Automorphic Forms on Feit's Hermitian Lattices

Neil Dummigan[1] and Sebastian Schönnenbeck* [2]

[1]School of Mathematics and Statistics , University of Sheffield , Hicks Building , Hounsfield Road , Sheffield, S3 7RH , United Kingdom , n.p.dummigan@sheffield.ac.uk
[2] RWTH Aachen University, Lehrstuhl D für Mathematik, Pontdriesch 14/16, 52062 Aachen, Germany, sebastian.schoennenbeck@rwth-aachen.de

March 11, 2018

We consider the genus of 20 classes of unimodular Hermitian lattices of rank 12 over the Eisenstein integers. This set is the domain for a certain space of algebraic modular forms. We find a basis of Hecke eigenforms, and guess global Arthur parameters for the associated automorphic representations, which recover the computed Hecke eigenvalues. Congruences between Hecke eigenspaces, combined with the assumed parameters, recover known congruences for classical modular forms, and support new instances of conjectured Eisenstein congruences for $\mathbb{U}(2,2)$ automorphic forms.

1 Introduction

Nebe and Venkov [32] looked at formal linear combinations of the 24 Niemeier lattices, which represent classes in the genus of even, unimodular, Euclidean lattices of rank 24. They found a set of 24 eigenvectors for the action of an adjacency operator for Kneser 2-neighbours, with distinct integer eigenvalues. What they did was equivalent to computing a set of Hecke eigenforms in a space of scalar-valued modular forms for a definite orthogonal group $\mathbb{O}_{24}$. They made, and proved most of, a conjecture on the degrees in which the Siegel theta series of these eigenvectors are first non-vanishing.

*The author is supported by the DFG collaborative research center TRR 195

Chenevier and Lannes [7] reconsidered the results of Nebe and Venkov, in the light of work of Arthur [1], and found (with proof) the endoscopic type of the automorphic representation of $\mathbb{O}_{24}(\mathbb{A}_{\mathbb{Q}})$ generated by each eigenvector. Each of these automorphic representations is a "lift" built out of automorphic representations of smaller rank groups, related to elliptic modular forms, and certain vector-valued Siegel modular forms of genus 2. They looked at various easily-proved congruences of Hecke eigenvalues between pairs of eigenvectors. Writing the eigenvalues in terms of the endoscopic decompositions, they obtained, after much cancellation from both sides, not only well-known congruences such as Ramanujan's $\tau(p) \equiv 1 + p^{11} \pmod{691}$, but also the first proved instance of a conjecture of Harder on congruences between Hecke eigenvalues of vector-valued Siegel cusp forms of genus 2 and cusp forms of genus 1, modulo large primes occurring in critical values of the L-functions of the latter [20]. The same method has subsequently been employed by Mégarbané to prove several similar congruences, and also some involving automorphic forms for SO_7 [30].

In this paper we replace the Niemeier lattices by the genus of 20 classes of unimodular Hermitian lattices of rank 12 over the Eisenstein integers. Thus the orthogonal group $\mathbb{O}_{24}$ is replaced by a definite unitary group $\mathbb{U}_{12}$. These classes were enumerated, and given explicit representatives, by Feit [15]. Using $\mathfrak{P}$-neighbours, in particular for $\mathfrak{P} = (2)$ (but also for $\mathfrak{P} = (\sqrt{-3})$), we obtain a basis of 20 eigenvectors in a space of scalar-valued algebraic modular forms. Using the computed Hecke eigenvalues, in tandem with the clues provided by various congruences between pairs of eigenvectors, we make compelling guesses for the endoscopic type of the automorphic representation of $\mathbb{U}_{12}(\mathbb{A}_{\mathbb{Q}})$ generated by each eigenvector.

Assuming these guesses, after cancellation we recover various congruences involving elliptic modular forms of levels 1 or 3, including Ramanujan's congruence, Eisenstein congruences "of local origin" [14] and Ribet-Diamond level-raising congruences [10, 34]. But we also obtain instances of conjectured congruences involving automorphic representations of $\mathbb{U}(2, 2)$, analogous to those of Harder, again with the moduli coming from critical L-values [13]. Indeed, the motivation for this work was to prove such congruences, following the work of Chenevier and Lannes on Harder's conjecture. However, because there is now a "bad" prime 3, it appears that it is not yet technically feasible to do something similar here. One of the alternative methods they employed was to use Arthur's multiplicity formula to prove the occurrence of the endoscopic types listed in their paper, and though work of Mok and of Kaletha et. al. provides such a formula in our case [27, 31], it appears that not enough is currently known about representations of ramified unitary groups to compute the terms in the formula. Similarly, there are problems in trying to imitate their use of explicit formulas of analytic number theory, to limit the possible components in the endoscopic decompositions. We are grateful to Chenevier for his comments on this. We thank him also for pointing out that it may be possible to prove, following what Ikeda did in the Niemeier lattices setting [25], some of our guesses for global Arthur parameters (see §7 below). However, this would not cover those cases involved in the congruences for $\mathbb{U}(2, 2)$ which we wish to prove.

In Section 2 we review some background on algebraic modular forms. In Section 3 we describe how to use $\mathfrak{P}$-neighbours of Hermitian lattices to compute Hecke operators for definite unitary groups, giving the matrices for $T_{(2)}$ and $T_{(\sqrt{-3})}$ on the 20-dimensional space of primary interest in this paper. Section 4 starts with a table of Hecke eigenvalues and conjectured global Arthur parameters, and proceeds to show how to recover the former from the latter. In Section 5 we consider congruences of Hecke eigenvalues between pairs of eigenspaces, and how they may be explained using the global Arthur parameters, then concentrating in Section 6 on congruences involving $\mathbb{U}(2,2)$. Section 7 contains some guesses on Hermitian theta series and Hermitian Ikeda lifts.

