, g}$ . In particular, if  $\mu_G(p)$  is the eigenvalue for  $T(p)$  on  $G$  (defined as in §2.1 of [Ar], replacing  $\text{Sp}_4(\mathbb{Z})$  by  $\Gamma_0^{(2)}(N)$ ), then

$$(4) \quad \mu_G(p) \equiv \alpha_p(f) + p^{(k'-k)/2} \alpha_p(g) \pmod{\lambda}, \text{ for all } p \nmid N.$$

Under certain additional hypotheses, we prove in §9 below, the existence of such a  $G$ . (We enlarge  $K$  if necessary, to contain the Hecke eigenvalues of  $G$ .)

Let  $\Pi_G$  be an automorphic representation of  $\text{GSp}_4(\mathbb{A})$  associated to  $G$  as in 3.2 of [Sc] and 3.5 of [AS]. (This  $\Pi_G$  is not necessarily uniquely determined by  $G$ , but its local components at  $p \nmid N$  are.) By Theorem I of [We2], there is an associated continuous, linear representation

$$\rho_G : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_4(\bar{\mathbb{Q}}_\ell).$$

By enlarging  $K$  if necessary, we may assume that it takes values in  $\text{GL}_4(K_\lambda)$ .

**Lemma 4.1.** *Suppose that there exists a  $G$  as above. Suppose also that  $\lambda$  is not a congruence prime for  $f$  in  $S_{k'}(\Gamma_0(N))$  or  $g$  in  $S_k(\Gamma_0(N))$ , that  $\ell > k'$ , and that  $\bar{\rho}_f$  and  $\bar{\rho}_g$  are irreducible representations of  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ .*

- (1)  $\Pi_G$  is not a weak endoscopic lift.
- (2)  $\Pi_G$  is not CAP.

By  $\lambda$  not being a congruence prime for  $f$  in  $S_{k'}(\Gamma_0(N))$ , we mean that there does not exist a different Hecke eigenform  $h \in S_{k'}(\Gamma_0(N))$ , and a prime  $\lambda'$  dividing  $\lambda$  in a sufficiently large extension, such that  $\alpha_p(h) \equiv \alpha_p(f) \pmod{\lambda'}$  for all primes  $p \nmid \ell N$ .

*Proof.* (1) If  $\Pi_G$  were a weak endoscopic lift then there would have to exist newforms  $f' \in S_{k'}(\Gamma_0(N))$ ,  $h \in S_k(\Gamma_0(N))$  such that  $\mu_G(\mathfrak{p}) = \mathfrak{a}_{\mathfrak{p}}(f') + \mathfrak{p}^{(k'-k)/2}\mathfrak{a}_{\mathfrak{p}}(h)$  for almost all primes  $\mathfrak{p}$ . (See the introduction of [We2] for a precise definition of weak endoscopic lift, and (3) of Hypothesis A of [We2] for this consequence.) We have then

$$\mathfrak{a}_{\mathfrak{p}}(f') + \mathfrak{p}^{(k'-k)/2}\mathfrak{a}_{\mathfrak{p}}(h) \equiv \mathfrak{a}_{\mathfrak{p}}(f) + \mathfrak{p}^{(k'-k)/2}\mathfrak{a}_{\mathfrak{p}}(g) \pmod{\lambda},$$

for almost all primes  $\mathfrak{p}$ . Consequently, using  $\ell > 4$  and the Brauer-Nesbitt theorem,

$$\bar{\rho}_f \oplus \bar{\rho}_g((k-k')/2) \simeq \bar{\rho}_{f'} \oplus \bar{\rho}_h((k-k')/2).$$

Now  $\bar{\rho}_{f'}$  could not be isomorphic to  $\bar{\rho}_g((k-k')/2)$ , since the restrictions to  $I_{\ell}$  give different characters (using  $\ell > k'$ ). The only way to reconcile the two sides of the above isomorphism is for  $\bar{\rho}_f \simeq \bar{\rho}_{f'}$ . Given that  $\lambda$  is not a congruence prime for  $f$  in  $S_{k'}(\Gamma_0(N))$ , we must have  $f' = f$ , and similarly  $h = g$ . It follows from (4) and (6) of Hypothesis A of [We2] that  $\Pi_G$  must be associated to some Yoshida lift  $F'_{f,g}$  of  $f$  and  $g$ . (Those  $\mathfrak{p} \mid N$  for which the local component is  $\Pi_{\mathfrak{v}}^+$  rather than  $\Pi_{\mathfrak{v}}^-$  are the divisors of  $N_1$ .) By (6) of Hypothesis A of [We2], the multiplicity of  $\Pi_G$  in the discrete spectrum is one. By Lemmes 1.2.8 and 1.2.10 of [SU], the local representation  $\Pi_{\mathfrak{p}}$  of  $\mathrm{GSp}(4, \mathbb{Q}_{\mathfrak{p}})$ , for  $\mathfrak{p} \mid N$ , is that labelled VIa in [Sc]. By Table 3 of [Sc], the spaces of  $\Gamma_0^{(2)}(\mathbb{Z}_{\mathfrak{p}})$ -fixed vectors in  $\Pi_{\mathfrak{p}}$  are 1-dimensional. It follows that (up to scaling),  $G = F'_{f,g}$ , contrary to hypothesis.

(2) By Corollary 4.5 of [PS],  $\Pi_G$  could only be CAP for a Siegel parabolic subgroup, but then, as on p.74 of [We2], we would have  $k = 2$  and

$$\mu_G(\mathfrak{p}) = \mathfrak{a}_{\mathfrak{p}}(f') + \chi(\mathfrak{p})\mathfrak{p}^{k'/2} + \chi(\mathfrak{p})\mathfrak{p}^{(k'/2)-1},$$

for some newform  $f' \in S_{k'}(\Gamma_0(N))$  and  $\chi$  a quadratic or trivial character. This is incompatible with  $\mu_G(\mathfrak{p}) \equiv \mathfrak{a}_{\mathfrak{p}}(f) + \mathfrak{p}^{(k'-k)/2}\mathfrak{a}_{\mathfrak{p}}(g) \pmod{\lambda}$  and the irreducibility of  $\bar{\rho}_f$  and  $\bar{\rho}_g$ .  $\square$

Note that the proof of Hypothesis A (on which Theorem I also depends) is not in [We2], but has now appeared in [We3].

**Lemma 4.2.** *Let  $G$  be as in Lemma 4.1. Then the representation  $\rho_G$  is irreducible.*

*Proof.* Suppose that  $\rho_G$  is reducible. It cannot have any 1-dimensional composition factor, since  $\bar{\rho}_G$  has 2-dimensional irreducible composition factors  $\bar{\rho}_f$  and  $\bar{\rho}_g((k-k')/2)$ . (The factors are well-defined, even though  $\bar{\rho}_G$  isn't.) Looking at the list, in 3.2.6 of [SU], of possibilities for the composition factors of  $\rho_G$ , we must be in Cas B, (iv) or (v). But as in 3.2.6 of [SU],  $\Pi_G$  would be CAP in one case, a weak endoscopic lift in the other, and both of these are ruled out by Lemma 4.1.  $\square$

Let  $V$ , a 4-dimensional vector space over  $K_{\lambda}$ , be the space of the representation  $\rho_G$ . Choose a  $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -invariant  $\mathcal{O}_{\lambda}$ -lattice  $T$  in  $V$ , and let  $W := V/T$ . Let  $\bar{\rho}_G$  be the representation of  $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  on  $W[\lambda] \simeq T/\lambda T$ . This depends on the choice of  $T$ , but we may choose  $T$  in such a way that  $\bar{\rho}_G$  has  $\bar{\rho}_g((k-k')/2)$  as a submodule and  $\bar{\rho}_f$  as a quotient. Assume that this has been done.



**Lemma 4.3.**  $T$  may be chosen in such a way that furthermore  $\bar{\rho}_f$  is not a submodule of  $\bar{\rho}_G$ , i.e. so that the extension of  $\bar{\rho}_f$  by  $\bar{\rho}_g((k-k')/2)$  is not split.

*Proof.* We argue as in the proof of Proposition 2.1 of [R1]. Choose an  $\mathcal{O}_\lambda$ -basis for  $T$ , so that  $\rho_G(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})) \subset \text{GL}_4(\mathcal{O}_\lambda)$ . Assuming the lemma is false, we prove by induction that for all  $i \geq 1$  there exists  $M_i = \begin{pmatrix} I_2 & S_i \\ 0_2 & I_2 \end{pmatrix} \in \text{GL}_4(\mathcal{O}_\lambda)$  such that  $M_i \rho_G(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})) M_i^{-1}$  consists of matrices of the form  $\begin{pmatrix} A & \lambda^i B \\ \lambda C & D \end{pmatrix}$ , with  $A, B, C, D \in M_2(\mathcal{O}_\lambda)$ , and with  $S_i \equiv S_{i-1} \pmod{\lambda^{i-1}}$ . Then letting  $S = \lim S_i$  and  $M = \begin{pmatrix} I_2 & S \\ 0_2 & I_2 \end{pmatrix}$ ,  $M \rho_G(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})) M^{-1}$  consists of matrices of the form  $\begin{pmatrix} A & 0_2 \\ \lambda C & D \end{pmatrix}$ , contradicting the irreducibility of  $\rho_G$ .

By assumption,  $\bar{\rho}_f$  is a submodule of  $\bar{\rho}_G$  (i.e.  $\bar{\rho}_G$  is semi-simple), so we have  $M_1$ . This is the base step. Now suppose that we have  $M_i$ . We must try to produce  $M_{i+1}$ . Let  $P = \begin{pmatrix} I_2 & 0_2 \\ 0_2 & \lambda I_2 \end{pmatrix}$ . Then  $P^i M_i \rho_G(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})) M_i^{-1} P^{-i}$  consists of matrices of the form  $\begin{pmatrix} A & B \\ \lambda^{i+1} C & D \end{pmatrix}$ . Now let  $U$  be a matrix of the form  $\begin{pmatrix} I_2 & B' \\ 0_2 & I_2 \end{pmatrix}$  such that  $U P^i M_i \rho_G(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})) M_i^{-1} P^{-i} U^{-1}$  consists of matrices of the form  $\begin{pmatrix} \tilde{A} & \lambda \tilde{B} \\ \lambda^{i+1} \tilde{C} & \tilde{D} \end{pmatrix}$ . This exists because we are assuming that not only  $\bar{\rho}_G$ , but any other reduction with submodule  $\bar{\rho}_g((k-k')/2)$ , is semi-simple. Now just let  $M_{i+1} = P^{-i} U P^i M_i$ . Note that since  $P^{-i} U P^i = \begin{pmatrix} I_2 & \lambda^i B' \\ 0_2 & I_2 \end{pmatrix}$ , it is clear that  $M_{i+1}$  is of the form  $\begin{pmatrix} I_2 & S_{i+1} \\ 0_2 & I_2 \end{pmatrix}$ , with  $S_{i+1} \equiv S_i \pmod{\lambda^i}$ .  $\square$

We remark that, though the first  $T$  chosen may give semi-simple  $\bar{\rho}_G$ , the lemma shows that there will be another choice that gives a non-trivial extension. Compare with the situation for 5-torsion on elliptic curves in the isogeny class of conductor 11.

## 5. A NON-ZERO ELEMENT IN A BLOCH-KATO SELMER GROUP

Let  $G$  be as in the previous section. Then by Lemma 4.3,  $\bar{\rho}_G$  is a non-trivial extension of  $\bar{\rho}_f$  by  $\bar{\rho}_g((k-k')/2)$ :

$$0 \longrightarrow \bar{\rho}_g((k-k')/2) \longrightarrow \bar{\rho}_G \longrightarrow \bar{\rho}_f \longrightarrow 0.$$

Applying  $\text{Hom}_{\mathbb{F}_\lambda}(\bar{\rho}_f, -)$  to the exact sequence, and pulling back the inclusion of the trivial module in  $\text{Hom}_{\mathbb{F}_\lambda}(\bar{\rho}_f, \bar{\rho}_f)$ , we get a non-trivial extension of the trivial module by  $\text{Hom}(\bar{\rho}_f, \bar{\rho}_g((k-k')/2))$ . Thus we get a non-zero class in  $H^1(\mathbb{Q}, \text{Hom}_{\mathbb{F}_\lambda}(\bar{\rho}_f, \bar{\rho}_g((k-k')/2)))$ , in the standard way. (Lifting the identity to a section  $s \in \text{Hom}_{\mathbb{F}_\lambda}(\bar{\rho}_f, \bar{\rho}_G)$ , a representing cocycle is  $g \mapsto g.s - s$ , where  $(g.s)(x) = g(s(g^{-1}(x)))$ .)

Now the dual of  $\bar{\rho}_f$  is  $\bar{\rho}_f(k'-1)$ , so

$$\text{Hom}_{\mathbb{F}_\lambda}(\bar{\rho}_f, \bar{\rho}_g((k-k')/2)) \simeq \bar{\rho}_f(k'-1) \otimes \bar{\rho}_g((k-k')/2) \simeq \bar{\rho}_f \otimes \bar{\rho}_g((k'+k-2)/2).$$

In the notation of §3, this is  $A[\lambda]((k'+k-2)/2)$ . So we have a non-zero class  $c \in H^1(\mathbb{Q}, A[\lambda]((k'+k-2)/2))$ . By Lemma 3.4,  $H^0(\mathbb{Q}, A_\lambda((k'+k-2)/2))$  is trivial, so we get a non-zero class  $d \in H^1(\mathbb{Q}, A_\lambda((k'+k-2)/2))$ , the image of  $c$  under the map induced by inclusion.

**Proposition 5.1.** *Let  $f \in S_{k'}(\Gamma_0(N))$ ,  $g \in S_k(\Gamma_0(N))$  be normalised newforms of square-free level  $N > 1$ , with  $k' > k \geq 2$ . Suppose that at each prime  $p \mid N$ ,  $f$  and  $g$  share the eigenvalue of the Atkin-Lehner involution. Let  $\lambda \mid \ell$  be a divisor of  $\frac{L_{f \otimes g}((k'+k-2)/2)}{\pi^{k'-1}(f,f)}$ , with  $\ell \nmid N$  and  $\ell > \frac{3k'+k-2}{2}$ . Suppose also that  $\lambda$  is not a congruence prime for  $f$  in  $S_{k'}(\Gamma_0(N))$  or  $g$  in  $S_k(\Gamma_0(N))$ , and that  $\bar{\rho}_f$  and  $\bar{\rho}_g$  are irreducible representations of  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ . Assume, for each  $p \mid N$ , the conditions of Lemma 3.2(1). Finally, suppose that there exists  $G \in S_p(\Gamma_0^{(2)}(N))$  as in the second paragraph of §4. Then the Bloch-Kato Selmer group  $H_f^1(\mathbb{Q}, A_\lambda((k'+k-2)/2))$  is non-zero.*

**Remark 5.2.** *Note that Corollary 9.2 gives sufficient conditions for the existence of  $G$ .*

*Proof.* We will show that the non-zero element  $d \in H^1(\mathbb{Q}, A_\lambda((k'+k-2)/2))$  satisfies  $\text{res}_p(d) \in H_f^1(\mathbb{Q}_p, A_\lambda((k'+k-2)/2))$  for each prime  $p$ .

- (1) If  $p \nmid \ell N$  then  $\rho_G|_{I_p}$  is trivial, so certainly

$$0 \longrightarrow \bar{\rho}_g((k-k')/2)|_{I_p} \longrightarrow \bar{\rho}_G|_{I_p} \longrightarrow \bar{\rho}_f|_{I_p} \longrightarrow 0$$

splits, showing that  $\text{res}_p(d) \in \ker(H^1(\mathbb{Q}_p, A[\lambda]((k'+k-2)/2)) \rightarrow H^1(I_p, A[\lambda]((k'+k-2)/2)))$ , hence that  $\text{res}_p(d) \in \ker(H^1(\mathbb{Q}_p, A_\lambda((k'+k-2)/2)) \rightarrow H^1(I_p, A_\lambda((k'+k-2)/2)))$ . Since  $A_\lambda^{I_p}$  is divisible (in this case the whole of  $A_\lambda$ ), this shows that  $\text{res}_p(d) \in H_f^1(\mathbb{Q}_p, A_\lambda((k'+k-2)/2))$ , as in Lemma 7.4 of [Br].

- (2) If  $p = \ell$  then we may prove  $\text{res}_p(d) \in H_f^1(\mathbb{Q}_p, A_\lambda((k'+k-2)/2))$  just as in Lemma 7.2 of [Du1]. Since  $\ell \nmid N$ ,  $\rho_G|_{D_\ell}$  is crystalline; see Theorem 3.2(ii) of [U1], which refers to [Fa] and [CF]. It is for this case that we need the condition  $\ell > \frac{3k'+k-2}{2}$ . This  $\frac{3k'+k-2}{2}$  arises as the span of the “weights”  $\{1-k', 0\}$  of  $\bar{\rho}_f^*$  and  $\{(k'-k)/2, (k'+k-2)/2\}$  of  $\bar{\rho}_g((k-k')/2)$ . See the proof of Lemma 7.2 of [Du1] for comparison.

- (3) Now consider the case that  $p \mid N$ . As in the proof of Lemma 3.2(1), the action of  $I_p$  on  $\mathfrak{M}_{f,\lambda}/\lambda\mathfrak{M}_{f,\lambda}$  and  $\mathfrak{M}_{g,\lambda}/\lambda\mathfrak{M}_{g,\lambda}$  is non-trivial and unipotent. Hence we may choose a basis for  $W[\lambda]$  (notation as in the previous section) such that for any  $\sigma \in I_p$ ,  $\bar{\rho}_G(\sigma)$  is represented by  $\exp(\mathfrak{t}_\ell(\sigma)\tilde{N})$ , with  $\mathfrak{t}_\ell : I_p \rightarrow \mathbb{Z}_\ell(1)$  the standard tamely ramified character and  $\tilde{N}$  of the form  $\tilde{N} = \begin{pmatrix} A & B \\ 0_2 & A \end{pmatrix}$ , with  $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ . (Note that  $A$  plays the rôle of  $\tilde{N}$  for the 2-dimensional representations  $\bar{\rho}_f|_{I_p}$  and  $\bar{\rho}_g|_{I_p}$ .) By Theorem 2.2.5(1) of [GT],  $\tilde{N}^2 = 0$ . To see that the conditions of that theorem are satisfied here, firstly  $\rho_G$  is irreducible by Lemma 4.2, secondly  $\rho_G$  is symplectic by Theorem 2 of [We4]. Lastly, given that the local component  $\Pi_p$  of  $\Pi_G$  has a non-zero vector fixed by  $\Gamma_0^{(2)}(\mathbb{Z}_p)$  but none fixed by  $\text{GSp}_4(\mathbb{Z}_p)$ , an inspection of Table 3 in [Sc] reveals that it is always the case that either the subspace of  $\Pi_p$  fixed by the Siegel parahoric  $\Gamma_0^{(2)}(\mathbb{Z}_p)$ , or that fixed by a Klingen parahoric, is 1-dimensional. (Note that if  $\Pi_p$  had a non-zero vector fixed by  $\text{GSp}_4(\mathbb{Z}_p)$  then, by Theorem I of [We2],  $\rho_G$  would be unramified at  $p$ , contrary to  $\bar{\rho}_G$  having  $\bar{\rho}_f$  as a quotient.)

Since  $\tilde{N}^2 = 0$ ,  $B$  must be of the form  $B = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ . Writing elements of  $\text{Hom}_{\mathbb{F}_\lambda}(\bar{\rho}_f, \bar{\rho}_g((k-k')/2))$  as 2-by-2 matrices in the obvious way, a short

calculation shows that  $c|_{I_p}$  is represented by the cocycle  $\sigma \mapsto \begin{pmatrix} 0 & t_\ell(\sigma)\mathfrak{b} \\ 0 & 0 \end{pmatrix}$ , which is the coboundary  $\sigma \mapsto \sigma \left( \begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{b} \end{pmatrix} \right) - \begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{b} \end{pmatrix}$ . Since  $c|_{I_p} = 0$ ,  $d|_{I_p} = 0$ . As already noted in the proof of Lemma 3.2,  $A_\lambda^{I_p}$  is divisible, so we may deduce as in (1) that  $\text{res}_p(d) \in H_f^1(\mathbb{Q}_p, A_\lambda((k' + k - 2)/2))$ .  $\square$

**Remark 5.3.** *We could have used a different formulation of the Bloch-Kato conjecture, for the incomplete L-function with Euler factors at  $\mathfrak{p} \mid N$  missing, as in (59) of [DFG], similarly using the exact sequence in their Lemma 2.1. This would have involved a Selmer group with no local restrictions at  $\mathfrak{p} \mid N$ , and eliminated the Tamagawa factors at  $\mathfrak{p} \mid N$ . Hence we could have avoided the related difficulties of showing triviality of  $\lambda$ -parts of Tamagawa factors (at  $\mathfrak{p} \mid N$  but not at  $\mathfrak{p} = \ell$ ) and proving that local conditions at  $\mathfrak{p} \mid N$  are satisfied. However, we chose to assume a little more than necessary (i.e. the conditions of Lemma 3.2(1)), then use it to prove something a bit stronger.*

## 6. THE DOUBLING METHOD WITH DIFFERENTIAL OPERATORS

We mainly recall some properties of the doubling method in the setting of holomorphic Siegel modular forms (with invariant differential operators). As long as one does not insist on explicit constants and explicit  $\Gamma$ -factors, everything works more generally for arbitrary polynomial representations as automorphy factors, see [BS3, section 2], [I1].