2 Preliminaries and notation

We want to take the time to establish the notation for the rest of the article and provide the necessary background on algebraic modular forms.

2.1 Algebraic modular forms

The primary reference for this subsection is Gross's original article [19]. In addition we also refer to the more algorithmically oriented article by Greenberg and Voight [17].

Let k be a totally real number field with ring of integers $\mathcal{O}_k$ and let

$$k_\infty := \mathbb{R} \otimes_{\mathbb{Q}} k \cong \mathbb{R}^{[k:\mathbb{Q}]}.$$

Looking at the finite places instead of the infinite ones we set $\hat{k}$ the ring of finite adeles of k:

$$\hat{k} = \left\{ (x_\mathfrak{p})_\mathfrak{p} \in \prod_{\mathfrak{p} \subset \mathcal{O}_k \text{ maximal}} k_\mathfrak{p} \mid x_\mathfrak{p} \in \mathcal{O}_\mathfrak{p} \text{ f.a.a. } \mathfrak{p} \right\} \tag{1}$$

where $k_\mathfrak{p}$ denotes the completion of k at $\mathfrak{p}$ and $\mathcal{O}_\mathfrak{p}$ its ring of integers. Finally we denote the (full) ring of adeles of k by $\mathbb{A}_k := k_\infty \times \hat{k}$.

Let $\mathbb{G}$ be a connected, reductive linear algebraic group over k such that $\mathbb{G}(k_\infty)$ is compact. Let $\rho : \mathbb{G} \to \mathrm{GL}_V$ be an irreducible finite-dimensional rational representation of $\mathbb{G}$ defined over some extension of k and let K be an open compact subgroup of $\mathbb{G}(\hat{k})$.

Definition 2.1. *The space of algebraic modular forms of weight V and level K is defined as*

$$\begin{aligned} M(V,K) &= \left\{ f : \mathbb{G}(\hat{k})/K \to V \mid {\scriptstyle f(g\gamma) = gf(\gamma) \text{ for} \atop \scriptstyle \gamma \in \mathbb{G}(\hat{k}),\ g \in \mathbb{G}(k)} \right\} \\ &\cong \left\{ f : \mathbb{G}(\hat{k}) \to V \mid {\scriptstyle f(g\gamma\kappa) = gf(\gamma) \text{ for } \gamma \in \mathbb{G}(\hat{k}), \atop \scriptstyle g \in \mathbb{G}(k), \kappa \in K} \right\}. \end{aligned} \tag{2}$$

The structure of $M(V,K)$ is summarized in the following proposition.

Proposition 2.2 ([19, Prop. (4.3),(4.5)])**.** *Set $\Sigma_K := \mathbb{G}(k)\backslash\mathbb{G}(\hat{k})/K$. The following holds:*

1. *The set Σ_K is finite.*

2. If $\alpha_i, 1 \leq i \leq h$, is a system of representatives for Σ_K and

$$\Gamma_i := \mathbb{G}(k) \cap \alpha_i K \alpha_i^{-1}, \tag{3}$$

then

$$M(V,K) \to \bigoplus_{i=1}^{h} V^{\Gamma_i}, \;\; f \mapsto (f(\alpha_1), ..., f(\alpha_h)) \tag{4}$$

is an isomorphism of vector spaces, where V^{Γ_i} denotes the Γ_i fixed points in V. In particular $M(V,K)$ is finite-dimensional.

Remark 2.3. For $V = k$ the trivial representation and in the notation of the preceeding proposition we have $V^{\Gamma_i} = V$ for all i so there is a natural isomorphism between $M(V,K)$ and the space of k-valued functions on Σ_K.

The groups Γ_i are discrete subgroups of the compact group $\mathbb{G}(k_\infty)$ hence finite. Moreover, since $\mathbb{G}(k_\infty)$ is compact the space V carries a $\mathbb{G}(k)$-invariant (totally positive) inner product (taking values in the extension of k over which V is defined) which we denote by $\langle -,- \rangle$. This inner product can be used to define a Petersson scalar product on the space $M(V,K)$. To that end let $\alpha_i, 1 \leq i \leq h$, again be a system of representatives for Σ_K and $\Gamma_i = \mathbb{G}(k) \cap \alpha_i K \alpha_i^{-1}$. For $f, f' \in M(V,K)$ we define

$$\langle f, f' \rangle_M := \sum_{i=1}^{h} \frac{1}{|\Gamma_i|} \langle f(\alpha_i), f'(\alpha_i) \rangle. \tag{5}$$

The so defined map $\langle -,- \rangle_M$ is obviously a totally positive definite symmetric bilinear from on $M(V,K)$ and does not depend on the choice of our representatives α_i.

2.2 Hecke operators

We keep the notation from the previous subsection.

In addition to being a finite-dimensional k-vector space, the space of algebraic modular forms also carries the structure of a module over the Hecke algebra of $\mathbb{G}$ with respect to K.

Definition 2.4. *The Hecke algebra $H_K = H(\mathbb{G}, K)$ is the (k-)algebra of all locally constant, compactly supported functions $\mathbb{G}(\hat{k}) \to k$ which are K-bi-invariant. The multiplication in H_K is given by convolution with respect to the (unique) Haar measure $d\lambda_K$ giving the compact group K measure 1, i.e.*

$$(F \cdot F')(\gamma) = \int_{\mathbb{G}(\hat{k})} F(x) F'(x^{-1}\gamma) d\lambda_K(x) = \int_{\mathbb{G}(\hat{k})} F(\gamma y^{-1}) F'(y) d\lambda_K(y) \tag{6}$$

for $F, F' \in H_K$ and $\gamma \in \mathbb{G}(\hat{k})$.

The algebra H_K has a canonical basis given by the characteristic functions of the double cosets with respect to K, $\mathbb{1}_{K\gamma K}$, $\gamma \in \mathbb{G}(\hat{k})$.

Remark 2.5. Let $\gamma, \gamma' \in \mathbb{G}(\hat{k})$ then $\mathbb{1}_{K\gamma K}$ and $\mathbb{1}_{K\gamma' K}$ commute as elements of H_K whenever the support of γ and γ' is disjoint. Moreover if K decomposes as a product $K = \prod_{\mathfrak{p} \text{ prime}} K_{\mathfrak{p}}$ and γ and γ' are only supported at primes where $K_{\mathfrak{p}}$ is a hyperspecial maximal compact subgroup of $\mathbb{G}(k_{\mathfrak{p}})$ the corresponding elements of H_K also commute.

The action of $\mathbb{1}_{K\gamma K} \in H_K$ on $M(V, K)$ is given as follows: Decompose $K\gamma K = \sqcup_i \gamma_i K$ into disjoint K-left cosets, then $\mathbb{1}_{K\gamma K}$ acts via the operator $T(\gamma) = T(K\gamma K) \in \operatorname{End}(M(V, K))$ defined by

$$(T(\gamma)f)(x) = \sum_i f(x\gamma_i) \text{ for } f \in M(V, K), x \in \mathbb{G}(\hat{k}). \tag{7}$$

We can extend T linearly to H_K to obtain a homomorphism of k-algebras. Moreover, the action of H_K on $M(V, K)$ is compatible with the inner product on $M(V, K)$ in the following sense.

Proposition 2.6 ([19, Prop. (6.9)])**.** *The adjoint operator of $T(\gamma)$ is given by $T(\gamma^{-1})$ (as an element of $\operatorname{End}(M(V, K))$).*

In particular we can conclude that $M(V, K)$ is a semisimple H_K-module.

2.3 Open compact subgroups arising from lattices

The open, compact subgroups of $\mathbb{G}(\hat{k})$ which will play a role in this article all arise as stabilizers of lattices in the following way. Let $\mathbb{G} \hookrightarrow \mathrm{GL}_W$ be a faithful k-rational representation of $\mathbb{G}$ and $L \subset W$ a (full) $\mathcal{O}_k$-lattice in W. The group $\mathrm{GL}_W(\hat{k})$ (and thus also $\mathbb{G}(\hat{k})$) acts on the set of lattices in W and we obtain an open, compact subgroup $K_L = \operatorname{Stab}_{\mathbb{G}(\hat{k})}(L)$ with

$$K_L = \prod_{\mathfrak{p} \text{ prime}} K_{L,\mathfrak{p}}, \text{ where } K_{L,\mathfrak{p}} = \operatorname{Stab}_{\mathbb{G}(k_{\mathfrak{p}})}(L \otimes \mathcal{O}_{\mathfrak{p}}). \tag{8}$$

The group $K_{L,\mathfrak{p}}$ is a hyperspecial maximal compact subgroup for all but finitely many finite primes of $\mathcal{O}_k$ (cf. [8, Prop. 3.3]). Moreover, two open compact subgroups K and K' arising as stabilizers in this way coincide at all but finitely many places. Note that we do not need to fix the representation W for this since open compact subgroups fix lattices in any representation.

In this situation decomposing $\mathbb{G}(\hat{k})$ into $\mathbb{G}(k)$-K_L-double cosets amounts to the same as finding representatives for the isomorphism classes in the ($\mathbb{G}$-)genus of L, i.e. decomposing the $\mathbb{G}(\hat{k})$-orbit of L into $\mathbb{G}(k)$-orbits. The class number $|\Sigma_{K_L}|$ is then also called the class number of L and the complexity of Σ_{K_L} can in some sense be measured by the mass of L,

$$\operatorname{mass}(L) := \operatorname{mass}_{\mathbb{G}}(L) := \sum_{i=1}^{h} \frac{1}{|\Gamma_i|}, \tag{9}$$

where $\Gamma_i = \mathbb{G}(k) \cap \alpha_i K_L \alpha_i^{-1}$ and $\mathbb{G}(\hat{k}) = \bigsqcup_{i=1}^{h} \mathbb{G}(k)\alpha_i K_L$, which means that

$$\{L_i = \alpha_i L \mid 1 \leq i \leq h\} \tag{10}$$

is a system of representatives for the genus of L.

The mass of L depends only on local information on $\mathbb{G}$ and L and can be computed without writing down a system of representatives for the genus. Formulae to do so are readily available in the literature (see for example [16] for the case of classical groups and [8] for semisimple groups split at every prime).

3 Hecke operators for unitary groups

In this section we want to introduce the specific groups we are working with and explain how to compute the relevant Hecke operators.

Let E be an imaginary quadratic number field and $n \in \mathbb{N}$. By $\mathbb{U}_n$ we will denote the linear algebraic group arising as the stabilizer of the standard n-dimensional Hermitian form over E. In other words $\mathbb{U}_n$ is the linear algebraic group over $\mathbb{Q}$ whose group of A-rational points is given by

$$\mathbb{U}_n(A) = \{g \in \mathrm{GL}_n(A \otimes_{\mathbb{Q}} E) \mid g^\dagger g = I_n\} \tag{11}$$

for any commutative $\mathbb{Q}$-algebra A, where I_n is the $n \times n$-unit matrix and $g^\dagger$ denotes the entrywise conjugate of g^{tr}.