**6.1. Construction of holomorphic differential operators.** We construct holomorphic differential operators on  $\mathfrak{H}_{2n}$  with certain equivariance properties. We combine the constructions from [B1] and [BSY]; a similar strategy was also used by [Koz].

We decompose  $Z \in \mathfrak{H}_{2n}$  as

$$Z = (z_{ij}) = \begin{pmatrix} z_1 & z_2 \\ z_2^t & z_4 \end{pmatrix} \quad (z_1, z_4 \in \mathfrak{H}_n).$$

We also use the natural embedding  $\text{Sp}(n) \times \text{Sp}(n) \hookrightarrow \text{Sp}(2n)$ , defined by

$$(M_1, M_2) \mapsto M_1^\uparrow \cdot M_2^\downarrow := \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}, \quad M_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \in \text{Sp}(n).$$

The differential operator matrix  $\partial = (\partial_{ij})$  with  $\partial_{ij} = \frac{1+\delta_{ij}}{2} \frac{\partial}{\partial z_{ij}}$  will then be decomposed in block matrices of size  $n$ , denoted by

$$\partial = \begin{pmatrix} \partial_1 & \partial_2 \\ \partial_2^t & \partial_4 \end{pmatrix}.$$

We realize the symmetric tensor representation  $\sigma_\nu := \text{Sym}^\nu$  of  $\text{GL}(n, \mathbb{C})$  in the usual way on the space  $V_\nu := \mathbb{C}[X_1, \dots, X_n]_\nu$  (of homogeneous polynomials of degree  $\nu$ ). For  $V_\nu$ -valued functions  $f$  on  $\mathfrak{H}_n$ ,  $\alpha, \beta \in \mathbb{C}$  and  $M \in \text{Sp}(n, \mathbb{R})$  we define the slash-operator by

$$(f|_{\alpha, \beta, \sigma_\nu} M)(z) := \det(cz + d)^{-\alpha} \det(c\bar{z} + d)^{-\beta} \sigma_\nu(cz + d)^{-1} f(M(z)).$$

We may ignore the ambiguity of the powers  $\alpha, \beta \in \mathbb{C}$  most of the time. If  $\beta = 0$  or  $\nu = 0$  we just omit them from the slash operator.

**Proposition 6.1.** *For nonnegative integers  $\mu, \nu$  there is a (nonzero) holomorphic differential operator  $\mathbb{D}_\alpha(\mu, \nu)$  mapping scalar-valued  $C^\infty$  functions  $F$  on  $\mathfrak{H}_{2n}$  to  $V_\nu \otimes V_\nu$ -valued functions on  $\mathfrak{H}_n \times \mathfrak{H}_n$ , satisfying*

$$(5) \quad \mathbb{D}_\alpha(\mu, \nu)(F |_{\alpha, \beta} (M_1^\uparrow M_2^\downarrow)) = (\mathbb{D}_\alpha(\mu, \nu)(F)) |_{\alpha+\mu, \beta, \sigma_\nu}^{z_1} M_1 |_{\alpha+\mu, \beta, \sigma_\nu}^{z_4} M_2$$

for all  $M_1, M_2 \in \mathrm{Sp}(n, \mathbb{R})$ ; the upper index at the slash operator indicates, for which variables  $M_i$  is applied.

More precisely, there is a  $V_\nu \otimes V_\nu$ -valued nonzero polynomial  $Q(\alpha, \mathbf{T}) = Q_\alpha^{(\mu, \nu)}(\mathbf{T})$  in the variables  $\alpha$  and  $\mathbf{T}$  (where  $\mathbf{T}$  is a symmetric  $2n \times 2n$  matrix of variables), with rational coefficients, such that

$$\mathbb{D}_\alpha(\mu, \nu) = Q_\alpha^{(\mu, \nu)}(\partial_{ij}) |_{z_2=0}.$$

The differential operator  $\mathbb{D}_\alpha(\mu, \nu)$  has the additional symmetry property

$$\mathbb{D}_\alpha(\mu, \nu)(F | V) = \mathbb{D}_\alpha(\mu, \nu)(F)^*,$$

where  $V$  is the operator defined on functions on  $\mathfrak{H}_{2n}$  by

$$F \longmapsto (F | V) \left( \begin{pmatrix} z_1 & z_2 \\ z_2^\dagger & z_4 \end{pmatrix} \right) = F \left( \begin{pmatrix} z_4 & z_2^\dagger \\ z_2 & z_1 \end{pmatrix} \right)$$

and for a function  $g$  on  $\mathfrak{H}_n \times \mathfrak{H}_n$  we put  $g^*(z, w) := g(w, z)$ .

**Remark 6.2.** *We allow arbitrary “complex weights”  $\alpha$  here; note that there is no ambiguity in this as long as we use the same branch of  $\log \det(\mathrm{CZ} + \mathrm{D})$  to define the  $\det(\mathrm{CZ} + \mathrm{D})^s$  on both sides of (5).*

*Note also that the differential operators do not depend at all on  $\beta$ .*

*Proof.* We recall from [B1, Satz 2] the existence of an explicitly given differential operator

$$\mathcal{D}_\alpha = (-1)^n C_n \left( \alpha - n + \frac{1}{2} \right) \det(\partial_2) + \dots + \det(z_2) \cdot \det(\partial_{ij})$$

with

$$(6) \quad C_n(s) := s \left( s + \frac{1}{2} \right) \dots \left( s + \frac{n-1}{2} \right) = \frac{\Gamma_n(s + \frac{n+1}{2})}{\Gamma_n(s + \frac{n-1}{2})} \left( \Gamma_n(s) = \pi^{\frac{n(n-1)}{4}} \prod_{j=0}^{n-1} \Gamma\left(s - \frac{j}{2}\right) \right).$$

This operator is compatible with the action of  $\mathrm{Sp}(n, \mathbb{R}) \times \mathrm{Sp}(n, \mathbb{R}) \hookrightarrow \mathrm{Sp}(2n, \mathbb{R})$ , increasing the weight  $\alpha$  by one (without restriction!), i.e.

$$\mathcal{D}_\alpha(F |_{\alpha, \beta} M_1^\uparrow \cdot M_2^\downarrow) = (\mathcal{D}_\alpha F) |_{\alpha+1, \beta} M_1^\uparrow \cdot M_2^\downarrow, \quad (M_i \in \mathrm{Sp}(n, \mathbb{R}))$$

We put

$$\mathcal{D}_\alpha^\mu := \mathcal{D}_{\alpha+\mu-1} \circ \dots \circ \mathcal{D}_\alpha.$$

**Remark 6.3.** *The combinatorics of this operator is not known explicitly for general  $\mu$ .*

The second type of differential operators maps scalar-valued functions on  $\mathfrak{H}_{2n}$  to  $\mathbb{C}[X_1, \dots, X_n]_{\nu} \otimes \mathbb{C}[Y_1, \dots, Y_n]_{\nu}$ -valued functions on  $\mathfrak{H}_n \times \mathfrak{H}_n$ , changing the automorphy factor from  $\det^{\alpha}$  on  $\mathrm{GL}(2n, \mathbb{C})$  to  $(\det^{\alpha} \otimes \mathrm{Sym}^{\nu}) \boxtimes (\det^{\alpha} \otimes \mathrm{Sym}^{\nu})$  on  $\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$ . This operator was introduced in [BSY, section 2]; it is a special feature that we know the combinatorics in this case quite explicitly:

$$(7) \quad L_{\alpha}^{\nu} := \frac{1}{(2\pi i)^{\nu} \alpha^{[\nu]}} \left( \sum_{0 \leq 2j \leq \nu} \frac{1}{j!(\nu-2j)!(2-\alpha-\nu)^{[j]}} (D_{\uparrow} D_{\downarrow})^j (D - D_{\uparrow} - D_{\downarrow})^{\nu-2j} \right)_{z_2=0};$$

here we use the same notation as in [BSY]:

$$\begin{aligned} \alpha^{[j]} &= \alpha(\alpha+1)\dots(\alpha+j-1) = \frac{\Gamma(\alpha+j)}{\Gamma(\alpha)} \\ D &= \partial[(X_1, \dots, X_n, Y_1, \dots, Y_n)^{\dagger}] \\ D_{\uparrow} &= \partial[(X_1, \dots, X_n, 0, \dots, 0)^{\dagger}] \\ D_{\downarrow} &= \partial[(0, \dots, 0; Y_1, \dots, Y_n)^{\dagger}], \end{aligned}$$

where  $A[x] := x^{\dagger} A x$ ; we remark that

$$D - D_{\uparrow} - D_{\downarrow} = (X_1, \dots, X_n; 0, \dots, 0) \cdot \partial_2 \cdot (0, \dots, 0; Y_1, \dots, Y_n)^{\dagger}.$$

In [BSY] the weight was a natural number  $k$ , but everything works also for arbitrary complex  $\alpha$  instead. (Due to the normalization of [BSY], we have to omit certain finitely many  $\alpha$ .)

We put

$$\mathbb{D}_{\alpha}(\mu, \nu) := L_{\alpha+\mu}^{\nu} \circ \mathcal{D}_{\alpha}^{\mu}.$$

This operator has all the requested properties, except for the fact that the coefficients are not polynomials in  $\alpha$  but rational functions.  $\square$

**6.2. Some combinatorics.** Then we consider the function  $h_{\alpha, \beta}$  defined on  $\mathbb{H}_{2n}$  by

$$h_{\alpha, \beta}(Z) := \det(z_1 + z_2 + z_2^{\dagger} + z_4)^{-\alpha} \overline{\det(z_1 + z_2 + z_2^{\dagger} + z_4)}^{-\beta}$$

and we note that (following [BCG, (1.25)])

$$\mathcal{D}_{\alpha}^{\mu} h_{\alpha, \beta} = A_{\alpha, \mu} \cdot h_{\alpha+\mu, \beta}$$

with

$$A_{\alpha, \mu} = \frac{\Gamma_n(\alpha + \mu)}{\Gamma_n(\alpha)} \frac{\Gamma_n(\alpha + \mu - \frac{n}{2})}{\Gamma_n(\alpha - \frac{n}{2})}$$

and also

$$L_{\alpha}^{\nu} h_{\alpha, \beta} = B_{\alpha, \nu} \sigma_{\nu}(z_1 + z_4)^{-1} \left( \sum X_i Y_i \right)^{\nu} \det(z_1 + z_4)^{-\alpha} \overline{\det(z_1 + z_4)}^{-\beta}$$

with

$$B_{\alpha, \nu} = \frac{1}{(-2\pi i)^{\nu} \nu!} \frac{\Gamma(2\alpha - 2 + \nu)}{\Gamma(2\alpha - 2)} \frac{\Gamma(\alpha - 1)}{\Gamma(\alpha + \nu - 1)},$$

following [BSY, Lemma 4.2].

For later purposes we summarize here some additional properties of these differential operators:

First we note that  $\mathbb{D}_\alpha(\mu, \nu)$  is a homogeneous polynomial (of degree  $n\mu + \nu$ ) in the partial derivatives; we decompose it as

$$\mathbb{D}_\alpha(\mu, \nu) = \mathcal{M} + \mathcal{R},$$

where the “main term”  $\mathcal{M}$  denotes the part free of derivatives w.r.t.  $z_1$  or  $z_4$ .

**Lemma 6.4.** *a) All the monomials occurring in the “remainder term”  $\mathcal{R}$  have positive degree in the partial derivatives w.r.t.  $z_1$  and  $z_4$ .*

*b) The “main term”  $\mathcal{M}$  is of the form*

$$\mathcal{M} = C_\alpha(\mu, \nu) (D - D^\uparrow - D^\downarrow)^\nu \det(\partial_2)^\mu$$

with

$$C_\alpha(\mu, \nu) = \frac{1}{(\alpha + \mu)^{[\nu]!} \nu!} \prod_{j=0}^{\mu-1} C_n \left( \alpha - n + \frac{\mu + \nu' + j}{2} \right) \quad \left( \nu' := \frac{\nu}{n} \right),$$

where  $C_n(s)$  is as in equation (6).

*c) For the polynomial  $Q_\alpha^{\mu, \nu}(\mathbf{T})$  with the symmetric matrix  $\mathbf{T} = \begin{pmatrix} \mathbf{T}_1 & \mathbf{T}_2 \\ \mathbf{T}_2^\dagger & \mathbf{T}_4 \end{pmatrix}$  of size  $2n$  this means*

$$(8) \quad Q_\alpha^{\mu, \nu}(\mathbf{T}) = C_\alpha(\mu, \nu) (2(X_1, \dots, X_n) \mathbf{T}_2 (Y_1, \dots, Y_n)^\dagger)^\nu \det(\mathbf{T}_2)^\mu + (*),$$

where  $(*)$  contains only contributions with positive degree in  $\mathbf{T}_1$  and  $\mathbf{T}_4$ .

*Proof.* a) The formula (12) in [B1] shows that in  $\mathcal{D}_\alpha$  an entry of  $\partial_1$  always appears together with an entry of  $\partial_4$ . The same is then true for  $\mathcal{D}^\mu$ . Furthermore, the explicit formula (7) for  $L_{\alpha+\mu}^\nu$  shows that only the contribution of  $j = 0$  is free of partial derivatives w.r.t.  $z_1$ ; it is at the same time the only contribution free of derivatives w.r.t.  $z_4$ .

b) We define an element  $M = M(X_1, \dots, X_n; Y_1, \dots, Y_n)$  of  $V_\nu \otimes V_\nu$  by

$$M := \mathbb{D}_\alpha(\mu, \nu)(\text{exptr}(z_2)) = \mathcal{M}(\text{exptr}(z_2)).$$

The transformation properties of  $\mathbb{D}_\alpha(\mu, \nu)$ , applied for

$$\begin{pmatrix} A^t & 0 \\ 0 & A^{-1} \end{pmatrix}^\uparrow, \quad \begin{pmatrix} A & 0 \\ 0 & A^{-t} \end{pmatrix}^\downarrow \quad (A \in \text{GL}(n, \mathbb{R}))$$

yield

$$M((X_1, \dots, X_n) \cdot A; Y_1, \dots, Y_n) = M(X_1, \dots, X_n; (Y_1, \dots, Y_n) A^t) \quad (A \in \text{GL}(n, \mathbb{C})).$$

Such a vector in  $V_\nu \otimes V_\nu$  is unique up to constants and is therefore a scalar multiple of  $(\sum X_i Y_i)^\nu$ , i.e.  $M = c \cdot (2 \sum_i X_i Y_i)^\nu$  for an appropriate constant  $c = C_\alpha(\mu, \nu)$ . To understand  $\mathcal{M}$  we study its action on those functions on  $\mathbb{H}_{2n}$ , which depend only on  $z_2$ ; it is enough to look at functions of type  $f_T(z_2) := \text{exptr}(Tz_2)$  with

$T \in \mathbb{R}^{(n,n)}$ ,  $\det(T) \neq 0$ . Then

$$\begin{aligned} \mathbb{D}_\alpha(\mu, \nu) f_T &= \det(T)^{-\alpha} \mathbb{D}_\alpha(\mu, \nu)(f_{1_n} |_\alpha \begin{pmatrix} T & 0 \\ 0 & T^{-t} \end{pmatrix}) \\ &= \det(T)^{-\alpha} (\mathbb{D}_\alpha(\mu, \nu) f_{1_n}) |_{\alpha+\mu, \nu}^z \begin{pmatrix} T & 0 \\ 0 & T^{-t} \end{pmatrix} \\ &= \det(T)^\mu c \cdot (2 \sum_i X_i T^t Y_i)^\nu \\ &= c(D - D^\uparrow - D^\downarrow)^\nu \det(\partial_2)^\mu f_T. \end{aligned}$$

It remains to determine the coefficient  $C_\alpha(\mu, \nu)$ ; we compute  $\mathbb{D}_s(\mu, \nu) \det(z_2)^s$  in two ways, using the standard formulas (see e.g. [BCG, Section 1])

$$\det(\partial_2) \det(z_2)^s = C_n \left( \frac{s}{2} \right) \det(z_2)^{s-1}$$

$$D_\alpha \det(z_2)^s = (-1)^n C_n \left( \frac{s}{2} \right) C_n \left( \alpha - n + \frac{s}{2} \right) \det(z_2)^{s-1}.$$

Then

$$\mathbb{D}_\alpha(\mu, \nu) \det(z_2)^s = C_\alpha(\mu, \nu) \left( \prod_{j=0}^{\mu-1} C_n \left( \frac{s-j}{2} \right) \right) \{(D - D^\uparrow - D^\downarrow)^\nu \det(z_2)^{s-\mu}\} |_{z_2=0}$$

and on the other hand

$$\begin{aligned} \mathbb{D}_\alpha(\mu, \nu) \det(z_2)^s &= L_{\alpha+\mu}^\nu (D_\alpha^\mu \det(z_2)^s) \\ &= \prod_{j=0}^{\mu-1} C_n \left( \frac{s-j}{2} \right) C_n \left( \alpha - n + \frac{s+j}{2} \right) \{L_{\alpha+\mu}^\nu \det(z_2)^{s-\mu}\} |_{z_2=0} \\ &= \prod_{j=0}^{\mu-1} C_n \left( \frac{s-j}{2} \right) C_n \left( \alpha - n + \frac{s+j}{2} \right) \frac{1}{(\alpha + \mu)^{[\nu] \nu}!} \{(D - D^\uparrow - D^\downarrow)^\nu \det(z_2)^{s-\mu}\} |_{z_2=0} \end{aligned}$$

If  $\nu = n\nu'$  is a multiple of  $n$ , then  $s := \mu + \nu'$  gives nonzero contributions and we get

$$C_\alpha(\mu, \nu) = \frac{1}{(\alpha + \mu)^{[\nu] \nu}!} \prod_{j=0}^{\mu-1} C_n \left( \alpha - n + \frac{\mu + \nu' + j}{2} \right).$$

Actually, this formula makes sense (and is also valid) for arbitrary  $\nu$ .  $\square$

For the special case  $n = 2$  considerations similar to the above appear in [DIK, Lemma 7.5., Corollary 7.6].

**6.3. Doubling method with the differential operators**  $\mathbb{D}_\alpha(\mu, \nu)$ . The inner product  $(\sum a_i X_i, \sum b_i X_i) = \sum a_i \bar{b}_i$  on  $V_1 := \mathbb{C}[X_1, \dots, X_n]_1$  induces a “produit scalaire adapté” (see [Go]) on the  $\nu$ -fold symmetric tensor product  $V_\nu = \text{Sym}^\nu(V_1) = \mathbb{C}[X_1, \dots, X_n]_\nu$  by

$$\{\alpha_1 \cdots \alpha_\nu, \beta_1 \cdots \beta_\nu\} = \frac{1}{\nu!} \sum_{\tau} \prod_{j=1}^{\nu} (\alpha_{\tau(j)}, \beta_j) \quad (\alpha_i, \beta_j \in V_1),$$

where  $\tau$  runs over the symmetric group of order  $\nu$ . This inner product is invariant under the action of unitary matrices via  $\text{Sym}^\nu$ .

Note that for all  $\mathbf{v} \in \mathbb{C}[X_1, \dots, X_n]_\nu$  we have

$$\left\{ \mathbf{v}, \left( \sum X_i Y_i \right)^\nu \right\} = \tilde{\mathbf{v}},$$

where  $\tilde{\mathbf{v}}$  denotes the same polynomial as  $\mathbf{v}$ , but with the variables  $Y_i$  instead of the  $X_i$ .

We describe here the general pullback formula for level  $N$  Eisenstein series ( $N$  square free).

We put

$$G_k^{(2n)}(Z, s) = \sum_{M \in \Gamma_0^{(2n)}(N)_\infty \backslash \Gamma_0^{(2n)}(N)} \det(CZ + D)^{-k-s} \det(C\bar{Z} + D)^{-s}.$$

For a cusp form  $F \in S_\rho(\Gamma_0^{(n)}(N))$  with  $\rho = \det^{k+\mu} \otimes \sigma_\nu$  and  $z = x + iy, w = u + iv \in \mathfrak{H}_n$  we get

$$\begin{aligned} & \int_{\Gamma_0^{(n)}(N) \backslash \mathbb{H}_n} \left\{ \rho(\sqrt{y}) F(z), \rho(\sqrt{y}) \mathbb{D}_{s+k}(\mu, \nu) G_k^{(2n)} \left( \begin{pmatrix} z & 0 \\ 0 & -\bar{w} \end{pmatrix}, \bar{s} \right) \det(y)^s \det(\mathbf{v})^s \right\} d\omega_n \\ (9) \quad & = \gamma_n(k, \mu, \nu, s) \sum_M F(w) |T_N(M) \det(M)^{-k-2s}. \end{aligned}$$

Here  $d\omega_n = \det(y)^{-n-1} dx dy$ ,  $M$  runs over all (integral) elementary divisor matrices of size  $n$  with  $M \equiv 0 \pmod{N}$ , and  $T_N(M)$  denotes the Hecke operator associated to the double coset  $\Gamma_0^{(n)}(N) \begin{pmatrix} 0 & -M^{-1} \\ M & 0 \end{pmatrix} \Gamma_0^{(n)}(N)$ .

To compute the archimedean factor  $\gamma$  one should keep in mind that the unfolding of the integral leads to an integration over  $\mathfrak{H}_n$  involving  $\mathbb{D}_{k+s}(\mu, \nu) h_{k+s, s}$ . Then  $\gamma$  is naturally a product of (essentially) three factors

$$\gamma_n(k, \mu, \nu, s) = i^{nk+n\mu+\nu} 2^{n(n-k-\mu-2s-\nu+1)} A_{k+s, \mu} B_{k+\mu+s, \nu} I(s+k+\mu-n-1, \nu)$$

with a Hua type integral

$$I(\alpha, \nu) = \frac{\pi^{\frac{n(n+1)}{2}}}{\alpha + n + \nu} \prod_{j=1}^{n-1} \frac{(2\alpha + 2j + 1)(n + j + 2\alpha)^{[\nu]}}{(\alpha + j) \Gamma(\nu + n + j + 2\alpha + 1)}.$$

We refer to [BSY, Sect.3], see also [B3, 2.2] for details.



**6.4. Doubling method with the differential operators  $\mathbb{D}_k(\mu, \nu)$ .** There are two ways to obtain holomorphic Siegel Eisenstein series of degree  $n$  and low weight after analytic continuation (sometimes called ‘‘Hecke summation’’): One is by evaluating at  $s = 0$ , the other by considering  $s_1 = \frac{n+1}{2} - k$ ; both are connected by a complicated functional equation involving all Siegel Eisenstein series. We need the case of weight 2 and degree 4, where only the Hecke summation for  $s_1$  is available. We first consider the general case: In (9) the differential operator  $\mathbb{D}_{k+s}(\mu, \nu)$  was applied directly to the Eisenstein series of ‘‘weight’’  $k + s$ . If we use the Hecke summation not in  $s = 0$  but in  $s_1 := \frac{2n+1}{2} - k$  for an Eisenstein series of degree  $2n$ , we should better use a differential operator acting on the weight  $k$  Eisenstein series  $E_k^{(2n)} := G_k^{(2n)} \cdot (\det \operatorname{Im} Z)^s$  to get holomorphic modular forms (in particular theta series) after evaluating in  $s = s_1$ . One might try to use the calculations of Takayanagi [Tak]. Note however that the results of [Tak] are applicable only for the case  $\mu = 0$ ; to incorporate the differential operator  $\mathcal{D}_k^\mu$  there is quite complicated, see also [Koz]. We avoid this difficulty by observing that the two types of differential operators are actually not that different:

By

$$F \longmapsto \mathcal{D}_{k,s}(\mu, \nu)(F) := \det(\mathbf{y})^s \det(\mathbf{v})^s \mathbb{D}_{k+s}(\mu, \nu)(\det(Y)^{-s} \times F)$$

we can define a new (nonholomorphic) differential operator mapping functions  $F$  on  $\mathfrak{H}_{2n}$  to  $\mathbb{C}[X_1, \dots, X_n]_{\mathbf{v}} \otimes \mathbb{C}[Y_1, \dots, Y_n]_{\mathbf{v}}$  valued functions on  $\mathfrak{H}_n \times \mathfrak{H}_n$ ; this operator has exactly the same transformation properties as  $\mathbb{D}_k(\mu, \nu)$ .

Starting from the observation that  $\mathcal{D}_{k,s}(\mu, \nu)$  maps holomorphic functions on  $\mathfrak{H}_{2n}$  to nearly holomorphic functions on  $\mathfrak{H}_n \times \mathfrak{H}_n$  we get from the theory of Shimura [Sh2, Sh3] in the same way as in [BCG, section 1] an operator identity

$$(10) \quad \mathcal{D}_{k,s}(\mu, \nu) = \sum_{\rho_i, \rho_j} \delta_{\rho_j}^{(z_1)} \otimes \delta_{\rho_i}^{(z_4)} \circ \mathcal{D}_s(\rho_i, \rho_j).$$

Here the  $\rho_i, \rho_j$  run over finitely many polynomial representations of  $\operatorname{GL}(n, \mathbb{C})$  and  $\mathcal{D}_s(\rho_i, \rho_j)$  denotes a  $V_{\rho_i} \otimes V_{\rho_j}$ -valued holomorphic differential operator (a polynomial in the  $\partial_{i,j}$ , evaluated at  $z_2 = 0$ ; it changes the automorphy factor  $\det^k$  on  $\operatorname{GL}(2n, \mathbb{C})$  to  $(\det^k \otimes \rho_1) \boxtimes (\det^k \otimes \rho_2)$  on  $\operatorname{GL}(n, \mathbb{C}) \times \operatorname{GL}(n, \mathbb{C})$ ). As is usual in the theory of nearly holomorphic functions, we have to avoid finitely many weights  $k$  here. Furthermore the  $\delta_{\rho_i}, \delta_{\rho_j}$  are non-holomorphic differential operators on  $\mathfrak{H}_n$ , changing automorphy factors from  $\det^k \otimes \rho$  to  $\det^{k+\mu} \otimes \operatorname{Sym}^{\nu}$ . In the simplest case (i.e.  $\rho = \det^\mu, \nu = 2$ ), the operator  $\delta_\rho$  has the explicit form

$$\delta_\rho = (X_1, \dots, X_n) \cdot ((\partial_{ij}) - 2i(k + \mu)\operatorname{Im}(Z)^{-1}) \cdot \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}.$$

Furthermore we mention that, by invariant theory, holomorphic differential operators  $\mathcal{D}_s(\rho_i, \rho_j)$  with the transformation properties described above only exist in the case  $\rho_i = \rho_j$  see [II].

If  $\delta_\rho^{(z_1)} \otimes \delta_\rho^{(z_4)}$  is the identity, then  $\rho = \det^{k+\mu} \otimes \operatorname{Sym}^{\nu}$  and (at least for  $k \geq n$ )  $\mathcal{D}_s(\rho, \rho)$  is a scalar multiple of  $\mathbb{D}_k(\mu, \nu)$ , because the space of such differential operators is one-dimensional. The decomposition (10) can then be rewritten as

$$(11) \quad p_s(k)\mathcal{D}_{k,s}(\mu, \nu) = d_s(k)\mathbb{D}_k(\mu, \nu) + \mathcal{K}$$

where  $p_s(k)$  and  $d_s(k)$  are polynomials in  $k$  and  $\mathcal{K}$  is a nonholomorphic differential operator with the same transformation properties as  $\mathbb{D}_k(\mu, \nu)$  and with the additional property that  $\mathcal{K}(F)$  is orthogonal to all holomorphic cusp forms in the variables  $z_1$  or  $z_4$  (for any  $C^\infty$  automorphic form on  $\mathbb{H}_{2n}$  with suitable growth properties). Note that (11) holds now for *all* weights  $k$ , if we request the finitely many exceptions from (10) to be among the zeroes of  $p_s(k)$ . We also observe that  $\mathcal{D}_{k,s}(\mu, \nu)$  is a homogeneous polynomial of degree  $n\mu + \nu$  in the variables  $(\partial_{ij})|_{z_2=0}$  and the entries of  $y_1^{-1}$  and  $y_4^{-1}$  and  $\mathcal{K}$  consists only of monomials whose joint degree in  $\partial_1$  and  $y_1^{-1}$  as well as in  $\partial_4$  and  $y_4^{-1}$  are both positive, in particular,  $\mathcal{K}$  cannot contribute monomials that only involve entries of  $\partial_2$ .

Therefore (as in [BCG, (1.31)]) we may compare the coefficients of  $\det(\partial_2)^\mu \left( \sum_{i,j} \frac{\partial}{\partial_{i,n+j}} X_i Y_j \right)^\nu$  on both sides: We get

$$p_s(k) C_{k+s}(\mu, \nu) = d_s(k) C_k(\mu, \nu).$$

From this we obtain a version of the pullback formula (9)

$$\begin{aligned} & \int_{\Gamma_0^{(n)}(\mathbb{N}) \backslash \mathbb{H}_n} \left\{ \rho(\sqrt{y}) F(z), \rho(\sqrt{y}) \mathbb{D}_k(\mu, \nu) E_k^{(2n)} \left( \begin{pmatrix} z & 0 \\ 0 & -\bar{w} \end{pmatrix}, \bar{s} \right) \right\} d\omega_n \\ (12) \quad & = \frac{p_s(k)}{d_s(k)} \cdot \gamma_n(k, \mu, \nu, s) \sum_M F | T_N(M) \det(M)^{-k-2s}. \end{aligned}$$

We need the result above for the pullback formula applied for a degree 4, weight 2 Eisenstein series at  $s_1 = \frac{1}{2}$ : we consider the holomorphic modular form

$$\mathcal{E}_2^{(4)} := \text{Res}_{s=s_1} E_2^{(4)}(Z, s)$$

Then we get for a cusp form  $F \in S_\rho(\Gamma_0^{(2)}(\mathbb{N}))$ , with  $\rho = \det^{2+\mu} \otimes \text{Sym}^\nu$ ,

$$\begin{aligned} \langle F, \mathbb{D}_2(\mu, \nu) \mathcal{E}_2^{(4)}(*, -\bar{w}) \rangle &= \text{Res}_{s=s_1} \langle F, \mathbb{D}_2(\mu, \nu) E_2^{(4)}(*, -\bar{w}) \rangle \\ &= \text{Res}_{s=s_1} \frac{p_s(2)}{d_s(2)} \langle F, \mathbb{D}_{2+s}(\mu, \nu) G_2^{(4)} \det(y)^s \det(v)^s \rangle \\ (13) \quad &= c \cdot \text{Res}_{s=s_1} \left( \sum_M F(w) | T_N(M) \det(M)^{-2-2s} \right). \end{aligned}$$

The relevant constant is then

$$(14) \quad c = \frac{C_2(\mu, \nu)}{C_{2+\frac{1}{2}}(\mu, \nu)} \gamma_2 \left( 2, \mu, \nu, \frac{1}{2} \right).$$

## 6.5. Standard-L-functions at $s = 1$ and $s = 2$ , in particular for Yoshida lifts of degree 2.

6.5.1. *An Euler product.* If  $F \in S_\rho(\Gamma_0^{(n)}(\mathbb{N}))$  is an eigenform of all the Hecke operators  $T_N(M)$  with eigenvalues  $\lambda_N(M)$ , then the Dirichlet series of these eigenvalues can be written in terms of the (good part of) the standard L-function  $D_F^{(\mathbb{N})}(s)$ :

$$\sum \lambda_N(M) \det(M)^{-s} = \left( \sum_{\det(M)|N^\infty} \det(M)^{-s} \right) \times \frac{1}{\zeta^{(N)}(s) \prod_{i=1}^n \zeta^{(N)}(2s-2i)} D_F^{(N)}(s-n).$$

The integral representations studied above allow us to investigate (for degree 2) the behavior of such a standard L-function at  $s = 1$  and  $s = 2$ ; we remark that  $s = 1$  is not a critical value for the standard L-function! Note that in the formula above, we get  $D_F(1)$  for degree  $n = 2$  for  $s = 3$ . In the formula (13) this corresponds to  $s = s_1 = \frac{1}{2}$  due to several shifts ( $2s_1 + 2 - 2 = 1$  for this  $s_1$ ).

If  $F$  is actually a Yoshida lift of level  $N$  associated to two elliptic cuspidal newforms  $f \in S_{k'}(\Gamma_0(N))$ ,  $g \in S_k(\Gamma_0(N))$ , with  $k' \geq k$ , then  $F \in S_\rho(\Gamma_0^{(2)}(N))$  with  $\rho = \det^{2+\frac{k'-k}{2}} \otimes \text{Sym}^{k-2}$  is indeed an eigenform of all the Hecke operators  $T_N(M)$ :

$$\sum_M F | T_N(M) \det(M)^{-s} = \frac{\lambda}{N^{ns}} \zeta^{(N)}(s-2) L^{(N)} \left( f \otimes g, s + \frac{k'+k}{2} - 3 \right) \Lambda_N(s-2) \cdot F$$

where  $\lambda = \pm N^{n(n-1)/2} = \pm N$  (with the sign depending only on  $N$ ),

$$\Lambda_N(s) = \prod_{p|N} \prod_{j=1}^2 (1 - p^{-s-2+j})^{-1}$$

and

$$L^{(N)}(f_1 \otimes f_2, s) := \prod_{p \nmid N} (1 - \alpha_p \beta_p p^{-s})(1 - \alpha_p \beta'_p p^{-s})(1 - \alpha'_p \beta_p p^{-s})(1 - \alpha'_p \beta'_p p^{-s}).$$

Moreover  $F|_p \left( \begin{smallmatrix} 0_2 & -1_2 \\ N \cdot 1_2 & 0_2 \end{smallmatrix} \right)$  is also an eigenfunction of all the  $T_N(M)$  with the same eigenvalues as  $F$ ; for details on the facts mentioned above we refer to [BS1, BS3].

6.5.2. *A version of the pullback formula for the Eisenstein series attached to the cusp zero.* We can consider the same doubling method using the Eisenstein series

$$\begin{aligned} \mathfrak{F}_k^{(2n)}(Z, s) &:= \sum_{C, D} \det(CZ + D)^{-k-s} \det(C\bar{Z} + D)^{-s}, \\ \mathbb{F}_k^{(2n)}(Z, s) &:= \mathfrak{F}_k^{(2n)}(Z, s) \times \det(Y)^s, \end{aligned}$$

where  $(C, D)$  runs over non-associated coprime symmetric pairs with the additional condition “ $\det(C)$  coprime to  $N$ ” (this is the Eisenstein series “attached to the cusp zero”). The reason for using both versions is that in our previous papers [BS1, BS3] we mainly worked with  $E_k^{(2n)}$ , whereas the Fourier expansion is more easily accessible for the Eisenstein series  $\mathbb{F}_k^{(2n)}$ .

The two doubling integrals are linked to each other by the elementary relation

$$E_k^{(2n)}(Z, s) |_k \left( \begin{smallmatrix} 0_{2n} & -1_{2n} \\ N \cdot 1_{2n} & 0_{2n} \end{smallmatrix} \right) = N^{-kn-2ns} \mathbb{F}_k^{(2n)}(Z, s).$$

Due to this relation, substituting  $\mathfrak{F}$  for  $E$  in the doubling method just means (for Yoshida-lifts) a modification by a power of  $N$  (the factor  $N^{-ns}$  in (15) goes away). For the case of arbitrary cusp forms we refer to [BCG, BKS].

We write down the relevant cases explicitly for the Yoshida lift  $F$  from above:

The residue of the standard L-function at  $s = 1$  corresponds to a near center value for  $L(f_1 \otimes f_2, s)$ :

The equation (13) then becomes (with  $\mathcal{F}_2^{(4)} := \text{Res}_{s=\frac{1}{2}} \mathbb{F}_2^{(4)}$ )

$$(16) \quad \left\langle \mathbb{F}, \mathbb{D}_2 \left( \frac{k' - k}{2}, k - 2 \right) \mathcal{F}_2^{(4)}(*, -\bar{w}) \right\rangle \\ = c \lambda \prod_{p|N} (1 - p^{-1}) \Lambda_N(1) \frac{1}{\zeta^{(N)}(3) \zeta^{(N)}(4) \zeta^{(N)}(2)} L^{(N)} \left( f \otimes g, \frac{k' + k}{2} \right) \cdot F(w)$$

with

$$c = \frac{C_2\left(\frac{k' - k}{2}, k - 2\right)}{C_{2+\frac{1}{2}}\left(\frac{k' - k}{2}, k - 2\right)} \cdot \gamma_2 \left( \frac{k' - k}{2}, k - 2, \frac{1}{2} \right).$$

To treat the critical value of the standard L-function at  $s = 2$ , we can directly use the formula (9), taking tacitly into account that  $\mathcal{F}_4^{(4)}(Z) := \mathbb{F}_4^{(4)}(Z, s)|_{s=0}$  defines a holomorphic modular form (see [Sh1, Prop.10.1]) by Hecke summation.

This yields

$$(17) \quad \left\langle \mathbb{F}, \mathbb{D}_4 \left( \frac{k' - k}{2} - 2, k - 2 \right) \mathcal{F}_4^{(4)}(*, -\bar{w}) \right\rangle \\ = \gamma_2 \left( 4, \frac{k' - k}{2} - 2, k - 2, 0 \right) \times \\ (\pm N) \Lambda_N(2) \frac{\zeta^{(N)}(2)}{\zeta^{(N)}(4) \zeta^{(N)}(6) \zeta^{(N)}(4)} L^{(N)} \left( f \otimes g, \frac{k' + k}{2} + 1 \right) \cdot F(w).$$

In the case of a general cusp form  $F \in S_\rho(\Gamma_0^{(2)}(N))$ , which we assume to be an eigenfunction of the Hecke operators “away from  $N$ ”, we can write

$$\left\langle \mathbb{F}, \mathbb{D}_4 \left( \frac{k' - k}{2} - 2 \right) \mathcal{F}_4^{(4)}(*, -\bar{w}) \right\rangle \\ = \gamma_2 \left( 4, \frac{k' - k}{2} - 2, k - 2, 0 \right) \times \frac{D_f^{(N)}(2)}{\zeta^{(N)}(4) \zeta^{(N)}(6) \zeta^{(N)}(4)} \mathcal{T}(F)(w)$$

where  $\mathcal{T}$  is an (infinite) sum of Hecke operators at the bad places.

## 7. INTEGRALITY PROPERTIES

The known results about integrality of Fourier coefficients of Eisenstein series are not sufficient for our purposes because they deal only with level one and large weights. We do not aim at the most general case, but just describe how to adapt the reasoning in [B4, section 5] to the cases necessary for our purposes.

**7.1. The Eisenstein series.** We collect some facts about the Fourier coefficients of Eisenstein series

$$F_k^m(Z) := \mathbb{F}_k^m(Z, s)|_{s=0}$$

for even  $m = 2n$  with  $k \geq \frac{m+4}{2}$

This function is known to define a holomorphic modular form with Fourier expansion

$$F_k^m(Z) = \sum_{T \geq 0} a_k^m(T, N) \exp(2\pi i \operatorname{tr}(TZ)).$$

We first treat  $T$  of maximal rank. We denote by  $d(T) := (-1)^n \det(2T)$  the discriminant of  $T$  and by  $\chi_T$  the corresponding quadratic character, defined by  $\chi_T(\cdot) := \left(\frac{d(T)}{\cdot}\right)$ .

Then  $a_k^m(T, N) = 0$  unless  $T > 0$ , see e.g. [BCG, prop.5.2].

If  $T > 0$  then the Fourier coefficient is of type

$$a_k^m(T) = A_k^m \det(T)^{k - \frac{m+1}{2}} \prod_{p \nmid N} \alpha_p(T, k)$$

where  $\alpha_p(T, k)$  denotes the usual local singular series and

$$A_k^m = (-1)^{\frac{mk}{2}} \frac{2^m}{\Gamma_m(k)} \pi^{mk}.$$

We can express the nonarchimedean part by a normalizing factor and polynomials in  $p^{-k}$ :

$$\begin{aligned} \prod_{p \nmid N} \alpha_p(T, k) &= \frac{1}{\zeta^{(N)}(k) \prod_{j=1}^n \zeta^{(N)}(2k-2j)} \times \\ &\sum_{G} \det(G)^{-2k+m-1} L^{(N)}(k-n, \chi_{T[G^{-1}]}) \prod_{p \nmid N} \beta_p(T[G^{-1}], k). \end{aligned}$$

Here  $G$  runs over

$$\operatorname{GL}(n, \mathbb{Z}) \setminus \{M \in \mathbb{Z}^{(n,n)} \mid \det(M) \text{ coprime to } N\}$$

and the  $\beta_p(T)$  denote the “normalized primitive local densities”. In general they are polynomials in  $p^{-k}$  with integer coefficients and they are equal to one for all  $p$  coprime to  $d(T)$ , see e.g. [B4, section 2].