The group $\mathbb{U}_n(\mathbb{R})$ is the usual unitary group of degree n over the complex numbers and hence compact. In particular, we are in the general setup of section 2 with $k = \mathbb{Q}$ and $\mathbb{G} = \mathbb{U}_n$. Finally we set

$$\mathrm{U}_n := \mathbb{U}_n(\mathbb{Q}) = \{g \in \mathrm{GL}_n(E) \mid E^\dagger E = I_n\}. \tag{12}$$

3.1 Hermitian lattices

Let $\mathcal{O}_E$ be the ring of integers of E. We denote by V_n the n-dimensional E-space E^n endowed with the standard Hermitian form

$$\langle v, w\rangle = v^\dagger w, \tag{13}$$

and by a Hermitian lattice we will always mean a full $\mathcal{O}_E$-lattice in V_n.

Definition 3.1. *Let $L \subset V_n$ be a Hermitian lattice. The dual of L is defined as*

$$L^\# := \{v \in V_n \mid \langle v, L\rangle \subset \mathcal{O}_E\}. \tag{14}$$

Let $I_1, ..., I_n$ be the invariant factors of $L^\#$ and L. Then

$$\partial(L) := \prod_{i=1}^{n} I_i \tag{15}$$

is called the discriminant of L. We call L integral if $L \subset L^\#$. For a fractional ideal $I \subset E$ we call L I-modular if $L^\# = I \cdot L$. If L is $\mathcal{O}_E$-modular (so $L = L^\#$) we call L unimodular.

Definition 3.2. *Let L be a Hermitian lattice. The group*

$$\mathrm{Aut}(L) := \mathrm{Stab}_{\mathrm{U}_n}(L) = \{g \in \mathrm{U}_n \mid gL = L\} \tag{16}$$

is called the automorphism group of L.

Remark 3.3. The lattice $L_0 := \mathcal{O}_E^n$ is unimodular and its ($\mathbb{U}_n$-)genus consists exactly of all unimodular lattices. In particular, the group $K_{L_0,p} \subset \mathbb{U}_n(\mathbb{Q}_p)$ is a hyperspecial maximal compact subgroup, whenever p decomposes in E. Moreover, for $E = \mathbb{Q}(\sqrt{-3})$ the unimodular lattices in dimension at most 12 are fully classified, see [15].

3.2 Neighbours and Hecke operators

We now want to describe how one can compute certain Hecke operators.

Definition 3.4. *Let $L \subset V_n$ be an integral Hermitian lattice and let $\mathfrak{p} \subset \mathcal{O}_E$ be a nonzero prime ideal of $\mathcal{O}_E$ such that $\mathfrak{p} \nmid \partial(L)$. A lattice $L' \in \mathrm{genus}(L)$ is called a $\mathfrak{p}$-neighbour of L if $L/(L \cap L') \cong \mathcal{O}_E/\mathfrak{p}$ and $L'/(L \cap L') \cong \mathcal{O}_E/\overline{\mathfrak{p}}$. We denote the (finite) set of $\mathfrak{p}$-neighbours of L by $N(L, \mathfrak{p})$.*

Remark 3.5. The stabilizer K_L of L in $\mathbb{U}_n(\hat{\mathbb{Q}})$ acts transitively on the set of $\mathfrak{p}$-neighbours of L for every $\mathfrak{p}$ (cf. [17, Thm. 5.10]). Taking an element $\gamma \in \mathbb{U}_n(\hat{\mathbb{Q}})$ such that $\gamma L \in N(L, \mathfrak{p})$, it follows that $N(L, \mathfrak{p}) = (K_L \gamma K_L)\{L\}$. Clearly γ can be chosen to have non-trivial support only at $\mathfrak{p}$. Moreover, all lattices in the set

$$M(L, \mathfrak{p}) := \{N \cap L \mid N \in N(L, \mathfrak{p})\} \tag{17}$$

belong to the same genus. We call $T(K_L \gamma K_L)$ the neighbouring operator at $\mathfrak{p}$ and denote it by $T_{\mathfrak{p}}$.

An algorithm for computing all $\mathfrak{p}$-neighbours of a given lattice L is described in [28]. In addition we can test Hermitian lattices for isometry by employing the Plesken-Souvignier algorithm [33]. Knowing this we can compute the Hecke operator corresponding to the the $\mathfrak{p}$-neighbours of a given lattice as follows.

Proposition 3.6. *Let L be an integral Hermitian lattice with adelic stabilizer $K_L < \mathbb{U}_n(\hat{\mathbb{Q}})$ and $\mathfrak{p} \subset \mathcal{O}_E$ a nonzero prime ideal such that $\mathfrak{p} \nmid \partial(L)$. Moreover let $\gamma \in \mathbb{U}_n(\hat{\mathbb{Q}})$ be such that $\gamma L \in N(L, \mathfrak{p})$, and choose a system $\alpha_i, 1 \le i \le h$ for $\mathrm{U}_n \backslash \mathbb{U}_n(\hat{\mathbb{Q}})/K_L$. Then $L_i := \alpha_i L, 1 \le i \le h$, forms a system of representatives, and with respect to the natural basis of $M(\mathrm{triv}, K_L)$ the operator $T(K_L \gamma K_L)$ has the matrix representation $(t_{i,j})_{i,j=1}^h$ with*

$$t_{i,j} = |\{M \in N(L_i, \mathfrak{p}) \mid M \cong L_j\}|. \tag{18}$$

Proof. The natural basis for $M(\mathrm{triv}, K_L)$ consists of the maps $F_i : \mathbb{U}_n(\hat{\mathbb{Q}}) \to \mathbb{Q}$ with $F_i(\alpha_j) = \delta_{i,j}$. We decompose