Let  $f_T$  be the conductor of the quadratic character  $\chi_T$  and  $\eta_T$  the corresponding primitive character. Then

$$\begin{aligned} L^{(N)}(k-n, \chi_T) &= \prod_{p \mid N} (1 - \chi_T(p) p^{-k+n}) L(k-n, \chi_T) \\ &= \prod_{p \mid N} (1 - \chi_T(p) p^{-k+n}) \prod_{p \mid d(T)} (1 - \eta(p) p^{n-k}) L(k-n, \eta_T) \end{aligned}$$

We quote from [B4] that

$$\left(\frac{d(T)}{f_T}\right)^{k - \frac{m}{2}} \prod_{p \nmid N} (1 - \eta_T(p) p^{n-k}) \beta_p(T, k) \in \mathbb{Z}.$$

We may therefore just ignore this factor. Then as in [B4] we use the functional equation of the Riemann zeta function and the Dirichlet L-functions attached to quadratic characters.

We get (for  $4 \mid k$ ) that

$$a_k^m(\mathbb{T}, N) \in \prod_{p \mid N} \left( (1 - p^{-k}) \prod_{j=1}^n (1 - p^{-2k+2j}) \right) 2^n \frac{k}{B_k} \frac{1}{\mathcal{N}_{2k-m}^*} \prod_{j=1}^n \frac{k-j}{B_{2k-2j}} \cdot \frac{1}{N^{k-n}} \cdot \mathbb{Z}$$

Here the factor  $N^{k-n}$  takes care of the possible denominator arising from  $\prod_{p \mid N} (1 - \chi_{\mathbb{T}}(p) p^{-k+n})$  and

$$\mathcal{N}_{2k-m}^* := \prod_{p \mid \mathcal{N}_{2k-m}} p^{1+\nu_p(k-n)},$$

where  $\mathcal{N}_{2k-m}$  is the denominator of the Bernoulli number  $B_{2k-m}$ .

If  $k \equiv 2 \pmod{4}$  there is a similar formula, see [B4].

We have to assure that nonzero Fourier coefficients of lower rank do not occur. This is a classical fact in the range of absolute convergence (i.e.  $k > m+1$ ), see e.g. the calculations in [Ma, section 18]. It is also true for small weight  $k \geq \frac{m+4}{2}$  and level one, as shown by Haruki [Har, Theorem 4.14]; his result relies on calculations by Shimura [Sh1] and Mizumoto [Miz]. The basic ingredient for Haruki is an expression [Har, (1.1)] for Fourier coefficients  $\mathbb{T}$  of rank  $r < m$  as finite sums of products of  $\Gamma$ -factors, singular series, confluent hypergeometric functions and Eisenstein series for  $\mathrm{GU}(n)$  evaluated at  $s = 0$ . Haruki's procedure remains valid for level  $N > 1$  as long as it is based on individual vanishing of the products mentioned above (the modification for level  $N > 1$  means to omit the local singular series for primes dividing  $N$ , i.e. for all  $p \mid N$  one has to multiply the level one expression by a polynomial in  $p^{-s-2k}$ , evaluated at  $s = 0$ ). Indeed, as shown in the proof of Theorem 4.14 [Har], such individual vanishing occurs for all  $\mathbb{T}$  of rank  $r < m$  and all  $k \geq \frac{m+4}{2}$  except possibly for the case  $k = \frac{m+4}{2}$  and  $r = m-4 > 0$ ; in this exceptional case the vanishing for level one depends on cancellations for some  $\mathbb{T}$ .

*In summary, the Fourier coefficients  $a_k^m(\mathbb{T}, N)$  all vanish for  $\mathrm{rank}(\mathbb{T}) < m$  and  $k > \frac{m+4}{2}$  and also for  $m = k = 4$ .*

**Remark 7.1.** *The Fourier coefficients of  $F_4^4(N)$  are in*

$$\prod_{p \mid N} ((1 - p^{-4})^2 (1 - p^{-6})) \frac{9}{2N^2} \cdot \mathbb{Z} \subseteq \frac{9}{N^{16}} \cdot \mathbb{Z} \left[ \frac{1}{2} \right].$$

**7.2. The differential operators.** By definition, the coefficients of the differential operator  $\mathcal{D}_k^\mu$  are in  $\mathbb{Z}[1/2]$ ; here we view  $\mathcal{D}_k^\mu$  as a polynomial in the variables  $z_2$  and  $\partial_{ij}$ .

Concerning the integrality properties of  $L_k^\nu$ , we just remark that because of

$$(2 - k - \nu)^{[j]} = (-1)^j (k + \nu - j - 1)^{[j]} = (-1)^j \frac{(k + \nu - 2)!}{(k + \nu - j - 1)!}$$

it is sufficient to look at

$$\frac{(k + \nu - j - 1)!}{k^{[\nu]j} (\nu - 2j)! (k + \nu - 2)!} \quad (0 \leq j \leq \lfloor \frac{\nu}{2} \rfloor).$$

Taking into account that  $\frac{\nu!}{j!(\nu-2j)!} \in \mathbb{Z}$  and

$$\frac{(k + \nu - j - 1)!}{(k + \nu - 2)!} \in \frac{1}{(k + \nu - \lfloor \frac{\nu}{2} \rfloor) \dots (k + \nu - 2)} \cdot \mathbb{Z}$$

we see that the coefficients of  $L_k^\nu$  are in

$$\frac{1}{k^{\lfloor \nu \rfloor} \nu! (k + \nu - 2) \dots (k + \nu - \lfloor \frac{\nu}{2} \rfloor)} \cdot \mathbb{Z}.$$

Putting things together, we see that  $\mathbb{D}_k(\mu, \nu)$  has coefficients in

$$\frac{1}{(k + \mu)^{\lfloor \nu \rfloor} \nu! (k + \mu + \nu - 2) \dots (k + \mu + \nu - \lfloor \frac{\nu}{2} \rfloor)} \cdot \mathbb{Z} [1/2].$$

**Remark 7.2.** *The Fourier coefficients of  $\mathbb{D}_4(\mu, \nu) \mathbf{F}_4^4$  are in*

$$\frac{1}{(4 + \mu)^{\lfloor \nu \rfloor} \nu! (4 + \mu + \nu - 2) \dots (4 + \mu + \nu - \lfloor \frac{\nu}{2} \rfloor)} \times \frac{9}{N^{16}} \mathbb{Z} [1/2].$$

This remark does not claim, that the denominator given there is the best possible one, there may be additional cancellations of denominators coming from the restriction.

## 8. THE PETERSSON NORM OF THE YOSHIDA LIFT

Take  $f = \sum a_n q^n, g = \sum b_n q^n$  as in the introduction, of weights  $k'$  and  $k$  respectively and assume that for all primes  $p$  dividing the common (square-free) level  $N$  of  $f, g$  both functions have the same Atkin-Lehner eigenvalue  $\epsilon_p$ . Let  $k' = 2\nu_1 + 2, k = 2\nu_2 + 2$ . Choose a factorization  $N = N_1 N_2$ , where  $N_1$  is the product of an odd number of prime factors, and let  $D = D(N_1, N_2)$  be the definite quaternion algebra over  $\mathbb{Q}$ , ramified at  $\infty$  and the primes dividing  $N_1$ . Let  $R = R(N_1, N_2)$  be an Eichler order of level  $N = N_1 N_2$  in  $D(N_1, N_2)$  with (left) ideal class number  $h$ .

We recall (and slightly modify) some notation from §1 of [BS3]: For  $\nu \in \mathbb{N}$  let  $U_\nu^{(0)}$  be the space of homogeneous harmonic polynomials of degree  $\nu$  on  $\mathbb{R}^3$  and view  $P \in U_\nu^{(0)}$  as a polynomial on  $D_\infty^{(0)} = \{x \in D_\infty | \text{tr}(x) = 0\}$  by putting  $P(\sum_{i=1}^3 x_i e_i) = P(x_1, x_2, x_3)$  for an orthonormal basis  $\{e_i\}$  of  $D_\infty^{(0)}$  with respect to the norm form  $n$  on  $D$ . The representations  $\tau_\nu$  of  $D_\infty^\times / \mathbb{R}^\times$  of highest weight  $(\nu)$  on  $U_\nu^{(0)}$  given by  $(\tau_\nu(y))(P)(x) = P(y^{-1}xy)$  for  $\nu \in \mathbb{N}$  give all the isomorphism classes of irreducible rational representations of  $D_\infty^\times / \mathbb{R}^\times$ .

For an irreducible rational representation  $(V_\tau, \tau)$  (with  $\tau = \tau_\nu$  as above) of  $D_\infty^\times / \mathbb{R}^\times$  we denote by  $\mathcal{A}(D_\mathbb{A}^\times, R_\mathbb{A}^\times, \tau)$  the space of functions  $\phi : D_\mathbb{A}^\times \rightarrow V_\tau$  satisfying  $\phi(\gamma x u) = \tau(u_\infty^{-1})\phi(x)$  for  $\gamma \in D_\mathbb{Q}^\times$  and  $u = u_\infty u_f \in R_\mathbb{A}^\times$ , where  $R_\mathbb{A}^\times = D_\infty^\times \times \prod_p R_p^\times$  is the adelic group of units of  $R$ . Let  $D_\mathbb{A}^\times = \cup_{i=1}^r D^\times y_i R_\mathbb{A}^\times$  be a double coset decomposition with  $y_{i,\infty} = 1$  and  $n(y_i) = 1$ . A function in  $\mathcal{A}(D_\mathbb{A}^\times, R_\mathbb{A}^\times, \tau)$  is then determined by its values at the  $y_i$ . We put  $I_{ij} = y_i R y_j^{-1}$ ,  $R_i = I_{ii}$  and let  $e_i$  be the number of units of the order  $R_i$ . On the space  $\mathcal{A}(D_\mathbb{A}^\times, R_\mathbb{A}^\times, \tau)$  we have for  $p \nmid N$  Hecke operators  $\tilde{T}(p)$  defined by  $\tilde{T}(p)\phi(x) = \int_{D_p^\times} \phi(xy^{-1}) \chi_p(y) dy$  where  $\chi_p$  is the characteristic function of  $\{y \in R_p | n(y) \in p\mathbb{Z}_p^\times\}$ . They commute with the

involutions  $\tilde{w}_p$  and are given explicitly by  $\tilde{T}(p)\phi(y_i) = \sum_{j=1}^r B_{ij}^{(\nu)}(p)\phi(y_j)$ , where the Brandt matrix entry  $B_{ij}^{(\nu)}(p)$  is given as

$$B_{ij}(p) = B_{ij}^{(\nu)}(p) = \frac{1}{e_j} \sum_{\substack{x \in y_j R y_i^{-1} \\ n(x)=p}} \tau(x),$$

hence is itself an endomorphism of the representation space  $U_{\nu}^{(0)}$  of  $\tau$ .

From [Ei, H-S, Shz, J-L] we know then that the essential part  $\mathcal{A}_{\text{ess}}(D_{\mathbb{A}}^{\times}, R_{\mathbb{A}}^{\times}, \tau)$  consisting of functions  $\phi$  that are orthogonal (under the natural inner product) to all  $\psi \in \mathcal{A}(D_{\mathbb{A}}^{\times}, (R'_{\mathbb{A}})^{\times}, \tau)$  for orders  $R'$  strictly containing  $R$  is invariant under the  $\tilde{T}(p)$  for  $p \nmid N$  and the  $\tilde{w}_p$  for  $p \nmid N$  and hence has a basis of common eigenfunctions of all the  $\tilde{T}(p)$  for  $p \nmid N$ . Moreover in  $\mathcal{A}_{\text{ess}}(D_{\mathbb{A}}^{\times}, R_{\mathbb{A}}^{\times}, \tau)$  strong multiplicity one holds, i.e., each system of eigenvalues of the  $\tilde{T}(p)$  for  $p \nmid N$  occurs at most once, and the eigenfunctions are in one to one correspondence with the newforms in the space  $S^{2+2\nu}(N)$  of elliptic cusp forms of weight  $2 + 2\nu$  for the group  $\Gamma_0(N)$  that are eigenfunctions of all Hecke operators (if  $\tau$  is the trivial representation and  $R$  is a maximal order one has to restrict here to functions orthogonal to the constant function 1 on the quaternion side in order to obtain cusp forms on the modular forms side).

Let  $\phi_1 = \phi_1^{(N_1, N_2)} : D_{\mathbb{A}}^{\times} \rightarrow U_{\nu_1}^{(0)}$  and  $\phi_2 = \phi_2^{(N_1, N_2)} : D_{\mathbb{A}}^{\times} \rightarrow U_{\nu_2}^{(0)}$  correspond to  $f$  and  $g$  respectively with respect to the choice of  $N_1, N_2$  and hence of  $D = D(N_1, N_2)$ . Let  $F = F_{f, g} = F_{\phi_1, \phi_2}$  (which of course also depends on the choice of  $N_1, N_2$ ) be the Yoshida lift; it takes values in the space  $W_{\rho}$  of the symmetric tensor representation  $\rho = \det^{\kappa} \otimes \text{Sym}^j(\mathbb{C}^2)$ ,  $j = k - 2$ ,  $\kappa = 2 + \frac{k' - k}{2}$  and is a Siegel cusp form  $F \in S_{\rho}(\Gamma_0^{(2)}(N))$ . To describe it explicitly we notice that the group of proper similitudes of the quadratic form  $q(x) = n(x)$  on  $D$  (with associated symmetric bilinear form  $B(x, y) = \text{tr}(x\bar{y})$ , where  $\text{tr}$  denotes the reduced trace on  $D$ ) is isomorphic to  $(D^{\times} \times D^{\times})/Z(D^{\times})$  (as algebraic group) via  $(y, y') \mapsto \sigma_{y, y'}$  with  $\sigma_{y, y'}(x) = yx(y')^{-1}$ , the special orthogonal group is then the image of  $\{(y, y') \in D^{\times} \times D^{\times} \mid n(y) = n(y')\}$ .

We denote by  $H$  the orthogonal group of  $(D, n)$ , by  $H^+$  the special orthogonal group and by  $K$  (resp.  $K^+$ ) the group of isometries (resp. isometries of determinant 1) of the lattice  $R$  in  $D$ . It is well known that the  $H^+(\mathbb{R})$ -space  $U_{\nu_1}^{(0)} \otimes U_{\nu_2}^{(0)}$  is isomorphic to the  $H^+(\mathbb{R})$ -space  $U_{\nu_1, \nu_2}$  of  $\mathbb{C}[X_1, X_2]$ -valued harmonic forms on  $D_{\infty}^2$  transforming according to the representation of  $GL_2(\mathbb{R})$  of highest weight  $(\nu_1 + \nu_2, \nu_1 - \nu_2)$ ; an intertwining map  $\Psi$  has been given in [BS5, Section 3]. It is also well known [KV] that the representation  $\lambda_{\nu_1, \nu_2}$  of  $H^+(\mathbb{R})$  on  $U_{\nu_1, \nu_2}$  is irreducible of highest weight  $(\nu_1 + \nu_2, \nu_1 - \nu_2)$ . If  $\nu_1 > \nu_2$  it can be extended in a unique way to an irreducible representation of  $H(\mathbb{R})$  on the space  $U_{\nu_1, \nu_2, s} := (U_{\nu_1}^{(0)} \otimes U_{\nu_2}^{(0)}) \oplus (U_{\nu_2}^{(0)} \otimes U_{\nu_1}^{(0)}) =: U_{\lambda}$  which we denote by  $(\tau_1 \otimes \tau_2) =: \lambda$  for simplicity, on this space  $\sigma_{y, y'} \in H^+(\mathbb{R})$  acts via  $\tau_1(y) \otimes \tau_2(y')$  on the summand  $U_{\nu_1}^{(0)} \otimes U_{\nu_2}^{(0)}$  and via  $\tau_2(y) \otimes \tau_1(y')$  on the summand  $U_{\nu_2}^{(0)} \otimes U_{\nu_1}^{(0)}$ . For  $\nu_1 = \nu_2$  there are two possible extensions to representations  $(\tau_1 \otimes \tau_2)_{\pm}$  on  $U_{\nu_1, \nu_2}$ ; we denote this space with the representation  $(\tau_1 \otimes \tau_2)_{+} =: \lambda$  on it by  $U_{\lambda}$  again (and don't consider the minus variant in the sequel).



We recall then from [KV, We1, BS3] that the space  $\mathcal{H}_q(\rho)$  consisting of all  $q$ -pluriharmonic polynomials  $P : M_{4,2}(\mathbb{C}) \rightarrow W_\rho$  such that  $P(xg) = (\rho(g^\dagger))P(x)$  for all  $g \in GL_2(\mathbb{C})$  is isomorphic to  $(U_\lambda, \lambda)$  as a representation space of  $H(\mathbb{R})$ . The space  $\mathcal{H}_q(\rho)$  carries an essentially unique  $H(\mathbb{R})$ -invariant scalar product  $\langle \cdot, \cdot \rangle_{\mathcal{H}_q(\rho)}$ , and in the usual way we can find a reproducing  $H(\mathbb{R})$  invariant kernel  $P_{\text{Geg}} \in \mathcal{H}_q(\rho) \otimes \mathcal{H}_q(\rho)$  (generalized Gegenbauer polynomial), i. e.,  $P_{\text{Geg}}$  is a polynomial on  $D_\infty^2 \oplus D_\infty^2$  taking values in  $W_\rho \otimes W_\rho$  which as function of each of the variables

- i) is a  $q$ -pluriharmonic polynomial in  $\mathcal{H}_q(\rho)$ ,
- ii) is symmetric in both variables
- iii) satisfies  $P_{\text{Geg}}(\mathbf{h}\mathbf{x}, \mathbf{h}\tilde{\mathbf{x}}) = P_{\text{Geg}}(\mathbf{x}, \tilde{\mathbf{x}})$  for  $\mathbf{h} \in H(\mathbb{R})$
- iv) satisfies  $\langle P_{\text{Geg}}(\mathbf{x}, \cdot), P(\cdot) \rangle_{\mathcal{H}_q(\rho)} = P(\mathbf{x})$  for all  $P \in \mathcal{H}_q(\rho)$ .

In fact, since such a polynomial is characterized by the first three properties up to scalar multiples we can construct it (in a more general situation) with the help of the differential operator  $\mathbb{D}_\alpha(\mu, \nu)$  and the polynomial  $Q_\alpha^{\mu, \nu}$  from 6.1:

For  $k \in \mathbb{N}$  and nonnegative integers  $\mu, \nu$  we define a polynomial map

$$\widetilde{P_{\text{Geg}}}^{(k, \mu, \nu)} : \mathbb{C}^{2k, n} \times \mathbb{C}^{2k, n} \longrightarrow V_\nu \otimes V_\nu$$

by

$$\widetilde{P_{\text{Geg}}}^{(k, \mu, \nu)}(\mathbf{Y}_1, \mathbf{Y}_2) := Q_k^{(\mu, \nu)} \left( \begin{pmatrix} \mathbf{Y}_1^\dagger \mathbf{Y} & \mathbf{Y}_1^\dagger \mathbf{Y}_2 \\ \mathbf{Y}_2^\dagger \mathbf{Y}_1 & \mathbf{Y}_2^\dagger \mathbf{Y}_2 \end{pmatrix} \right)$$

Then  $\widetilde{P_{\text{Geg}}}^{(k, \mu, \nu)}$  is symmetric and pluriharmonic in  $\mathbf{Y}_1$  and  $\mathbf{Y}_2$ , see [I1]; moreover, for  $A, B \in GL(n, \mathbb{C})$  we have

$$\widetilde{P_{\text{Geg}}}^{(k, \mu, \nu)}(\mathbf{Y}_1 \cdot A, \mathbf{Y}_2 \cdot B) = \det(A)^\mu \det(B)^\mu \sigma_\nu(A) \otimes \sigma_\nu(B) \widetilde{P_{\text{Geg}}}^{(k, \mu, \nu)}(\mathbf{Y}_1, \mathbf{Y}_2).$$

For  $g \in O(2k, \mathbb{C})$  we get

$$\widetilde{P_{\text{Geg}}}^{(k, \mu, \nu)}(g\mathbf{Y}_1, \mathbf{Y}_2) = \widetilde{P_{\text{Geg}}}^{(k, \mu, \nu)}(\mathbf{Y}_1, g^{-1}\mathbf{Y}_2).$$

If we consider a  $2k$ -dimensional positive definite real quadratic space with positive definite quadratic form  $q$  and associated bilinear form  $B$  (so that  $B(x, x) = 2q(x)$ ) we write  $q(x_1, \dots, x_{2n}) = (B(x_i, x_j)/2)_{i,j}$  for (half) the  $2n \times 2n$  Gram matrix associated to the  $2n$ -tuple of vectors  $(x_1, \dots, x_{2n})$  and put in a similar way as above for  $(\mathbf{y}, \mathbf{y}') \in V^{2n}$

$$P_{\text{Geg}}^{(k, \mu, \nu)}(\mathbf{y}, \mathbf{y}') = Q_k^{\mu, \nu}(q(\mathbf{y}, \mathbf{y}')),$$

this gives a nonzero polynomial with values in  $V_\nu \otimes V_\nu$  which is symmetric in the variables  $\mathbf{y}, \mathbf{y}'$ , is  $q$ -pluriharmonic in each of the variables with the proper transformation under the right action of  $GL_n$  and is invariant under the diagonal action of the orthogonal group of  $q$ ; it is hence a scalar multiple of the  $V_\nu \otimes V_\nu$ -valued Gegenbauer polynomial on this space.

If we apply the differential operator  $\mathbb{D}_k(\mu, \nu)$  to a degree  $2n$  theta series  $\Theta_S^{2n}(Z) := \sum_{R \in \mathbb{Z}^{2k, 2n}} \exp 2\pi i \text{tr}(R^\dagger SRZ)$  written in matrix notation we get

$$\begin{aligned} & (\mathbb{D}_k(\mu, \nu)\Theta_S^{2n})(z_1, z_4) \\ &= \sum_{R_1, R_2 \in \mathbb{Z}^{(2k, n)}} (2\pi i)^{n\mu} Q_k^{(\mu, \nu)} \left( \begin{pmatrix} S[R_1] & R_1^\dagger SR_2 \\ R_2^\dagger SR_1 & S[R_2] \end{pmatrix} \right) \exp 2\pi i \text{tr}(S[R_1]z_1 + S[R_2]z_4); \end{aligned}$$

writing the theta series in lattice notation as the degree  $2n$  theta series

$$\theta_{\Lambda}^{(2n)}(Z) = \sum_{\mathbf{x} \in \Lambda^{2n}} \exp(2\pi i \operatorname{tr}(\mathbf{q}(\mathbf{x})Z))$$

of a lattice  $\Lambda$  on  $V$  we obtain in the same way

$$\begin{aligned} (18) \quad \mathbb{D}_k(\mu, \nu) \theta_{\Lambda}^{(2n)}(z_1, z_4) &= (2\pi i)^{n\mu} \sum_{(\mathbf{y}, \mathbf{y}') \in \Lambda^{2n}} P_{\text{Geg}}(\mathbf{y}, \mathbf{y}') \exp(2\pi i \operatorname{tr}(\mathbf{q}(\mathbf{y})z_1 + \mathbf{q}(\mathbf{y}')z_4)) \\ &= (2\pi i)^{n\mu} \sum_{(\mathbf{y}) \in \Lambda^n} \theta_{\Lambda}^{(n, \nu)}(z_4)(\mathbf{y}) \exp(2\pi i \operatorname{tr}(\mathbf{q}(\mathbf{y})z_1)), \end{aligned}$$

where we have written

$$(19) \quad \theta_{\Lambda}^{(n, \nu)}(z_4)(\mathbf{y}) := \sum_{(\mathbf{y}') \in \Lambda^n} P_{\text{Geg}}(\mathbf{y}, \mathbf{y}') \exp(2\pi i \operatorname{tr}(\mathbf{q}(\mathbf{y}')z_4)).$$

Going through the construction above in our quaternionic situation with  $V_{\nu} = W_{\rho}$  we see that we can normalize the scalar product on  $\mathcal{H}_q(\rho)$  in such a way that the polynomial  $P_{\text{Geg}}$  obtained in the way just described is indeed the reproducing kernel for this space. We choose this normalization in what follows and write

$$\theta_{ij, \rho}(Z)(\tilde{\mathbf{x}}) := \sum_{\mathbf{x} \in (\mathbf{y}_i \mathbf{R} \mathbf{y}_j^{-1})^2} P_{\text{Geg}}(\mathbf{x}, \tilde{\mathbf{x}}) \exp(2\pi i \operatorname{tr}(\mathbf{q}(\tilde{\mathbf{x}})Z)) \in W_{\rho} \otimes W_{\rho}$$

(so that  $\theta_{ij, \rho}(Z)$  is (for each  $Z$  in the Siegel upper half space  $\mathfrak{H}_2$ ) an element of  $\mathcal{H}_q(\rho) \otimes W_{\rho}$ .) For an arbitrary lattice  $\Lambda$  on  $D$  the theta series  $\theta_{\Lambda, \rho}$  is defined analogously as given in equation (19).

We denote by  $\mathcal{P}$  the (essentially unique) isomorphism from  $\mathbf{U}_{\lambda}$  to  $\mathcal{H}_q(\rho)$ . With the help of the map  $\Psi$  from [BS5] mentioned above we can fix a normalization and write  $\mathcal{P}(\mathbf{R}_1 \otimes \mathbf{R}_2)$  for  $\mathbf{R}_j \in \mathbf{U}_{\nu_j}^{(0)}$  as

$$(20) \quad \begin{aligned} \mathcal{P}(\mathbf{R}_1 \otimes \mathbf{R}_2)(\mathbf{d}_1, \mathbf{d}_2)(X_1, X_2) &= (\mathcal{D}(\mathbf{n}(\mathbf{d}_1 X_1 + \mathbf{d}_2 X_2))^{\nu_2} \tau_2(\mathbf{d}_1 X_1 + \mathbf{d}_2 X_2) \mathbf{R}_2) \mathbf{R}_1 (\operatorname{Im}(\mathbf{d}_1 \bar{\mathbf{d}}_2)), \end{aligned}$$

where we associate as usual to a polynomial  $\mathbf{R} \in \mathbb{C}[t_1, t_2, t_3]$  the differential operator  $\mathcal{D}(\mathbf{R}) = \mathbf{R}(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \frac{\partial}{\partial t_3})$ , set  $\operatorname{Im}(\mathbf{d}) = \mathbf{d} - \bar{\mathbf{d}}$  and write all vectors as coordinate vectors with respect to an orthonormal basis.

**Definition 8.1.** *With notation as above we define the Yoshida lift of  $(\phi_1, \phi_2)$ , or also of  $(f, g)$  with respect to  $(\mathbf{N}_1, \mathbf{N}_2)$ , to be given by*

$$F(Z) := Y^{(2)}(\phi_1, \phi_2)(Z) := \sum_{i, j=1}^r \frac{1}{e_i e_j} \langle \mathcal{P}(\phi_1(\mathbf{y}_i) \otimes \phi_2(\mathbf{y}_j)), \theta_{ij, \rho}(Z) \rangle_{\mathcal{H}_q(\rho)} \in W_{\rho}.$$

**Lemma 8.2.** (1) *One has  $\theta_{ij, \rho}(Z)(\mathbf{x}) = \theta_{ji, \rho}(Z)(\bar{\mathbf{x}})$  (where  $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2)$  denotes the quaternionic conjugate of the pair  $\mathbf{x} = (x_1, x_2)$ ).*

$$\begin{aligned}
(2) \quad 2F(Z) &= 2Y^{(2)}(\phi_1, \phi_2, Z) \\
&= \sum_{i,j=1}^r \frac{1}{e_i e_j} \sum_{\mathbf{x} \in (\mathbf{y}_i R \mathbf{y}_j^{-1})^2} \mathcal{P}(\phi_1(\mathbf{y}_i) \otimes \phi_2(\mathbf{y}_j) + \phi_2(\mathbf{y}_i) \otimes \phi_1(\mathbf{y}_j))(\mathbf{x}_1, \mathbf{x}_2) \times \\
&\quad \times \exp(2\pi i \operatorname{tr}(\mathbf{q}(\mathbf{x})Z)).
\end{aligned}$$

(3) Denote by  $\langle F, \theta_{ij, \rho} \rangle_{\text{Pet}}$  the Petersson product of the vector valued Siegel modular forms  $F$  and  $\theta_{ij, \rho}$ . Then the function  $\xi : (\mathbf{y}_i, \mathbf{y}_j) \mapsto \langle F, \theta_{ij, \rho} \rangle_{\text{Pet}} \in \mathcal{H}_{\mathbf{q}}(\rho)$  has the symmetry property  $\xi(\mathbf{y}_i, \mathbf{y}_j)(\mathbf{x}) = \xi(\mathbf{y}_j, \mathbf{y}_i)(\bar{\mathbf{x}})$ . It induces a unique function, denoted by  $\tilde{\xi}$ , on  $H(\mathbb{A})$  satisfying  $\tilde{\xi}(\sigma_{\mathbf{y}_i, \mathbf{y}_j}) = \xi(\mathbf{y}_i, \mathbf{y}_j)$  and

$$\tilde{\xi}(\gamma \sigma \mathbf{k}) = \lambda(\mathbf{k}_{\infty}^{-1}) \tilde{\xi}(\sigma) \text{ for } \sigma \in H(\mathbb{A}), \gamma \in H(\mathbb{Q}), \mathbf{k} = (\mathbf{k}_v)_v \in H(\mathbb{R}_{\mathbb{A}}),$$

where we denote by  $H(\mathbb{R}_{\mathbb{A}})$  the group of adelic isometries of the lattice  $\mathbb{R}$  on  $D$ .

*Proof.* This is easily seen to be a consequence of the fact that the lattice  $I_{ij} = \mathbf{y}_i R \mathbf{y}_j^{-1}$  is the quaternionic conjugate of the lattice  $I_{ji} = \mathbf{y}_j R \mathbf{y}_i^{-1}$  and that quaternionic conjugation is an element of the (global) orthogonal, but not of the special orthogonal group of  $(D, \mathfrak{n})$ .  $\square$

As in [BS1] we need to show that  $\xi$  is proportional to the function  $\xi_{\phi_1, \phi_2} : (\mathbf{y}_i, \mathbf{y}_j) \mapsto \phi_1(\mathbf{y}_i) \otimes \phi_2(\mathbf{y}_j) + \phi_2(\mathbf{y}_i) \otimes \phi_1(\mathbf{y}_j)$  that appears in our formula for the Yoshida lifting, and to determine the factor of proportionality occurring.

**Lemma 8.3.** *With notations as in the previous lemma one has*

$$\langle F, \theta_{ij, \rho} \rangle_{\text{Pet}} = c_5 \mathcal{P}(\phi_1(\mathbf{y}_i) \otimes \phi_2(\mathbf{y}_j) + \phi_2(\mathbf{y}_i) \otimes \phi_1(\mathbf{y}_j))$$

and

$$\langle F, F \rangle_{\text{Pet}} = c_5 \langle \mathcal{P}(\phi_1 \otimes \phi_2), \mathcal{P}(\phi_1 \otimes \phi_2) \rangle,$$

with some constant  $c_5 \neq 0$ , where the latter inner product is the natural inner product on  $\mathcal{H}_{\mathbf{q}}(\rho)$ -valued functions on  $D_{\mathbb{A}}^{\times} \times D_{\mathbb{A}}^{\times}$  satisfying the usual invariance properties under  $\mathbb{R}_{\mathbb{A}}^{\times}$  and  $D_{\mathbb{Q}}^{\times}$ , which is defined by

$$\langle \mathcal{P}(\phi_1 \otimes \phi_2), \mathcal{P}(\phi_1 \otimes \phi_2) \rangle = \sum_{i,j=1}^r \frac{1}{e_i e_j} \langle \mathcal{P}(\phi_1(\mathbf{y}_i) \otimes \phi_2(\mathbf{y}_j)), \mathcal{P}(\phi_1(\mathbf{y}_i) \otimes \phi_2(\mathbf{y}_j)) \rangle_{\mathcal{H}_{\mathbf{q}}(\rho)}.$$

*Proof.* The proof proceeds in essentially the same way as in [BS1]: We notice first that the space of all  $\xi$  with the symmetry property mentioned (or equivalently the space of functions  $\tilde{\xi}$  on  $H(\mathbb{A})$  with the invariance property given) has a basis consisting of the  $\xi_{\phi_1, \phi_2} = \xi_{\phi_2, \phi_1}$ , where  $(\phi_1, \phi_2)$  runs through the pairs of eigenforms in  $(\mathcal{A}(D_{\mathbb{A}}^{\times}, (\mathbb{R}_{\mathbb{A}})^{\times}, \tau_1) \times (\mathcal{A}(D_{\mathbb{A}}^{\times}, (\mathbb{R}_{\mathbb{A}})^{\times}, \tau_2))$  and where the pairs are unordered if  $\nu_1 = \nu_2$ .

The Hecke operators  $T'_i(\mathfrak{p})$  on the spaces  $\mathcal{A}(D_{\mathbb{A}}^{\times}, \mathbb{R}_{\mathbb{A}}^{\times}, \tau_i)$  (for  $i = 1, 2$ ) via Brandt matrices described above induce Hecke operators  $\hat{T}(\mathfrak{p})$  on the space of  $\xi$ , as above that are given by

$$\xi|\hat{T}(\mathfrak{p})(\mathbf{y}_i, \mathbf{y}_j) = \sum_{k=1}^r \tilde{B}_{jk}^{(\text{right})}(\mathfrak{p}) \xi(\mathbf{y}_i, \mathbf{y}_k) + \sum_{l=1}^r \tilde{B}_{il}^{(\text{left})}(\mathfrak{p}) \xi(\mathbf{y}_l, \mathbf{y}_j),$$

where for  $\nu_1 > \nu_2$  we let  $\tilde{B}_{jk}^{(\text{right})}(\mathfrak{p})$  act on  $\mathbf{U} = \mathbf{U}_{\nu_1}^{(0)} \otimes \mathbf{U}_{\nu_2}^{(0)} \oplus \mathbf{U}_{\nu_2}^{(0)} \otimes \mathbf{U}_{\nu_1}^{(0)}$  via  $\text{id} \otimes B_{jk}^{\nu_2}(\mathfrak{p}) \oplus \text{id} \otimes B_{jk}^{\nu_1}(\mathfrak{p})$  and  $\tilde{B}_{il}^{(\text{left})}(\mathfrak{p})$  as  $B_{jk}^{\nu_1}(\mathfrak{p}) \otimes \text{id} \oplus B_{jk}^{\nu_2}(\mathfrak{p}) \otimes \text{id}$ , and where for  $\nu_1 = \nu_2$  the action of  $\tilde{B}^{(\text{left})}, \tilde{B}^{(\text{right})}$  on  $\mathbf{U} = \mathbf{U}_{\nu_1}^{(0)} \otimes \mathbf{U}_{\nu_2}^{(0)}$  is simply the action of the Brandt matrix on the respective factor of the tensor product.

In the same way as sketched in [BS1, 10 b)] we obtain then (using the calculations of Hecke operators from [Y1, Y2]) first

$$\langle F, \theta_{ij, \rho} | T(\mathfrak{p}) \rangle_{\text{Pet}} = \xi | \hat{T}(\mathfrak{p})(\mathbf{y}_i, \mathbf{y}_j).$$

Since, again by Yoshida's computations of Hecke operators (see also [BS3]), we know that  $F$  is an eigenfunction of  $T(\mathfrak{p})$  with eigenvalue  $\lambda_{\mathfrak{p}}(f) + \lambda_{\mathfrak{p}}(g)$ , this implies that  $\xi$  is an eigenfunction with the same eigenvalue for  $\hat{T}(\mathfrak{p})$ . A computation that uses the eigenfunction property of  $\phi_1, \phi_2$  for the action of the Hecke operators on the spaces  $\mathcal{A}(D_{\mathbb{A}}^{\times}, (\mathbb{R}_{\mathbb{A}})^{\times}, \tau_1), \mathcal{A}(D_{\mathbb{A}}^{\times}, (\mathbb{R}_{\mathbb{A}})^{\times}, \tau_2)$  shows that the same is true for the function  $\xi_{\phi_1, \phi_2}$ .

Since  $\phi_1, \phi_2$  are in the essential parts of  $\mathcal{A}(D_{\mathbb{A}}^{\times}, (\mathbb{R}_{\mathbb{A}})^{\times}, \tau_1), \mathcal{A}(D_{\mathbb{A}}^{\times}, (\mathbb{R}_{\mathbb{A}})^{\times}, \tau_2)$ , their eigenvalue systems occur with strong multiplicity one in these spaces, and as in Section 10 of [BS1] we can conclude that that  $\xi$  and  $\xi_{\phi_1, \phi_2}$  are indeed proportional, i.e., we have

$$\langle F, \theta_{ij, \rho} \rangle_{\text{Pet}} = c_5 \mathcal{P}(\phi_1(\mathbf{y}_i) \otimes \phi_2(\mathbf{y}_j) + \phi_2(\mathbf{y}_i) \otimes \phi_1(\mathbf{y}_j))$$

with some constant  $c_5 \neq 0$ .

From this we see:

$$\begin{aligned} \langle F, F \rangle_{\text{Pet}} &= \langle F, \sum_{i,j=1}^r \frac{1}{e_i e_j} \langle \mathcal{P}(\phi_1(\mathbf{y}_i) \otimes \phi_2(\mathbf{y}_j)), \theta_{ij, \rho} \rangle_{\mathcal{H}_q(\rho)} \rangle_{\text{Pet}} \\ &= \sum_{i,j=1}^r \frac{c_5}{e_i e_j} \langle \mathcal{P}(\phi_1(\mathbf{y}_i) \otimes \phi_2(\mathbf{y}_j)), \mathcal{P}(\phi_1(\mathbf{y}_i) \otimes \phi_2(\mathbf{y}_j) + \phi_2(\mathbf{y}_i) \otimes \phi_1(\mathbf{y}_j)) \rangle_{\mathcal{H}_q(\rho)} \\ &= c_5 \langle \mathcal{P}(\phi_1 \otimes \phi_2), \mathcal{P}(\phi_1 \otimes \phi_2) \rangle. \end{aligned}$$

□

In order to compute the constant  $c_5$  we will first need the generalization of Lemma 9.1 of [BS1] to the present situation:

- Lemma 8.4.** (1) *If  $\Lambda$  is a lattice on some quaternion algebra  $D'$  with  $\mathfrak{n}(\Lambda) \subseteq \mathbb{Z}$ , of level dividing  $N$ , and with  $\text{disc}(\Lambda) \neq N^2$  the theta series  $\theta_{\Lambda, \rho}$  is orthogonal to all Yoshida lifts  $Y^{(2)}(\phi_1, \phi_2)$  of level  $N$ .*
- (2) *If  $\Lambda$  is a lattice on some quaternion algebra  $D' \neq D$  with  $\mathfrak{n}(\Lambda) \in \mathbb{Z}$ , of level  $N$ , and with  $\text{disc}(\Lambda) = N^2$  the theta series  $\theta_{\Lambda, \rho}$  is orthogonal to all Yoshida lifts  $Y^{(2)}(\phi_1, \phi_2)$  of level  $N$  associated to  $D$ .*

*Proof.* The proof of Lemma 9.1 of [BS1] unfortunately contains some misprints: In line 4 on p. 81 the minus sign in front of the whole factor should not be there and the exponent at  $\mathfrak{p}$  should be  $\mathfrak{n}(\mathfrak{n} + 1)/2$  (which is equal to 3 in our present situation), in line 5 the exponent at  $\mathfrak{p}$  should be  $\mathfrak{n}(\mathfrak{n} - 1)/2$  (hence 1 in our case), in line 9 the factor  $\mathfrak{p}$  in the right hand side of the equation should be omitted, and in line 14 the exponent at  $\mathfrak{p}$  should be 1 instead of 3.

Apart from these corrections the argument given there carries over to our situation unchanged. In particular, the results from Section 7 of [BS1] that were used

in the proof of that lemma remain true and their proof carries over if one uses the reformulation of Evdokimov's result from [Ev] sketched in Section 4 of [BS3].  $\square$

We recall from [BS1] that we have

$$\mathcal{E}_2^{(4)}(Z_1, Z_2) = \sum_{r=1}^t \alpha_r \sum_{\{K_r\}} \frac{1}{|\mathcal{O}(K_r)|} \theta_{K_r}^{(2)}(Z_1) \theta_{K_r}^{(2)}(Z_2),$$

where we denote by  $L_1, \dots, L_t$  representatives of the genera of lattices of rank 4, square discriminant and level dividing  $N = N_1 N_2$ , the summation over  $\{K_r\}$  runs over a set of representatives of the isometry classes in the genus of  $L_r$  and  $\alpha_r$  are some constants that are explicitly determined in [BS1].

Hence by (19) we obtain

$$\begin{aligned} & \left( \mathbb{D}_2 \left( \frac{k' - k}{2} - 2, k - 2 \right) (\mathcal{E}_2^{(4)}) \right) (Z_1, Z_2) \\ &= c_3 \sum_{r=1}^t \alpha_r \sum_{\{K_r\}} \frac{1}{|\mathcal{O}(K_r)|} \sum_{(\mathbf{x}_1, \mathbf{x}_2) \in K_r^2 \times K_r^2} P_{\text{Geg}}(\mathbf{x}_1, \mathbf{x}_2) \exp(2\pi i \text{tr}(q(\mathbf{x}_1)Z_1 + q(\mathbf{x}_2)Z_2)) \end{aligned}$$

with  $c_3 = (2\pi i)^{k' - k}$ , and similarly for the Eisenstein series  $\mathcal{F}_2^{(4)}$  attached to the cusp zero, with the  $\alpha_r$  replaced by  $\beta_r$  as in [BS1].