$$K_L \gamma K_L = \bigsqcup_{j=1}^{r} \gamma_r K_L \tag{19}$$

such that

$$N(L,\mathfrak{p}) = \{\gamma_j L \mid 1 \le j \le r\}. \tag{20}$$

Then the (i,j)-entry of $T(K_L\gamma K_L)$ with respect to the given basis is the coefficient of F_i in $T(K_L\gamma K_L)F_j$ which is

$$\begin{aligned} T(K_L\gamma K_L)F_j(\alpha_i) &= \sum_{a=1}^{r} F_j(\alpha_i\gamma_a) \\ &= |\{1 \le a \le r \mid \alpha_i\gamma_a \in \mathrm{U}_n\alpha_j K_L\}| \\ &= |\{1 \le a \le r \mid \alpha_i\gamma_a L \cong \alpha_j L\}| \\ &= |\{M \in N(L,\mathfrak{p}) \mid \alpha_i M \cong L_j\}| \\ &= |\{M \in N(L_i,\mathfrak{p}) \mid M \cong L_j\}|. \end{aligned} \tag{21}$$

This proves the assertion. □

While this algorithm works perfectly well in the cases we are primarily interested in, we also want to describe an alternative method, which often takes significantly less time. This alternative method has the added benefit of computing a system of representatives of a second genus of lattices along the way.

Proposition 3.7. *Let L and $\mathfrak{p}$ be as in Proposition 3.6, where we in addition assume that $\mathfrak{p}$ corresponds to a prime of $\mathbb{Q}$ that is either inert or ramified in E, and $M \in \{L \cap N \mid N \in N(L,\mathfrak{p})\}$. Choose representatives $L_i, 1 \le i \le h$ and $M_j, 1 \le j \le h'$, for* genus(L) *and* genus(M)*, respectively. Set $S_{\mathfrak{p}} = (s_{i,j}) \in \mathbb{Z}^{h\times h'}$ to be the matrix with entries*

$$s_{i,j} = |\{X \subset L_i \mid X \cong M_j\}|. \tag{22}$$

In addition let $d := |\{L \cap N \mid N \in N(L,\mathfrak{p})\}|$. Set $S'_{\mathfrak{p}}$ to be the matrix

$$S'_{\mathfrak{p}} := \mathrm{diag}(|\mathrm{Aut}(M_1)|, ..., |\mathrm{Aut}(M_{h'})|) \cdot S^{tr} \cdot \mathrm{diag}(|\mathrm{Aut}(L_1)|^{-1}, ..., |\mathrm{Aut}(L_h)|^{-1}). \tag{23}$$

Then the operator $T(K_L\gamma K_L)$ from Proposition 3.6 (with respect to the natural basis) can be computed as

$$S_{\mathfrak{p}} \cdot S'_{\mathfrak{p}} - d \cdot I_h. \tag{24}$$

Proof. The proof works analogously to that of Proposition 3.6. One only needs to see that the (j,i)-entry of

$$\mathrm{diag}(|\mathrm{Aut}(M_1)|, ..., |\mathrm{Aut}(M_{h'})|) \cdot S^{tr} \cdot \mathrm{diag}(|\mathrm{Aut}(L_1)|^{-1}, ..., |\mathrm{Aut}(L_h)|^{-1}) \tag{25}$$

counts the lattices above M_j which are isomorphic to L_i. This however is a simple counting argument (cf. [2] and [37, La. (4.2)]). □

Remark 3.8. If one wants to employ Proposition 3.7 in practice it is not necessary to have a system of representatives of genus(M) already at hand. Instead such a system can be found along the way by computing the relevant sublattices of the representatives for genus(L).

3.3 Computational results

In this subsection we present the results of our computations of Hecke operators for certain genera of Hermitian lattices for $E = \mathbb{Q}(\sqrt{-3})$.

We start by computing the neighbouring operator T_2 acting on the space $M(\mathrm{triv}, K_L)$ where L is a so-called Eisenstein lattice in dimension 12. The genus in question consists of all $\langle\sqrt{-3}\rangle$-modular lattices in V_{12} and was classified in [22]. The genus decomposes into 5 isometry classes which we take in the order of [22, Thm. 2]. In particular, the cardinalities of the automorphism groups are (in this order)

$$22568879259648000, 8463329722368, 206391214080, 101016305280, \text{ and } 2690072985600. \tag{26}$$

Following Proposition 3.6 one computes the basis representation of T_2 as

$$\begin{pmatrix} 65520 & 3888000 & 1640250 & 0 & 0 \\ 1458 & 516285 & 3956283 & 1119744 & 0 \\ 15 & 96480 & 2467899 & 2998272 & 31104 \\ 0 & 13365 & 1467477 & 3935781 & 177147 \\ 0 & 0 & 405405 & 4717440 & 470925 \end{pmatrix}. \tag{27}$$

Alternatively we employ Proposition 3.7. The second genus which we compute along the way consists of lattices which are of index 4 (elementary divisors $2 \cdot \mathcal{O}_E$) in lattices of the given genus. It decomposes into 25 isometry classes with corresponding automorphism group cardinalities (in the order in which we found the representatives)

$$\begin{aligned} &501530650214400, 4701849845760, 3715041853440, 11609505792, 27518828544, \\ &705277476864, 9795520512, 181398528, 967458816, 15116544, \\ &4478976, 95551488, 103195607040, 859963392, 23887872, \\ &20155392, 839808, 186624, 13271040, 524880, \\ &1530550080, 9447840, 233280, 1140480, \text{ and } 246343680. \end{aligned} \tag{28}$$

After computation we find

$$S_2' = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, \tag{29}$$

where we only write down S_2' for the sake of readability. In particular the number d from Proposition 3.7 is 3 (as it is the sum of the entries of any of the rows of S_2') and we obtain the same representation for T_2 as before.