The reproducing property of  $P_{\text{Geg}}$  implies then

$$\begin{aligned} & \sum_{(\mathbf{x}_1, \mathbf{x}_2) \in K_r^2 \times K_r^2} P_{\text{Geg}}(\mathbf{x}_1, \mathbf{x}_2) \exp(2\pi i \text{tr}(q(\mathbf{x}_1)Z_1 + q(\mathbf{x}_2)Z_2)) \\ &= \langle \langle \theta_{K, \rho}(Z_1)(\mathbf{u}_1) \otimes \theta_{K, \rho}(Z_2)(\mathbf{u}_2), P_{\text{Geg}}(\mathbf{u}_1, \mathbf{u}_2) \rangle_{\mathcal{H}_q(\rho)} \rangle_{\mathcal{H}_q(\rho)}. \end{aligned}$$

Using the fact that by Lemma 8.4 the Yoshida lifting  $F$  is orthogonal to all  $\theta_{K, \rho}$  where  $K$  is not in the genus of the given Eichler order of level  $N_1 N_2$  we see that the part of the sum for  $\mathbb{D}(\mathcal{F}_2^{(4)})(Z_1, Z_2)$  which contributes to the Petersson product with  $F$  can be written as

$$c_3 \beta_1 \sum_{i, j} \frac{1}{e_i e_j} \langle \langle \theta_{ij, \rho}(Z_1)(\mathbf{u}_1) \otimes \theta_{ij, \rho}(Z_2)(\mathbf{u}_2), P_{\text{Geg}}(\mathbf{u}_1, \mathbf{u}_2) \rangle_{\mathcal{H}_q(\rho)} \rangle_{\mathcal{H}_q(\rho)}.$$

We further recall that by (16) we have

$$\langle F, \mathbb{D}(\mathcal{F}_2^{(4)})(*, -\bar{w}) \rangle_{\text{Pet}} = c_4 L^{(N)} \left( f \otimes g, \frac{k + k'}{2} \right) F(w)$$

with  
(21)

$$c_4 = \lambda \prod_{p|N} (1 - p^{-1}) \Lambda_N(1) \frac{1}{\zeta^{(N)}(3) \zeta^{(N)}(4) \zeta^{(N)}(2)} \frac{C_2 \left( \frac{k' - k}{2}, k - 2 \right)}{C_{2 + \frac{1}{2}} \left( \frac{k' - k}{2}, k - 2 \right)} \cdot \gamma_2 \left( \frac{k' - k}{2}, k - 2, \frac{1}{2} \right).$$

**Proposition 8.5.** *With notations as above we have*

$$\langle F, F \rangle_{\text{Pet}} = \frac{c_4}{2c_3 \beta_1} L^{(N)} \left( f \otimes g, \frac{k + k'}{2} \right) \langle \mathcal{P}(\phi_1 \otimes \phi_2), \mathcal{P}(\phi_1 \otimes \phi_2) \rangle.$$

*Proof.* From what we saw above and using Lemma 8.3 we get

$$\begin{aligned}
& \langle \mathbb{F}, \mathbb{D}(\mathcal{F}_2^{(4)})(*, -\bar{Z}) \rangle_{\text{Pet}} \\
&= c_3 \beta_1 \sum_{i,j} \frac{1}{e_i e_j} \langle \langle \mathbb{F}(*), \theta_{ij,\rho}(-\bar{Z})(\mathbf{u}_1) \otimes \langle \theta_{ij,\rho}(*)(\mathbf{u}_2) \rangle_{\text{Pet}}, \mathbb{P}_{\text{Geg}}(\mathbf{u}_1, \mathbf{u}_2) \rangle_{\mathcal{H}_q(\rho)} \rangle_{\mathcal{H}_q(\rho)} \\
&= c_5 c_3 \beta_1 \sum_{i,j} \frac{1}{e_i e_j} \langle \overline{\langle \theta_{ij,\rho}(-\bar{Z}) \otimes \mathcal{P}(\phi_1(\mathbf{y}_i) \otimes \phi_2(\mathbf{y}_j) + \phi_2(\mathbf{y}_i) \otimes \phi_1(\mathbf{y}_j)) \rangle}, \mathbb{P}_{\text{Geg}} \rangle_{\mathcal{H}_q(\rho)} \rangle_{\mathcal{H}_q(\rho)} \\
&= c_5 c_3 \alpha_1 \sum_{i,j} \frac{1}{e_i e_j} \langle \theta_{ij,\rho}(Z), \mathcal{P}(\phi_1(\mathbf{y}_i) \otimes \phi_2(\mathbf{y}_j) + \phi_2(\mathbf{y}_i) \otimes \phi_1(\mathbf{y}_j)) \rangle_{\mathcal{H}_q(\rho)} \\
&= 2c_3 c_5 \beta_1 \mathbb{F}.
\end{aligned}$$

Comparing with

$$\langle \mathbb{F}, \mathbb{D}(\mathcal{F}_2^{(4)})(*, -\bar{Z}) \rangle_{\text{Pet}} = c_4 L^{(N)} \left( f \otimes g, \frac{k+k'}{2} \right) \mathbb{F}(Z)$$

we obtain

$$c_5 = \frac{c_4 L^{(N)}(f \otimes g, \frac{k+k'}{2})}{2\beta_1 c_3},$$

which together with Lemma 8.3 yields the assertion.  $\square$

In order to make use of the above proposition in the next section we will also need to compare  $\langle \mathcal{P}(\phi_1 \otimes \phi_2), \mathcal{P}(\phi_1 \otimes \phi_2) \rangle$  with  $\langle \phi_1, \phi_1 \rangle \langle \phi_2, \phi_2 \rangle$ , where we have  $\langle \phi_\mu, \phi_\mu \rangle = \sum_{i=1}^r \frac{\langle \phi_\mu(\mathbf{y}_i), \phi_\mu(\mathbf{y}_i) \rangle_\mu}{e_i}$  for  $\mu = 1, 2$ , with  $\langle \cdot, \cdot \rangle_\mu$  denoting the (suitably normalized, see below) scalar product on  $\mathbf{U}_{\nu_\mu}^{(0)}$ . As always we denote by  $B(x, y) = \text{tr}(x\bar{y})$  the symmetric bilinear form associated to the quaternionic norm form.

**Lemma 8.6.** *Write  $\widetilde{G}_a^{(\nu)}(x) = (B(a, x))^\nu$  for  $a, x \in D_{\mathbb{C}}^{(0)} := D_{\infty}^{(0)} \otimes \mathbb{C}$  with  $n(a) = 0$  and let  $\nu_1 \geq \nu_2$ . Then*

$$\begin{aligned}
(1) \quad & \mathcal{P}(\widetilde{G}_a^{(\nu_1)} \otimes \widetilde{G}_a^{(\nu_2)})(d_1, d_2)(X_1, X_2) \\
(22) \quad &= \frac{\nu_1!}{\nu_2!} (B(a, d_1)X_1 + B(a, d_2)X_2)^{2\nu_2} \widetilde{G}_a^{(\nu_1-\nu_2)}(\text{Im}(d_1 \bar{d}_2)).
\end{aligned}$$

(2) *For  $a \in D_{\mathbb{C}}^{(0)}$  as above there is  $b \in D_{\mathbb{C}} := D \otimes \mathbb{C}$  with  $ab = 0, a\bar{b} = a, n(b) = 0$ , and for such a  $b$  we have*

$$\begin{aligned}
(23) \quad & \lim_{\lambda \rightarrow 0} \frac{1}{\lambda^{\nu_1-\nu_2}} \mathbb{P}_{\text{Geg}}((a, a + \lambda b), (d_1, d_2))(Y_1, Y_2, X_1, X_2) \\
&= c_6 (B(a, d_1)X_1 + B(a, d_2)X_2)^{2\nu_2} \widetilde{G}_a^{(\nu_1-\nu_2)}(\text{Im}(d_1 \bar{d}_2))(Y_1 + Y_2)^{2\nu_2}.
\end{aligned}$$

with  $c_6 = C_2(\nu_1 - \nu_2, 2\nu_2)$  as in Lemma 6.4.

*Proof.* (1) From the formula for the map  $\mathcal{P}$  in equation (20) we get

$$\begin{aligned}
& \mathcal{P}(\widetilde{G}_a^{(\nu_1)} \otimes \widetilde{G}_a^{(\nu_2)})(d_1, d_2)(X_1, X_2) \\
&= \frac{\nu_1!}{\nu_2!} (B(a, (d_1 X_1 + d_2 X_2) a (\bar{d}_1 X_1 + \bar{d}_2 X_2)))^{\nu_2} \widetilde{G}_a^{(\nu_1-\nu_2)}(\text{Im}(d_1 \bar{d}_2)).
\end{aligned}$$

Using  $\bar{a} = -a$ ,  $a^2 = 0$  and  $x\bar{a} = a\bar{x} - B(a, x)$  for  $x \in D_{\mathbb{C}}$  we get  $B(a, y\bar{a}x) = B(a, x)B(a, y)$  for  $x, y \in D_{\mathbb{C}}$ . We extend this identity to the polynomial ring, insert for  $x, y$  one of  $d_1X_1, d_2X_2$  and obtain  $B(a, (d_1X_1 + d_2X_2)\overline{(d_1X_1 + d_2X_2)}) = (B(a, d_1)X_1 + B(a, d_2)X_2)^2$ , which yields the assertion.

- (2) For simplicity we identify  $D_{\mathbb{C}}$  with the matrix ring  $M_2(\mathbb{C})$  and fix  $\mathbf{a} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ,  $\mathbf{b} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  (we will need this Lemma only for one particular choice of  $\mathbf{a}, \mathbf{b}$ ). Equation (8) in Lemma 6.4 gives us

$$\begin{aligned} & P_{\text{Geg}}((\mathbf{a}, \mathbf{a} + \lambda\mathbf{b}), (d_1, d_2))(Y_1, Y_2, X_1, X_2) \\ &= c_6 \left( (Y_1, Y_2) \begin{pmatrix} B(\mathbf{a}, d_1) & B(\mathbf{a}, d_2) \\ B(\mathbf{a} + \lambda\mathbf{b}, d_1) & B(\mathbf{a} + \lambda\mathbf{b}, d_2) \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \right)^{2\nu_2} \\ & \times \det \left( \begin{pmatrix} B(\mathbf{a}, d_1) & B(\mathbf{a}, d_2) \\ B(\mathbf{a} + \lambda\mathbf{b}, d_1) & B(\mathbf{a} + \lambda\mathbf{b}, d_2) \end{pmatrix} \right)^{\nu_1 - \nu_2}. \end{aligned}$$

Dividing by  $\lambda^{\nu_1 - \nu_2}$  and taking the limit for  $\lambda \rightarrow 0$  we get

$$c_6 ((Y_1 + Y_2)(B(\mathbf{a}, d_1)X_1 + B(\mathbf{a}, d_2)X_2))^{2\nu_2} \det \left( \begin{pmatrix} B(\mathbf{a}, d_1) & B(\mathbf{a}, d_2) \\ B(\mathbf{b}, d_1) & B(\mathbf{b}, d_2) \end{pmatrix} \right)^{\nu_1 - \nu_2}.$$

Computing the determinant for our choice of  $\mathbf{a}, \mathbf{b}$ , writing  $d_1, d_2$  as matrices  $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}, \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$  and using that quaternionic conjugation sends a matrix  $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$  to its classical adjoint  $\begin{pmatrix} x_4 & -x_2 \\ -x_3 & x_1 \end{pmatrix}$  one checks that both  $\det(\dots)^{\nu_1 - \nu_2}$  and  $G_{\mathbf{a}}^{(\nu_1 - \nu_2)}(\text{Im}(d_1\bar{d}_2))$  evaluate to  $(x_3y_4 - x_4y_3)^{\nu_1 - \nu_2}$ , which proves the assertion.  $\square$

**Proposition 8.7.** *Let  $R_1 \in \mathcal{U}_{\nu_1}^{(0)}, R_2 \in \mathcal{U}_{\nu_2}^{(0)}$  be given and let the scalar products  $\langle \cdot, \cdot \rangle_{\mu}$  on  $\mathcal{U}_{\nu_{\mu}}^{(0)}$  for  $\mu = 1, 2$  be normalized such that the Gegenbauer polynomial*

$$G^{(\nu_{\mu})}(x, y) = \frac{2^{\nu_{\mu}}}{\Gamma(1/2)} \sum_{j=0}^{[\nu_{\mu}/2]} (-1)^j \frac{1}{j!} (\nu_{\mu} - 2j)! \Gamma(\nu_{\mu} - j + \frac{1}{2}) (\text{tr}(x\bar{y}))^{\nu_{\mu} - 2j} (n(x)n(y))^j$$

(see [BS5, p. 47]) is the reproducing kernel for  $\mathcal{U}_{\nu_{\mu}}^{(0)}$ . Then one has

$$\langle \mathcal{P}(R_1 \otimes R_2), \mathcal{P}(R_1 \otimes R_2) \rangle_{\mathcal{H}_q(\rho)} = c_7 \langle R_1, R_1 \rangle_1 \langle R_2, R_2 \rangle_2$$

with

$$(24) \quad c_7 = c_6 \frac{\nu_1!}{\nu_2!} \binom{2\nu_1}{\nu_1} \binom{2\nu_2}{\nu_2} = C_2(\nu_1 - \nu_2, 2\nu_2) \frac{\nu_1!}{\nu_2!} \binom{2\nu_1}{\nu_1} \binom{2\nu_2}{\nu_2}$$

with  $C_2(\nu_1 - \nu_2, 2\nu_2)$  explicitly given in Lemma 6.4.

*Proof.* Since  $\mathcal{P}$  is an intertwining map between finite dimensional irreducible unitary representations of the compact orthogonal group it is clear that the right hand side and the left hand side of the asserted equality are proportional. It suffices therefore to evaluate both sides for a particular choice of  $R_1, R_2$ . We choose  $R_1 = G_{\mathbf{a}}^{(\nu_1)}, R_2 = G_{\mathbf{a}}^{(\nu_2)}$  with  $G_{\mathbf{a}}^{(\nu_{\mu})}(\mathbf{y}) = G^{(\nu_{\mu})}(\mathbf{a}, \mathbf{y})$ . The reproducing property of the Gegenbauer polynomial gives

$$\langle P_{\text{Geg}}(\mathbf{a}, \mathbf{a} + \lambda\mathbf{b}), \mathcal{P}(G_{\mathbf{a}}^{(\nu_1)} \otimes G_{\mathbf{a}}^{(\nu_2)}) \rangle_{\mathcal{H}_q(\rho)} = (\mathcal{P}(G_{\mathbf{a}}^{(\nu_1)} \otimes G_{\mathbf{a}}^{(\nu_2)}))^c(\mathbf{a}, \mathbf{a} + \lambda\mathbf{b}),$$

where we denote by the exponent  $c$  at  $(\mathcal{P}(G_{\mathbf{a}}^{(\nu_1)} \otimes G_{\mathbf{a}}^{(\nu_2)}))$  complex conjugation of the coefficients of this polynomial (in order to avoid confusion with quaternionic

conjugation). With the particular choice of  $\mathbf{a}, \mathbf{b}$  from the previous lemma we obtain, using  $\mathbf{a}\bar{\mathbf{b}} = \mathbf{a}$  and  $\mathbf{a}^2 = 0$  and writing  $\mathbf{a}^c$  for the vector obtained from  $\mathbf{a}$  by complex conjugation of the coordinates with respect to an orthonormal basis of  $D_\infty$ ,

$$\begin{aligned} & \frac{1}{\lambda^{\nu_1 - \nu_2}} (\mathcal{P}(G_{\mathbf{a}}^{(\nu_1)} \otimes G_{\mathbf{a}}^{(\nu_2)}))^c(\mathbf{a}, \mathbf{a} + \lambda \mathbf{b})(Y_1, Y_2) \\ &= \frac{(Y_1 + Y_2)^{2\nu_2}}{\lambda^{\nu_1 - \nu_2}} \binom{2\nu_1}{\nu_1} \binom{2\nu_2}{\nu_2} (B(\mathbf{a}^c, \mathbf{a}))^{2\nu_2} (B(\mathbf{a}^c, \text{Im}(\mathbf{a}(\overline{\mathbf{a} + \lambda \mathbf{b}}))))^{\nu_1 - \nu_2} \\ &= \binom{2\nu_1}{\nu_1} \binom{2\nu_2}{\nu_2} (B(\mathbf{a}^c, \mathbf{a}))^{\nu_1 + \nu_2} (Y_1 + Y_2)^{2\nu_2} \\ &= (G_{\mathbf{a}}^{(\nu_1)})(\mathbf{a}^c)(G_{\mathbf{a}}^{(\nu_2)})(\mathbf{a}^c)(Y_1 + Y_2)^{2\nu_2} \\ &= \langle G_{\mathbf{a}}^{(\nu_1)}, G_{\mathbf{a}}^{(\nu_1)} \rangle_1 \langle G_{\mathbf{a}}^{(\nu_2)}, G_{\mathbf{a}}^{(\nu_2)} \rangle_2 (Y_1 + Y_2)^{2\nu_2}. \end{aligned}$$

Inserting the formulas from Lemma 8.6 proves the assertion.  $\square$

**Corollary 8.8.** *With notations as in Proposition 8.5 and  $\mathcal{P}$  normalized as above one has*

$$\begin{aligned} \langle F, F \rangle_{\text{Pet}} &= \frac{c_4 c_7}{2c_3 \beta_1} L^{(N)} \left( f \otimes g, \frac{k+k'}{2} \right) \langle \phi_1, \phi_1 \rangle_1 \langle \phi_2, \phi_2 \rangle_2 \\ &= \frac{c_4 c_7}{2(2\pi i)^{k'-k} \beta_1} L^{(N)} \left( f \otimes g, \frac{k+k'}{2} \right) \langle \phi_1, \phi_1 \rangle_1 \langle \phi_2, \phi_2 \rangle_2, \end{aligned}$$

with  $c_4$  as in (21),  $c_7$  as in (24), and  $\beta_1 = \beta_1^{(4)}$  as in Corollary 3.2 of [BS1] (with  $m = 4$ ,  $r_p(1) = 1$  for all  $p \mid N$ , and  $\mathbf{a}_4(N)$  as in Proposition 3.2 of [BS1]).

Let  $F$  be a Yoshida lift of  $f$  and  $g$  as above and define  $F_{\text{can}} = \frac{F}{\sqrt{\langle \mathcal{P}(\phi_1 \otimes \phi_2), \mathcal{P}(\phi_1 \otimes \phi_2) \rangle}}$ .

Any rescaling of  $\phi_1, \phi_2$  or  $\mathcal{P}$  affects the numerator and denominator in the same way, so this may be viewed as a canonical choice of scaling of  $F$ . We can now express this canonical choice of  $F$  explicitly.

**Proposition 8.9.** *Let  $\phi_1^{(0)}, \phi_2^{(0)}$  be normalized by  $\langle \phi_1^{(0)}, \phi_1^{(0)} \rangle_1 = \langle \phi_2^{(0)}, \phi_2^{(0)} \rangle_2 = 1$  and let  $\mathcal{P}$  be normalized as above. Then one has*

$$F_{\text{can}} = \frac{1}{c_7} Y^2(\phi_1^{(0)}, \phi_2^{(0)})$$

with  $c_7$  given explicitly in equation (24).

Note that the Fourier coefficients of  $F_{\text{can}}$  are algebraic. From the results of [BS5, BS4] it is clear that the square of the (scalar valued) average over matrices  $T$  of fixed fundamental discriminant  $-d$  of the Fourier coefficients  $A(F, T) \in W_\rho$  of the Yoshida lifting  $F_{\text{can}}$  is proportional to the product of the central critical values of the twists with the quadratic character  $\chi_{-d}$  of the L-functions of the elliptic modular forms  $f$  and  $g$ ; notice that the averaging procedure for the  $W_\rho$ -valued Fourier coefficients involves a scalar product of  $A(F, T)$  with the vector  $\rho(T^{-1/2})\mathbf{v}_0$ , where  $\mathbf{v}_0$  is an  $O_n(\mathbb{R})$ -invariant vector in  $W_\rho$ . We can now make this proportionality as explicit as the result of [BS2] for the scalar valued case.

**Proposition 8.10.** *Assume that  $\nu_1, \nu_2$  are even and that both  $f, g$  have a  $+$ -sign in the functional equation. Choose  $N_1, N_2$  such that the (common) Atkin-Lehner eigenvalue  $\epsilon_p$  of  $f, g$  at  $p$  is  $-1$  if and only if  $p \mid N_1$ . Let  $-d < 0$  be a fundamental*



discriminant with  $(\frac{-d}{p})\epsilon_p = 1$  for all primes  $p$  dividing  $N_d = N/\gcd(N, d)$ . We let  $F = F_{\text{can}}$  be the canonical Yoshida lifting of  $f, g$  with respect to  $N_1, N_2$  and put

$$\mathfrak{a}(F, d) = \frac{\sqrt{d}}{2} \sum_{\substack{\{\mathbb{T}\} \\ \text{disc } \mathbb{T} = -d}} \frac{1}{\epsilon(\mathbb{T})} \int_{\mathbb{T}[\mathbf{x}] \leq 1} A(F, \mathbb{T})(x_1, x_2) dx_1 dx_2$$

where  $A(F, \mathbb{T})$  is the Fourier coefficient at  $\mathbb{T}$  of  $F$ , the summation is over integral equivalence classes of  $\mathbb{T}$ , and  $\epsilon(\mathbb{T})$  is the number of automorphy (units) of  $\mathbb{T}$ , i.e., the number of  $g \in \text{GL}_2(\mathbb{Z})$  with  ${}^t g \mathbb{T} g = \mathbb{T}$ .

Then one has

$$(25) \quad (\mathfrak{a}(F, d))^2 = c_8 \frac{L(1 + \nu_1, f)L(1 + \nu_2, g)L(1 + \nu_1, f \otimes \chi_{-d})L(1 + \nu_2, g \otimes \chi_{-d})}{\langle f, f \rangle \langle g, g \rangle}$$

with  $c_8^{-1} = 2^6(\nu_2 + 1)^2 \pi^{2+2\nu_1+2\nu_2}$ .

*Proof.* Corollary 4.3 of [BS5] gives

$$\left(\frac{d}{4}\right)^{\frac{\nu_1 + \nu_2}{2}} \sigma_0(N_d) \mathfrak{a}(F, d) = \frac{c}{2} \mathfrak{a}(\mathcal{W}(\phi_1), d) \mathfrak{a}(\mathcal{W}(\phi_2), d),$$

where the  $\mathfrak{a}(\mathcal{W}(\phi_\mu), d)$  are the Fourier coefficients of the Waldspurger liftings  $\mathcal{W}(\phi_\mu) = \sum_{j=1}^r \frac{1}{e_j} \sum_{x \in L_j} \phi(y_j)(x) \exp(2\pi i n(x)z)$  associated to the lattices  $L_j = \mathbb{D}^{(0)} \cap (\mathbb{Z}1 + 2\mathbb{R}_j)$  and where  $c = \frac{(-1)^{\nu_2} 2\pi}{2\nu_2 + 2}$ . Inserting the explicit version of Waldspurger's theorem from [Koh, BS4] gives the assertion.  $\square$

**Remark 8.11.** (1) *The restrictive conditions on  $f, g, N_1, d$  in the proposition are chosen in order to prevent that  $\mathfrak{a}(F, d)$  becomes zero for trivial reasons.*

(2) *Since  $\frac{\sqrt{d}}{2} \int_{\mathbb{T}[\mathbf{x}] \leq 1} x_1^i x_2^j dx_1 dx_2$  is zero for  $i$  or  $j$  odd and equal to*

$$\int_0^{\frac{\pi}{2}} \cos^i(\alpha) \sin^j(\alpha) d\alpha = \frac{\Gamma(i_1 + \frac{1}{2})\Gamma(j_1 + \frac{1}{2})}{2\Gamma(i_1 + j_1 + 1)}$$

*for even  $i = 2i_1, j = 2j_1$ , we have:*

*If for a prime  $\lambda$  not dividing  $2\nu_2!$  and some  $j \in \mathbb{N}$  one has  $\lambda^j \nmid \mathfrak{a}(F, d)/\pi$ , then there is some  $\mathbb{T}$  of discriminant  $-d$  such that  $\lambda^j$  does not divide all coefficients of the polynomial  $A(F, \mathbb{T})$ .*

(3) *With the help of the above proposition for the case  $\nu_1 = \nu_2$  and  $f = g$  one could derive an explicit version of formula (5.7) of [BS5]. Such an explicit version has been given independently by Luo in [Lu, (8)].*

## 9. A CONGRUENCE OF HECKE EIGENVALUES

As above, let  $f$  and  $g$  be cuspidal Hecke eigenforms for  $\Gamma_0(N)$ , of weights  $k' > k \geq 2$ . For critical  $k \leq t < k'$ , define  $L_{\text{alg}}(f \otimes g, t) := \frac{L(f \otimes g, t)}{\pi^{2t - (k-1)} \langle f, f \rangle}$ . (Alternatively one could divide by a canonical Deligne period—it makes no difference to the proposition below.) Let  $K$  be a number field containing all the Hecke eigenvalues of  $f$  and  $g$ . Let  $F$  be a Yoshida lift of  $f$  and  $g$ , lying in  $S_\rho(\Gamma_0^{(2)}(N))$  say, and define as in the previous section  $F_{\text{can}} = \frac{F}{\sqrt{\langle \mathcal{P}(\phi_1 \otimes \phi_2), \mathcal{P}(\phi_1 \otimes \phi_2) \rangle}}$ . In fact we have such an  $F$  and  $F_{\text{can}}$

for each factorisation  $N = N_1 N_2$  with an odd number of prime factors in  $N_1$ , and we label these  $F_i$  and  $F_{i,\text{can}}$  for  $1 \leq i \leq u$ , say. Note that by Lemma 8.4, these different Yoshida lifts of the same  $f$  and  $g$  are mutually orthogonal with respect to the Petersson inner product. Let's say  $F = F_1$  arbitrarily.

As in §2.1 of [Ar] the operators  $T(\mathfrak{m})$ , for  $(\mathfrak{m}, N) = 1$  (generated over  $\mathbb{Z}$  by the  $T(\mathfrak{p})$  and  $T(\mathfrak{p}^2)$ , see (2.2) of [Ar]) are self-adjoint for the Petersson inner product, and commute amongst themselves, so  $S_\rho(\Gamma_0^{(2)}(N))$  has a basis of simultaneous eigenvectors for such  $T(\mathfrak{m})$ . Also, these  $T(\mathfrak{m})$ , acting on elements of  $S_\rho(\Gamma_0^{(2)}(N))$ , preserve integrality (at any given prime) of Fourier coefficients, by (2.13) of [Sa]. If  $G \in S_\rho(\Gamma_0^{(2)}(N))$  is an eigenform (for the  $T(\mathfrak{m})$ , with  $(\mathfrak{m}, N) = 1$ ), then the Hecke eigenvalues for  $G$  are algebraic integers. This follows from Theorem I of [We2], which says that the characteristic polynomial of  $\rho_G(\text{Frob}_p^{-1})$  (c.f. §4 above) is  $1 - \mu_G(p)X + (\mu_G(p)^2 - \mu_G(p^2) - p^{k'-2})X^2 - p^{k'-1}\mu_G(p)X^3 + p^{2(k'-1)}X^4$  (c.f. (2.2) of [Ar]), and that the eigenvalues of  $\rho_G(\text{Frob}_p^{-1})$  are algebraic integers. Moreover, as  $p$  varies for fixed  $G$ , the  $\mu_G(p)$  and  $\mu_G(p^2)$  generate a finite extension of  $\mathbb{Q}$ .

**Proposition 9.1.** *Suppose that  $k' - k \geq 6$ . Suppose that  $\lambda$  is a prime of  $K$  such that  $\text{ord}_\lambda \left( L_{\text{alg}} \left( f \otimes g, \frac{k'+k}{2} \right) \right) > 0$  but  $\text{ord}_\lambda \left( L_{\text{alg}} \left( f \otimes g, \frac{k'+k}{2} + 1 \right) \right) = 0$ , and let  $\ell$  be the rational prime that  $\lambda$  divides. Suppose that  $\ell \nmid N$  and  $\ell > k' - 2$ . Assume that there exist a half-integral symmetric 2-by-2 matrix  $A$ , and an integer  $0 \leq b \leq k - 2$  such that, if for  $1 \leq i \leq u$ ,  $a_i$  denotes the coefficient of the monomial  $x^b y^{k-2-b}$  in the  $A$ -Fourier coefficient in  $F_{i,\text{can}}$ , then  $\text{ord}_\lambda \left( \sum_{i=1}^u a_i^2 \right) \leq 0$ . Then there is a cusp form  $G \in S_\rho(\Gamma_0^{(2)}(N))$ , an eigenvector for all the  $T(\mathfrak{m})$ , with  $(\mathfrak{m}, N) = 1$ , not itself a Yoshida lift of the same  $f$  and  $g$ , such that there is a congruence of Hecke eigenvalues between  $G$  and  $F$ :*

$$\mu_G(\mathfrak{m}) \equiv \mu_F(\mathfrak{m}) \pmod{\lambda}, \text{ for all } (\mathfrak{m}, N) = 1.$$

(We make  $K$  sufficiently large to contain the Hecke eigenvalues of  $G$ .)

*Proof.* Since  $k' - k \geq 6$ ,  $\frac{k'-k}{2} - 2 > 0$ , so  $\mathbb{D}_4 \left( \frac{k'-k}{2} - 2, k - 2 \right) \mathcal{F}_4^{(4)}(Z, W)$  is a cusp form. Let  $\{F_1, F_2, \dots, F_r\}$  be a basis of  $S_\rho(\Gamma_0^{(2)}(N))$  consisting of eigenforms for all the local Hecke algebras at  $\mathfrak{p} \nmid N$ , with  $F_1, \dots, F_u$  the Yoshida lifts of  $f$  and  $g$ , as above.

It is easy to show that  $\mathbb{D}_4 \left( \frac{k'-k}{2} - 2, k - 2 \right) \mathcal{F}_4^{(4)}(Z, W) = \sum_{i,j=1}^r c_{i,j} F_i(Z) F_j(W)$ , for some  $c_{i,j}$ . By (17),  $c_{1,1}$  is equal to the right hand side of (17), divided by  $F(w)\langle F, F \rangle$ , and  $c_{1,j} = 0$  for  $j \neq 1$ . Similarly for all the  $c_{i,i}$  for  $1 \leq i \leq u$ . Using Proposition 8.5, we find

$$(26) \quad c_{1,1} = c' \frac{L_{\text{alg}} \left( f \otimes g, \frac{k'+k}{2} + 1 \right)}{L_{\text{alg}} \left( f \otimes g, \frac{k'+k}{2} \right) \langle \mathcal{P}(\phi_1 \otimes \phi_2), \mathcal{P}(\phi_1 \otimes \phi_2) \rangle},$$

where

$$(27) \quad c' = \gamma_2 \left( 4, \frac{k'-k}{2} - 2, k - 2, 0 \right) (\pm N) \wedge_N(2) \frac{\zeta^{(N)}(2) \pi^2}{\zeta^{(N)}(4) \zeta^{(N)}(6) \zeta^{(N)}(4)} \prod_{p|N} \frac{(1-p^{-3})}{(1-p^{-1})}.$$

(The last term takes into account the fact that we have passed from incomplete to complete L-functions.)

We now choose  $A$  and  $\mathbf{b}$  as in the statement of the proposition. Imitating §4 of [Ka], let  $\mathcal{F}_{4,\rho,A}(Z)$  be the coefficient of  $x_w^{\mathbf{b}} y_w^{k-2-\mathbf{b}}$  in the coefficient of  $\mathbf{e}(\mathrm{Tr}(AW))$  in  $\mathbb{D}_4\left(\frac{k'-k}{2}-2, k-2\right)\mathcal{F}_4^{(4)}(Z, W)$ . Then

$$(28) \quad \mathcal{F}_{4,\rho,A}(Z) = \sum_{i=1}^u e_i F_{i,\mathrm{can}}(Z) + \sum_{i \geq u+1} e'_i F_i(Z),$$

where, for  $1 \leq i \leq u$ ,  $e_i = c' \frac{L_{\mathrm{alg}}(f \otimes g, \frac{k'+k}{2}+1)}{L_{\mathrm{alg}}(f \otimes g, \frac{k'+k}{2})} a_i$ . Careful checking of all the things that go into  $c'$  shows that it is a rational number, and that it follows from  $\ell > k'-2$  that  $\mathrm{ord}_{\ell}(c') \leq 0$ . The coefficients of  $\mathcal{F}_{4,\rho,A}$  are integral at  $\lambda$ , by Remarks 7.1 and 7.2. Given all this, we can apply the method of Lemma 5.1 of [Ka], to deduce that there is a congruence  $(\bmod \lambda)$  of Hecke eigenvalues (for all  $T(\mathfrak{m})$ , with  $(\mathfrak{m}, N) = 1$ ) between  $F$  and some other  $F_i = G$ , say, with  $i \geq u+1$ .

In a little more detail, we suppose that no such  $G$  exists, so that for each  $u+1 \leq i \leq r$  there exists an  $\mathfrak{m}_i$ , with  $(\mathfrak{m}_i, N) = 1$ , such that if  $\mu_{F_i}(\mathfrak{m}_i)$  is the eigenvalue of  $T(\mathfrak{m}_i)$  on  $F_i$  then  $\mu_{F_i}(\mathfrak{m}_i) \not\equiv \mu_F(\mathfrak{m}_i) \pmod{\lambda}$ . (We may enlarge  $K$  to contain all the Hecke eigenvalues for all the  $F_i$ .) Applying  $\prod_{i=u+1}^r (T(\mathfrak{m}_i) - \mu_{F_i}(\mathfrak{m}_i))$  to both sides of (28), we get something on the left that is integral at  $\lambda$ . On the right all the  $F_i$  terms, for  $i \geq u+1$ , disappear, while the remaining terms get multiplied by  $\prod_{i=u+1}^r (\mu_F(\mathfrak{m}_i) - \mu_{F_i}(\mathfrak{m}_i))$ , which is not divisible by  $\lambda$ , so on the right-hand-side the coefficient of  $x_z^{\mathbf{b}} y_z^{k-2-\mathbf{b}}$  in the coefficient of  $\mathbf{e}(\mathrm{Tr}(AZ))$ , namely

$$c' \prod_{i=u+1}^r (\mu_F(\mathfrak{m}_i) - \mu_{F_i}(\mathfrak{m}_i)) \frac{L_{\mathrm{alg}}(f \otimes g, \frac{k'+k}{2}+1)}{L_{\mathrm{alg}}(f \otimes g, \frac{k'+k}{2})} \left( \sum_{i=1}^u a_i^2 \right),$$

is non-integral at  $\lambda$ , which is a contradiction.  $\square$

In this proposition,  $L\left(f \otimes g, \frac{k'+k}{2}+1\right)$  plays the rôle of any critical value further right than the near-central one. We chose this next-to-near-central value merely for definiteness. In fact, the further right the evaluation point, the less laborious is the calculation of the critical value using Theorem 2 of [Sh4], but we have managed without too much difficulty in Example 9.1(3) below. Using Proposition 8.10 and Remark 8.11(2), we obtain the following.

**Corollary 9.2.** *Suppose that  $k'-k \geq 6$ , with  $k/2$  and  $k'/2$  odd, that  $N$  is prime, and that the common eigenvalue  $\epsilon_N$  for  $f$  and  $g$  is  $-1$ . Suppose that  $\lambda$  is a prime of  $K$  such that  $\mathrm{ord}_{\lambda}\left(L_{\mathrm{alg}}\left(f \otimes g, \frac{k'+k}{2}\right)\right) > 0$  but  $\mathrm{ord}_{\lambda}\left(L_{\mathrm{alg}}\left(f \otimes g, \frac{k'+k}{2}+1\right)\right) = 0$ , with  $\ell \nmid N$  and  $\ell > k'-2$ , where  $\ell$  is the rational prime that  $\lambda$  divides. Suppose that there is some fundamental discriminant  $-d < 0$  such that  $\left(\frac{-d}{p}\right) = \epsilon_p$  for all primes  $p$  dividing  $N_d = N/\mathrm{gcd}(N, d)$ , such that*

$$\mathrm{ord}_{\lambda} \left( \frac{L(k'/2, f)L(k'/2, f \otimes \chi_{-d})}{\pi^{k'} \langle f, f \rangle} \frac{L(k/2, g)L(k/2, g \otimes \chi_{-d})}{\pi^k \langle g, g \rangle} \right) \leq 0.$$

*Then there is a cusp form  $G \in S_{\rho}(\Gamma_0^{(2)}(N))$ , an eigenvector for all the  $T(\mathfrak{m})$  with  $(\mathfrak{m}, N) = 1$ , not a multiple of  $F$ , such that there is a congruence of Hecke eigenvalues*

between  $G$  and  $F$ :

$$\mu_G(\mathfrak{m}) \equiv \mu_F(\mathfrak{m}) \pmod{\lambda}, \text{ for all } (\mathfrak{m}, N) = 1.$$

(We make  $K$  sufficiently large to contain the Hecke eigenvalues of  $G$ .)

### 9.1. Examples.

- (1) When  $k = 2$  and  $k' = 4$  (so  $j = 0$  and  $\kappa = 3$ ), one may check that, for  $N = 23, 29, 31, 37$  or  $43$ , the dimension of  $S_3(\Gamma_0^{(2)}(N))$  ( $2, 4, 4, 9, 14$  respectively, using Theorem 2.2 in [I2]) is the same as that of the subspace spanned by Yoshida lifts of  $f \in S_4(\Gamma_0(N))$  and  $g \in S_2(\Gamma_0(N))$ . This appears to leave no room for  $G$  (recall Lemma 4.1). However, we calculated  $L_{\text{alg}}(f \otimes g, 3)$  in the case  $N = 23$ , using Theorem 2 of [Sh4] and Stein's tables [St]. (The two choices for  $g$  are conjugate over  $\mathbb{Q}(\sqrt{5})$ .) For the near-central value, this calculation involves an Eisenstein series of weight 2, to which a non-holomorphic adjustment must be made. The result was that  $L_{\text{alg}}(f \otimes g, 3) = 32/3$ , so there is in fact no divisor  $\lambda$ , dividing a large prime  $\ell$ , for which a congruence with some  $G$  is required.

- (2) The previous paragraph leaves open the possibility that the condition  $k' - k \geq 6$ , in Proposition 9.1, is purely technical. However, the following example shows that it is essential. Let  $k = 2$  and  $k' = 6$  (so  $j = 0$  and  $\kappa = 4$ ) and  $N = 11$ . As is well-known,  $S_2(\Gamma_0(11))$  is 1-dimensional, spanned by  $g = q - 2q^2 - q^3 + \dots$ , for which  $\epsilon_{11} = -1$ . Using [St],  $\dim S_6(\Gamma_0(11)) = 4$ , with the  $\epsilon_{11} = -1$  eigenspace 3-dimensional, spanned by the embeddings of a newform  $f = q + \beta q^2 + \dots$ , where  $\beta^3 - 90\beta + 188 = 0$ . The discriminant of this polynomial is  $2^4 3^3 19 \cdot 239$ . Using Theorem 2 of [Sh4] we find that  $L_{\text{alg}}(f \otimes g, 4) = -\frac{4^5 \alpha}{3}$ , with  $\text{Norm}(\alpha) = -\frac{17 \cdot 76157}{2^4 3^4 5^2 11^2 19 \cdot 239}$ . In fact  $\alpha$  is divisible by the prime ideals  $(17, \beta + 1)$  and  $(76157, \beta + 74208)$ . We check that  $L_{\text{alg}}(f \otimes g, 5) = \frac{4^5 11 \gamma}{6}$ , with  $\gamma = \frac{1}{1648383} (784522 - 12341\beta - 3842\beta^2)$ , of norm  $\frac{2^8 3 \cdot 5^2}{11^3 9 \cdot 239}$ , in which 17 and 76157 do not appear.

The dimension of  $S_4(\Gamma_0^{(2)}(11))$  is 7, from the table in §2.4 of [I2]. This fact was also obtained by Poor and Yuen, who gave an explicit basis for this space using theta series, [PY]. We are indebted to D. Yuen for calculating for us a Hecke eigenbasis, which included the three Yoshida lifts, a non-lift with rational eigenvalues, and three conjugate non-lifts with eigenvalues and Fourier coefficients in the same cubic field as  $f$  and the Yoshida lifts. He looked for congruences modulo primes dividing 17 or 76157 (or any other large primes), but found that there were none, though it appears that each Yoshida lift has *Fourier coefficients* (not just Hecke eigenvalues) congruent mod 5 to those of a corresponding non-lift (suitably normalised).

- (3) We should expect that any example of  $f$  and  $g$  we look at, with prime level  $N$ , common  $\epsilon_N = -1$ , weights  $k' > k \geq 2$  with  $k' - k \geq 6$  and  $k'/2, k/2$  odd, is very likely to satisfy the remaining conditions of Corollary 9.2, for some  $\lambda$ . Here is an explicit example. Let  $N = 3, k = 6, k' = 14$ . We have  $S_6(\Gamma_0(3))$  spanned by  $g = q - 6q^2 + 9q^3 + \dots$ , and  $S_{14}(\Gamma_0(3))$  spanned by  $f = q + (-27 + 6\sqrt{1969})q^2 + 729q^3 + \dots$ ,  $\bar{f} = q + (-27 - 6\sqrt{1969})q^2 + 729q^3 + \dots$  and  $h = q - 12q^2 - 729q^3 - 8048q^4 + \dots$ . For both  $f$  and  $g$ ,  $\epsilon_3 = -1$ . Using Theorem 2 of [Sh4] we find that  $L_{\text{alg}}(f \otimes g, 10) = \frac{-4^{14}}{41913} \alpha$ , with  $\alpha = \frac{-467}{35640} - \frac{2119\sqrt{1969}}{140350320}$ ,  $\text{Norm}(\alpha) = \frac{7 \cdot 271 \cdot 461 \cdot 653}{2^8 3^7 5^2 11^3 179}$ . (Note that  $1969 = 11 \cdot 179$ .)