We now present the Hecke operators we are primarily interested in. To that end let $L \subset V_{12}$ be a unimodular Hermitian lattice of rank 12. The genus of L consists of 20 isometry classes which were classified in [15]. We consider them in the following order: The first 11 are the indecomposable unimodular lattices in the same order as [15, Table II], then the 12-dimensional standard lattice, then the 7 direct sums of lattices in [15, Table I] with standard lattices of appropriate rank (again in the same order as in the source table), and finally the direct sum of two copies of the lattice called U_6 in [15].

Employing Proposition 3.7 we managed to compute the Hecke operators T_2 and $T_{\sqrt{-3}}$ acting on $M(\mathrm{triv}, K_L)$ which are given by the following matrices:

$$
T_2 : \begin{pmatrix}
3389169 & 0 & 49 & 18088 & 18088 & 588 & 661920 & 278565 & 4116 & 215600 & 918750 & 0 & 0 & 0 & 70 & 1302 & 441 & 45570 & 41454 & 0 \\
0 & 72810 & 118098 & 1358127 & 1358127 & 1082565 & 1082565 & 0 & 449064 & 0 & 0 & 0 & 0 & 0 & 13365 & 0 & 0 & 59049 & 0 & 0 \\
473088 & 2048 & 129031 & 575488 & 575488 & 110880 & 1013760 & 2113188 & 473088 & 0 & 46200 & 1 & 0 & 990 & 0 & 12012 & 0 & 67584 & 0 & 924 \\
2217072 & 299 & 7306 & 262990 & 154297 & 37323 & 1602315 & 1003002 & 72072 & 0 & 150150 & 0 & 0 & 0 & 715 & 4290 & 0 & 45903 & 36036 & 0 \\
2217072 & 299 & 7306 & 154297 & 262990 & 37323 & 1602315 & 1003002 & 72072 & 0 & 150150 & 0 & 0 & 0 & 715 & 4290 & 0 & 45903 & 36036 & 0 \\
870912 & 2880 & 17010 & 451008 & 451008 & 174303 & 2099520 & 1197504 & 161280 & 0 & 85050 & 0 & 63 & 135 & 0 & 8505 & 1890 & 0 & 72576 & 126 \\
3063744 & 9 & 486 & 60507 & 60507 & 6561 & 1104750 & 523908 & 19656 & 120960 & 544320 & 0 & 0 & 0 & 153 & 1944 & 756 & 38637 & 46872 & 0 \\
2910720 & 0 & 2287 & 85504 & 85504 & 8448 & 1182720 & 720060 & 34304 & 102400 & 373800 & 0 & 2 & 60 & 0 & 2580 & 900 & 49152 & 35328 & 1 \\
1653372 & 324 & 19683 & 236196 & 236196 & 43740 & 1705860 & 1318761 & 196713 & 97200 & 0 & 0 & 0 & 0 & 810 & 0 & 6075 & 39366 & 39366 & 108 \\
3464208 & 0 & 0 & 0 & 0 & 0 & 419904 & 157464 & 3888 & 337437 & 1121931 & 0 & 0 & 0 & 0 & 0 & 1458 & 52488 & 34992 & 0 \\
3456000 & 0 & 18 & 4608 & 4608 & 216 & 442368 & 134568 & 0 & 262656 & 1199763 & 0 & 0 & 9 & 0 & 1296 & 108 & 41472 & 46080 & 0 \\
0 & 0 & 354294 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 27126 & 449064 & 2165130 & 0 & 2598156 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 183708 & 0 & 826686 & 0 & 0 & 0 & 567 & 59931 & 153090 & 725760 & 1377810 & 306180 & 0 & 1959552 & 486 \\
0 & 0 & 21870 & 0 & 0 & 19440 & 0 & 1224720 & 0 & 0 & 510300 & 135 & 7560 & 81045 & 207360 & 831060 & 340200 & 1866240 & 483840 & 0 \\
1417176 & 486 & 0 & 118098 & 118098 & 0 & 669222 & 0 & 40824 & 0 & 0 & 0 & 3402 & 19683 & 123111 & 551124 & 183708 & 1062882 & 1285956 & 0 \\
1714176 & 0 & 1638 & 46080 & 46080 & 7560 & 552960 & 325080 & 0 & 0 & 453600 & 1 & 420 & 5130 & 35840 & 275229 & 77112 & 1036800 & 1016064 & 0 \\
1679616 & 0 & 0 & 0 & 0 & 4860 & 622080 & 328050 & 57600 & 345600 & 109350 & 0 & 270 & 6075 & 34560 & 223074 & 129753 & 1119744 & 933120 & 18 \\
2577960 & 6 & 396 & 21186 & 21186 & 0 & 472230 & 266112 & 5544 & 184800 & 623700 & 0 & 0 & 495 & 2970 & 44550 & 16632 & 721215 & 634788 & 0 \\
2466936 & 0 & 0 & 17496 & 17496 & 2916 & 602640 & 201204 & 5832 & 129600 & 729000 & 0 & 27 & 135 & 3780 & 45927 & 14580 & 667764 & 688437 & 0 \\
0 & 0 & 964467 & 0 & 0 & 857304 & 0 & 964467 & 2709504 & 0 & 0 & 0 & 1134 & 0 & 0 & 0 & 47628 & 0 & 0 & 49266
\end{pmatrix} \tag{30}
$$

$$
T_{\sqrt{-3}}: \begin{pmatrix}
163898 & 0 & 0 & 693 & 693 & 14 & 31010 & 12348 & 98 & 9800 & 44100 & 0 & 0 & 0 & 0 & 70 & 0 & 1960 & 1764 & 0 \\
0 & 1322 & 59049 & 59049 & 59049 & 0 & 0 & 0 & 85932 & 0 & 0 & 1 & 0 & 0 & 660 & 0 & 0 & 0 & 0 & 1386 \\
0 & 1024 & 3808 & 57344 & 57344 & 29568 & 56320 & 59136 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 880 & 0 & 1024 & 0 & 0 \\
84942 & 13 & 728 & 6292 & 19604 & 1716 & 77935 & 66066 & 6006 & 0 & 0 & 0 & 0 & 0 & 0 & 286 & 0 & 2860 & 0 & 0 \\
84942 & 13 & 728 & 19604 & 6292 & 1716 & 77935 & 66066 & 6006 & 0 & 0 & 0 & 0 & 0 & 0 & 286 & 0 & 2860 & 0 & 0 \\
20736 & 0 & 4536 & 20736 & 20736 & 5288 & 53760 & 111132 & 26880 & 0 & 0 & 0 & 0 & 40 & 0 & 0 & 336 & 0 & 2240 & 28 \\
143532 & 0 & 27 & 2943 & 2943 & 168 & 57422 & 30618 & 1456 & 6720 & 17010 & 0 & 0 & 1 & 12 & 0 & 84 & 1944 & 1568 & 0 \\
129024 & 0 & 64 & 5632 & 5632 & 784 & 69120 & 37280 & 1024 & 0 & 14400 & 0 & 0 & 0 & 0 & 160 & 0 & 1024 & 2304 & 0 \\
39366 & 62 & 0 & 19683 & 19683 & 7290 & 126360 & 39366 & 9798 & 1800 & 0 & 0 & 2 & 0 & 120 & 0 & 0 & 0 & 2916 & 2 \\
157464 & 0 & 0 & 0 & 0 & 0 & 23328 & 0 & 72 & 11696 & 69984 & 0 & 0 & 0 & 16 & 0 & 0 & 0 & 3888 & 0 \\
165888 & 0 & 0 & 0 & 0 & 0 & 13824 & 5184 & 0 & 16384 & 61280 & 0 & 0 & 0 & 0 & 0 & 48 & 2304 & 1536 & 0 \\
0 & 6144 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2640 & 88704 & 0 & 168960 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 21504 & 0 & 0 & 112 & 5000 & 22680 & 53760 & 0 & 90720 & 0 & 72576 & 96 \\
0 & 0 & 0 & 0 & 0 & 5760 & 46080 & 0 & 0 & 0 & 0 & 0 & 1120 & 5168 & 23040 & 77760 & 0 & 0 & 107520 & 0 \\
0 & 24 & 0 & 0 & 0 & 0 & 52488 & 0 & 6048 & 20160 & 0 & 1 & 252 & 2187 & 9356 & 39366 & 17010 & 78732 & 40824 & 0 \\
92160 & 0 & 120 & 3072 & 3072 & 0 & 0 & 20160 & 0 & 0 & 0 & 0 & 0 & 480 & 2560 & 20600 & 6720 & 69120 & 48384 & 0 \\
0 & 0 & 0 & 0 & 0 & 864 & 69120 & 0 & 0 & 0 & 48600 & 0 & 80 & 0 & 3200 & 19440 & 7640 & 31104 & 86400 & 0 \\
110880 & 0 & 6 & 1320 & 1320 & 0 & 23760 & 5544 & 0 & 0 & 34650 & 0 & 0 & 0 & 220 & 2970 & 462 & 42812 & 42504 & 0 \\
104976 & 0 & 0 & 0 & 0 & 90 & 20160 & 13122 & 432 & 14400 & 24300 & 0 & 1 & 30 & 120 & 2187 & 1350 & 44712 & 40568 & 0 \\
0 & 25088 & 0 & 0 & 0 & 190512 & 0 & 0 & 50176 & 0 & 0 & 0 & 224 & 0 & 0 & 0 & 0 & 0 & 0 & 448
\end{pmatrix} \tag{31}
$$

4 Eigenvectors and automorphic representations

4.1 The eigenvectors

We consider still the genus of 20 classes of unimodular Hermitian lattices of rank 12. Bearing in mind the remark following Proposition 2.6, we seek a basis $\{v_1, v_2 \ldots, v_{20}\}$ of $M(\mathrm{triv}, K_L)$, simultaneous eigenvectors for all the $T(\gamma)$. We scale the v_i to have integral values with no common factor, and write $\lambda_i(T(\gamma))$ for the eigenvalue of $T(\gamma)$ acting on v_i. We order them as in the following table, which presents the eigenvalues for $T_{(2)}$ and $T_{(\sqrt{-3})}$. There is only one eigenvalue for $T_{(2)}$ whose eigenspace is not 1-dimensional. In fact $T_{(2)}$ and $T_{(\sqrt{-3})}$ have a common 2-dimensional eigenspace, though looking at the last column of the table (to be explained later) we would expect it to be broken up by $T_{(7)}$. The eigenvalues and eigenvectors were computed using the above 20-by-20 matrices, and the computer package Maple.