So we may take  $\lambda$  to be an appropriate divisor of  $\ell = 271,461$  or  $653$ . (All three of these primes split in  $\mathbb{Q}(\sqrt{1969})$ .) Also using Theorem 2 of [Sh4], we find that  $L_{\text{alg}}(f \otimes g, 11) = \frac{4^{14} 3!}{3 \cdot 10! 5!} \beta$ , with  $\beta = -25/(3\sqrt{1969})$ , so  $\lambda \nmid L_{\text{alg}}(f \otimes g, 11)$ . Finally, by direct application of Theorem 5.6 of [GZ], we calculate  $\frac{L(k/2, g)L(k/2, g \otimes \chi_{-4})}{\pi^k \langle g, g \rangle} = \frac{2^{12} 6}{4! 4^{5/2}}$  and  $\frac{L(k'/2, f)L(k'/2, f \otimes \chi_{-4})}{\pi^{k'} \langle f, f \rangle} = \frac{2^{27} 6!}{12! 4^{13/2}} \gamma$ , where  $\gamma = 13488 + \frac{256056}{\sqrt{1969}}$ , with  $\text{Norm}(\gamma) = \frac{2^6 3^3 5^2 7 \cdot 967751}{11 \cdot 179}$ . The product of these is not divisible by  $\lambda$  (for any of the three choices).

It seems though that finding an example where one can directly observe the congruence guaranteed by Corollary 9.2 would be difficult. Already for  $k = 2, k' = 10$  and  $N = 11$  we have  $\dim S_{\mathcal{G}}(\Gamma_0^{(2)}(11)) = 31$  (from the table in 7-11 of [Has]).

- (4) For us,  $f$  and  $g$  are of level  $N > 1$ , and Yoshida lifts do not exist at level 1. However, Bergström, Faber and van der Geer have found experimentally what appear to be eleven examples of congruences of exactly the same shape, but for  $f$  and  $g$  of level 1 [BFvdG]. For example, it appears that there is a genus-2 cusp form of level 1 and weight  $\text{Sym}^{20} \otimes \det^5$  such that

$$\mu_{\mathcal{G}}(\mathfrak{p}) \equiv \alpha_{\mathfrak{p}}(f) + \mathfrak{p}^3 \alpha_{\mathfrak{p}}(g) \pmod{\lambda},$$

with  $\lambda \mid 227$ , where  $f$  and  $g$  are cuspidal Hecke eigenforms of genus 1, level 1 and weights  $k' = 28, k = 22$  respectively. Bergström et. al. have checked this for  $\mathfrak{p} \leq 17$ . Using Theorem 2 of [Sh4], we have checked that  $L(f \otimes g, 25) = \frac{4^{27} \pi^{29}}{108(24!)} \cdot \alpha(f, f)$ , with  $\text{Norm}(\alpha) = \frac{7 \cdot 17 \cdot 227}{2 \cdot 3^6 \cdot 5^4 \cdot 131 \cdot 139}$ . In two more examples, with  $(k', k, \ell) = (28, 18, 223)$  and  $(28, 20, 2647)$ , we have likewise checked that the prime occurring in the modulus of an apparent congruence also appears in the near-central tensor-product L-value, in accord with the Bloch-Kato conjecture.

**9.2. Higher powers of  $\lambda$ .** A minor modification of the proof of Proposition 9.1 gives the following.

**Proposition 9.3.** *Suppose that  $k' - k \geq 6$ . Suppose that  $\lambda$  is a prime of  $K$  such that  $\text{ord}_{\lambda} \left( \frac{L_{\text{alg}}(f \otimes g, \frac{k'+k}{2})}{L_{\text{alg}}(f \otimes g, \frac{k'+k}{2} + 1)} \right) = n > 0$ , and let  $\ell$  be the rational prime that  $\lambda$  divides. Suppose that  $\ell \nmid N$  and  $\ell > k' - 2$ . Assume that there exist a half-integral symmetric 2-by-2 matrix  $A$ , and an integer  $0 \leq b \leq k - 2$  such that, if for  $1 \leq i \leq u$ ,  $\alpha_i$  denotes the coefficient of the monomial  $x^b y^{k-2-b}$  in the  $A$ -Fourier coefficient in  $F_{i, \text{can}}$ , then  $\text{ord}_{\lambda}(\sum_{i=1}^u \alpha_i^2) \leq 0$ . Then there are independent cusp forms  $G_1, \dots, G_r \in S_{\rho}(\Gamma_0^{(2)}(N))$ , eigenvectors for all the  $T(\mathfrak{m})$  with  $(\mathfrak{m}, N) = 1$ , not themselves Yoshida lifts of the same  $f$  and  $g$ , such that there are congruences of Hecke eigenvalues between the  $G_i$  and  $F$ :*

$$\mu_{G_i}(\mathfrak{m}) \equiv \mu_F(\mathfrak{m}) \pmod{\lambda^{s(i)}}, \text{ for all } (\mathfrak{m}, N) = 1,$$

with  $\sum_{i=1}^r s(i) \geq n$ . (We make  $K$  sufficiently large to contain the Hecke eigenvalues of  $G$ .)

Modifying the proof of Proposition 5.1, applying the main theorem of [U2], one may show (under similar conditions) that each  $G_i$  contributes an element of order  $\lambda^{s(i)}$  to  $H_r^1(\mathbb{Q}, A_{\lambda}((k' + k - 2)/2))$ , but it does not show that these elements are independent. However, using Hecke algebras as in [U1], it should be possible to show

that  $\lambda^n$  divides  $\#H_f^1(\mathbb{Q}, A_\lambda((k' + k - 2)/2))$ , and this is covered by the approach in [AK], so we leave it to them.

## REFERENCES

- [AK] M. Agarwal, K. Klosin, Yoshida lifts and the Bloch-Kato conjecture for the convolution L-function, submitted.
- [An] A. N. Andrianov, *Quadratic forms and Hecke operators*, Grundle. d. Math. Wiss. 286, Springer, Berlin 1987.
- [Ar] T. Arakawa, Vector valued Siegel's modular forms of degree two and the associated Andrianov L-functions, *Manuscripta Math.* **44** (1983), 155–185.
- [AS] M. Asgari, R. Schmidt, Siegel modular forms and representations, *Manuscripta Math.* **104** (2001), 173–200.
- [BFvdG] J. Bergström, C. Faber, G. van der Geer, Siegel modular forms of degree three and the cohomology of local systems, arXiv:1108.3731v1[math:AG].
- [BK] S. Bloch, K. Kato, L-functions and Tamagawa numbers of motives, The Grothendieck Festschrift Volume I, 333–400, Progress in Mathematics, 86, Birkhäuser, Boston, 1990.
- [B1] S. Böcherer, Über die Fourier-Jacobi-Entwicklung Siegelscher Eisensteinreihen II, *Math. Z.* **189** (1985), 81–110.
- [B2] S. Böcherer, Siegel modular forms and theta series, *AMS Proc. Symp. Pure Math.*, Vol. 49 (1989), part 2, 3–17.
- [B3] S. Böcherer, Über die Funktionalgleichung automorpher L-Funktionen zur Siegelschen Modulgruppe, *J. f. d. reine u. angew. Math.* **362** (1985), 146–168.
- [B4] S. Böcherer, Über die Fourierkoeffizienten der Siegelschen Eisensteinreihen, *Manuscripta math.* **45** (1984), 273–288.
- [BKS] S. Böcherer, H. Katsurada, R. Schulze-Pillot, On the basis problem for Siegel modular forms with level, pp.13–28 in: Modular forms on Schiermonnikoog. Cambridge University Press 2008.
- [BSY] S. Böcherer, T. Satoh, T. Yamazaki, On the pullback of a differential operator and its application to vector valued Eisenstein series, *Comment. Math. Univ. St. Paul.* **41** (1992), 1–22.
- [BCG] S. Böcherer, C. G. Schmidt, p-adic measures attached to Siegel modular forms, *Ann. Inst. Fourier* **50** (2000), 1375–1443.
- [BS1] S. Böcherer, R. Schulze-Pillot, Siegel modular forms and theta series attached to quaternion algebras, *Nagoya Math. J.* **121** (1991), 35–96.
- [BS2] S. Böcherer, R. Schulze-Pillot, The Dirichlet series of Koecher and Maaß and modular forms of weight  $\frac{3}{2}$ . *Math. Z.* **209** (1992), no. 2, 273–287.
- [BS3] S. Böcherer, R. Schulze-Pillot, Siegel modular forms and theta series attached to quaternion algebras, II, *Nagoya Math. J.* **147** (1997), 71–106.
- [BS4] S. Böcherer, R. Schulze-Pillot, Vector valued theta series and Waldspurger's theorem. *Abh. Math. Sem. Univ. Hamburg* **64** (1994), 211–233.
- [BS5] S. Böcherer, R. Schulze-Pillot, Mellin transforms of vector valued theta series attached to quaternion algebras, *Math. Nachr.* **169** (1994), 31–57.
- [Br] J. Brown, Saito-Kurokawa lifts and applications to the Bloch-Kato conjecture, *Compositio Math.* **143** (2007), 290–322.
- [CF] C.-L. Chai, G. Faltings, *Degeneration of Abelian Varieties*, Ergeb. Math. Grenzgeb. (3), Vol. 22, Springer, Berlin, 1990.
- [De] P. Deligne, Valeurs de Fonctions L et Périodes d'Intégrales, *AMS Proc. Symp. Pure Math.*, Vol. 33 (1979), part 2, 313–346.
- [Di] F. Diamond, The refined conjecture of Serre, in Elliptic Curves, Modular Forms and Fermat's Last Theorem, J. Coates, S.T. Yau, eds., 172–186, International Press, Cambridge MA, 1995.
- [DFG] F. Diamond, M. Flach, L. Guo, The Tamagawa number conjecture of adjoint motives of modular forms, *Ann. Sci. École Norm. Sup. (4)* **37** (2004), 663–727.
- [Du1] N. Dummigan, Symmetric square L-functions and Shafarevich-Tate groups, II, *Int. J. Number Theory* **5** (2009), 1321–1345.

- [Du2] N. Dummigan, Selmer groups for tensor product L-functions, pp. 37–46 in *Automorphic representations, automorphic L-functions and arithmetic*, R.I.M.S. Kokyuroku 1659, July 2009.
- [DIK] N. Dummigan, T. Ibukiyama, H. Katsurada, Some Siegel modular standard L-values, and Tate-Shafarevich-Tate groups, *J. Number Theory* **131**(2011), 1296–1330.
- [Ed] B. Edixhoven, The weight in Serre’s conjectures on modular forms, *Invent. Math.* **109** (1992), 563–594.
- [Ei] M. Eichler, The basis problem for modular forms and the traces of the Hecke operators, p. 76–151 in *Modular functions of one variable I*, Lecture Notes Math. 320, Berlin-Heidelberg-New York 1973.
- [Ev] S.A. Evdokimov, Action of the irregular Hecke operator of index  $p$  on the theta series of a quadratic form, *J. Sov. Math.* **38** (1987), 2078–2081.
- [Fa] G. Faltings, Crystalline cohomology and  $p$ -adic Galois representations, in *Algebraic analysis, geometry and number theory* (J. Igusa, ed.), 25–80, Johns Hopkins University Press, Baltimore, 1989.
- [Fl] M. Flach, A generalisation of the Cassels-Tate pairing, *J. f. d. reine u. angew. Math.* **412** (1990), 113–127.
- [Fo1] J.-M. Fontaine, Le corps de périodes  $p$ -adiques, *Astérisque* **223** (1994), 59–111.
- [Fo2] J.-M. Fontaine, Valeurs spéciales des fonctions L des motifs, Séminaire Bourbaki, Vol. 1991/92. *Astérisque* **206** (1992), Exp. No. 751, 4, 205–249.
- [FL] J.-M. Fontaine, G. Lafaille, Construction de représentations  $p$ -adiques, *Ann. Sci. E.N.S.* **15** (1982), 547–608.
- [Fu] M. Furusawa, On L-functions for  $\mathrm{GSp}(4) \times \mathrm{GL}(2)$  and their special values, *J. f. d. reine u. angew. Math.* **438** (1993), 187–218.
- [GT] A. Genestier, J. Tilouine, Systèmes de Taylor-Wiles pour  $\mathrm{GSp}(4)$ , *Astérisque* **302** (2005), 177–290.
- [Go] R. Godement, Séminaire Cartan 10 (1957/58) Exp.4-9.
- [GZ] B. H. Gross, D. B. Zagier, Heegner points and derivatives of L-series, *Invent. Math.* **84** (1986), 225–320.
- [Ha] G. Harder, A congruence between a Siegel and an elliptic modular form, manuscript, 2003, reproduced in *The 1-2-3 of Modular Forms* (J. H. Bruinier et. al.), 247–262, Springer-Verlag, Berlin Heidelberg, 2008.
- [Har] A. Haruki, Explicit formulae of Siegel Eisenstein series, *Manuscripta math.* **92** (1997), 107–134.
- [Has] K. Hashimoto, The dimension of the spaces of cusp forms on Siegel upper half-plane of degree two, I, *J. Fac. Sci. Univ. Tokyo, Sect. IA, Math.* **30** (1983), 403–488.
- [Hi1] H. Hida, *Modular forms and Galois cohomology*. Cambridge University Press, 2000.
- [Hi2] H. Hida, *Geometric Modular Forms and Elliptic Curves*. World Scientific, Singapore, 2000.
- [H-S] H. Hijikata, H. Saito, On the representability of modular forms by theta series, p. 13–21 in *Number Theory, Algebraic Geometry and Commutative Algebra*, in honor of Y. Akizuki, Tokyo 1973
- [I1] T. Ibukiyama, On differential operators on automorphic forms and invariant pluri-harmonic polynomials, *Comment. Math. Univ. Sanct. Paul.* **48** (1999), 103–118.
- [I2] T. Ibukiyama, Dimension formulas of Siegel modular forms of weight 3 and supersingular abelian varieties, in *Siegel Modular Forms and Abelian Varieties* (T. Ibukiyama, ed.), Proceedings of the Fourth Spring Conference on Modular Forms and Related Topics, Hamana Lake, Japan, 2007. Ryushido, 2007.
- [J-L] H. Jacquet, R. Langlands, Automorphic forms on  $\mathrm{GL}(2)$ , Lect. Notes in Math. 114, Berlin-Heidelberg-New York 1970.
- [Ji] J. Jia, Arithmetic of the Yoshida lift, preprint 2010, <http://www.math.lsa.umich.edu/~jxj/JohnsonJia/Research/Research.html>
- [KV] M. Kashiwara, M. Vergne, On the Segal-Shale-Weil representations and harmonic polynomials, *Invent. Math.* **44** (1987), 1–44
- [Ka] H. Katsurada, Congruence of Siegel modular forms and special values of their standard zeta functions, *Math. Z.* **259** (2008), 97–111.
- [Koh] W. Kohnen, Fourier coefficients of modular forms of half-integral weight. *Math. Ann.* **271** (1985), no. 2, 237–268.

- [Koz] N. Kozima, On special values of standard L-functions attached to vector valued Siegel modular forms, *Kodai Math. J.* **23** (2000), 255–265.
- [La] R. P. Langlands, Modular forms and  $\ell$ -adic representations, pp. 362–499 in *Modular functions of one variable II*, Lecture Notes in Math. Vol. 349, Springer, Berlin, 1973.
- [Lu] W. Luo, A note on the distribution of integer points on spheres, *Math. Z.* **267** (2011), 965–970.
- [Ma] H. Maaß, Siegel’s modular forms and Dirichlet series, Lecture Notes in Math., Vol. 216, Springer, Berlin, Heidelberg, New York 1971.
- [MW] B. Mazur, A. Wiles, Class fields of abelian extensions of  $\mathbb{Q}$ , *Invent. Math.* **76** (1984), 179–330.
- [PS] A. Pitale, R. Schmidt, Ramanujan-type results for Siegel cusp forms of degree 2, *J. Ramanujan Math. Soc.* **24** (2009), 87–111.
- [Miz] S. Mizumoto, Eisenstein series for Siegel modular groups, *Math. Annalen* **297**(1993), 581–625
- [PY] C. Poor, D.S. Yuen, Dimensions of cusp forms for  $\Gamma_0(p)$  in degree two and small weights, *Abh. Math. Sem. Univ. Hamburg.* **77** (2007), 59–80.
- [R1] K. Ribet, A modular construction of unramified  $p$ -extensions of  $\mathbb{Q}(\mu_p)$ , *Invent. Math.* **34** (1976), 151–162.
- [R2] K. Ribet, On modular representations of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  arising from modular forms, *Invent. Math.* **100** (1990), 431–476.
- [R3] K. Ribet, Report on  $(\text{mod } l)$  representations of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ , in *Motives*, A.M.S. Proc. Symp. Pure Math. **55:2** (1994), 639–676.
- [Sa] T. Satoh, On certain vector valued Siegel modular forms of degree two, *Math. Ann.* **274** (1986), 335–352.
- [Sc] R. Schmidt, Iwahori-spherical representations of  $\text{GSp}(4)$  and Siegel modular forms of degree 2 with square-free level, *J. Math. Soc. Japan* **57** (2005), 259–293.
- [Shz] H. Shimizu, Theta series and automorphic forms on  $\text{GL}_2$ , *J. Math. Soc. Japan* **24** (1972), 638–683.
- [Sh1] G. Shimura, On Eisenstein series, *Duke Math. J.* **50** (1983), 417–476.
- [Sh2] G. Shimura, On a class of nearly holomorphic automorphic forms, *Annals of Math.* **123** (1986), 347–406.
- [Sh3] G. Shimura, Nearly holomorphic functions on hermitian symmetric spaces, *Math. Ann.* **278** (1987), 1–28.
- [Sh4] G. Shimura, The special values of the zeta functions associated with cusp forms, *Comm. Pure Applied Math.* **29** (1976), 783–804.
- [SU] C. Skinner, E. Urban, Sur les déformations  $p$ -adiques de certaines représentations automorphes, *J. Inst. Math. Jussieu* **5** (2006), 629–698.
- [St] W. Stein, The Modular Forms Database: Tables, <http://modular.fas.harvard.edu/Tables/tables.html>
- [Tak] H. Takayanagi, Vector valued Siegel modular forms and their L-functions; application of a differential operator, *Japan J. Math.* **19** (1994), 251–297.
- [Tay] R. Taylor, On the  $\ell$ -adic cohomology of Siegel threefolds, *Invent. Math.* **114** (1993), 289–310.
- [U1] E. Urban, Selmer groups and the Eisenstein-Klingen ideal, *Duke Math. J.* **106** (2001), 485–525.
- [U2] E. Urban, On Residually Reducible Representations on Local Rings, *J. Algebra* **212** (1999), 738–742.
- [vdG] G. van der Geer, Siegel Modular Forms and Their Applications, in *The 1-2-3 of Modular Forms* (J. H. Bruinier et. al.), 181–245, Springer-Verlag, Berlin Heidelberg, 2008.
- [We1] R. Weissauer, Vektorwertige Siegelische Modulformen kleinen Gewichts, *J. f. d. reine u. angew. Math.* **343** (1983), 184–202
- [We2] R. Weissauer, Four dimensional Galois representations, *Astérisque* **302** (2005), 67–150.
- [We3] R. Weissauer, Endoscopy for  $\text{GSp}(4)$  and the Cohomology of Siegel Modular Threefolds, Lecture Notes in Math., Vol. 1968, 368 pp., Springer, Berlin, 2009.
- [We4] R. Weissauer, Existence of Whittaker models related to four dimensional symplectic Galois representations. Modular forms on Schiermonnikoog, 285–310, Cambridge University Press, Cambridge, 2008.
- [Y1] H. Yoshida, Siegel’s modular forms and the arithmetic of quadratic forms, *Invent. Math.* **60** (1980), 193–248.



- [Y2] H. Yoshida, The action of Hecke operators on theta series, p. 197-238 in: Algebraic and topological theories - to the memory of Dr. T. Miyata -, 1985.

KUNZENHOF 4B, 79117 FREIBURG, GERMANY  
*E-mail address:* boecherer@math.uni-mannheim.de

UNIVERSITY OF SHEFFIELD, SCHOOL OF MATHEMATICS AND STATISTICS, HICKS BUILDING, HOUNSFIELD ROAD, SHEFFIELD, S3 7RH, U.K.  
*E-mail address:* n.p.dummigan@shef.ac.uk

FR 6.1 MATHEMATIK, UNIVERSITÄT DES SAARLANDES, POSTFACH 151150, D-66041 SAARBRÜCKEN, GERMANY  
*E-mail address:* schulzep@math.uni-sb.de