i	$\lambda_i\left(T_{(2)}\right)$	$\lambda_i\left(T_{(\sqrt{-3})}\right)$	Global Arthur parameters (conjectural)
1	5593770	266448	$[12]$
2	1395945	89552	$\Delta_{11} \oplus [10]$
3	1401453	88328	$\Delta_{11}(3) \oplus [10]$
4	357525	30032	$\Delta_{10}[2] \oplus [8]$
5	348453	29528	$\Delta_{11} \oplus \Delta_9(3) \oplus [8]$
6	91845	9368	$\Delta_{10}[2] \oplus \Delta_7(3) \oplus [6]$
7	90873	10664	$\Delta_{11}(3) \oplus \Delta_8[2] \oplus [6]$
8	85365	11888	$\Delta_{11} \oplus \Delta_8[2] \oplus [6]$
9	23805	7568	$\Delta_8[4] \oplus [4]$
10	40005	1808	$\Delta_{10}[2] \oplus \Delta_6[2] \oplus [4]$
11	30933	1304	$\Delta_{11} \oplus \Delta_9(3) \oplus \Delta_6[2] \oplus [4]$
12	$23319 + 162\sqrt{193}$	$4148 + 36\sqrt{193}$	$\Delta_{11,5}^{(2)} \oplus \Delta_8[2] \oplus [4]$
13	$23319 - 162\sqrt{193}$	$4148 - 36\sqrt{193}$	"
14	46485	−4528	$\Delta_{11} \oplus \Delta_6[4] \oplus [2]$
15	51993	−5752	$\Delta_{11}(3) \oplus \Delta_6[4] \oplus [2]$
16	11925	−1072	$\Delta_{11} \oplus \Delta_{9,3} \oplus \Delta_6[2] \oplus [2]$
17	176085	−18928	$\Delta_6[6]$
18	−5355	728	$\Delta_{11} \oplus \Delta_{9,1} \oplus \Delta_5(3)[3]$
19	108693	−13312	$(\Delta_5(3) \otimes \psi_6) \oplus \psi_6[4] \oplus \overline{\psi}_6[6]$
20	108693	−13312	$(\Delta_5(3) \otimes \overline{\psi}_6) \oplus \overline{\psi}_6[4] \oplus \psi_6[6]$

Each v_i may be thought of as a complex-valued function on $\mathbb{U}_{12}(\mathbb{Q})\backslash\mathbb{U}_{12}(\mathbb{A}_\mathbb{Q})$, right-invariant under $K = K_\infty K_L$, where $K_\infty := \mathbb{U}_{12}(\mathbb{R})$. Under the right-translation action of $\mathbb{U}_{12}(\mathbb{A}_\mathbb{Q})$, it generates an infinite-dimensional automorphic representation π_i of $\mathbb{U}_{12}(\mathbb{A}_\mathbb{Q})$.

For each local Weil group $W_\mathbb{R}$ and $W_{\mathbb{Q}_p}$ of $\mathbb{Q}$ there is associated to π_i a Langlands parameter, a homomorphism from that group to the local L-group $\mathrm{GL}_{12}(\mathbb{C}) \rtimes W_\mathbb{R}$ or $\mathrm{GL}_{12}(\mathbb{C}) \rtimes W_{\mathbb{Q}_p}$ of $\mathbb{U}_{12}$. Restricting to the local Weil group $W_\mathbb{C}$ or $W_{E_\mathfrak{p}}$ of E, and projecting to $\mathrm{GL}_{12}(\mathbb{C})$, we obtain Langlands parameters

$$c_\infty(\widetilde{\pi}_i) : W_{\mathbb{C}} \to \mathrm{GL}_{12}(\mathbb{C}),\;\; c_{E_{\mathfrak{P}}}(\widetilde{\pi}_i) : W_{E_{\mathfrak{P}}} \to \mathrm{GL}_{12}(\mathbb{C}),$$

defined up to conjugation in $\mathrm{GL}_{12}(\mathbb{C})$, which is here playing the role of the Langlands dual of $\mathrm{GL}_{12,E}$. See [31] (following (2.2.3)) for this "standard base-change of L-parameters". Now $W_{\mathbb{C}} = \mathbb{C}^\times$, and it is a consequence of the fact that v_i is scalar-valued that (up to conjugation)

$$c_\infty(\widetilde{\pi}_i) : z \mapsto \mathrm{diag}\left((z/\overline{z})^{11/2}, (z/\overline{z})^{9/2}, \ldots, (z/\overline{z})^{-11/2}\right).$$

At a prime $\mathfrak{P}$ dividing p for which both $\mathbb{U}_{12}$ and π_i are unramified (i.e. $p \neq 3$, given our choice of E and K_L), $c_{E_{\mathfrak{P}}}(\widetilde{\pi}_i)$ is determined by $\mathrm{Frob}_{\mathfrak{P}} \mapsto t_{\mathfrak{P}}(\widetilde{\pi}_i)$ (the Satake parameter at $\mathfrak{P}$). This determines $\lambda_i(T_{\mathfrak{P}})$, by the formulas

$$\lambda_i(T_{\mathfrak{P}}) = \begin{cases} (N\mathfrak{P})^{11/2}\mathrm{Tr}(t_{\mathfrak{P}}(\widetilde{\pi}_i)) + \frac{p^{12}-1}{p+1} & \text{for } (p) = \mathfrak{P} \text{ inert;} \\ (N\mathfrak{P})^{11/2}\mathrm{Tr}(t_{\mathfrak{P}}(\widetilde{\pi}_i)) & \text{for } (p) = \mathfrak{P}\overline{\mathfrak{P}} \text{ split.} \end{cases}$$

In the split case, where $\mathbb{U}_{12}(\mathbb{Q}_p) \simeq \mathrm{GL}_{12}(\mathbb{Q}_p)$, this is a direct consequence of a formula of Tamagawa [38],[18, $i = 1$ in (3.14)]. In the inert case, where $\mathbb{U}_{12}(\mathbb{Q}_p) \simeq \mathbb{U}(6,6)(\mathbb{Q}_p)$, it may be justified assuming a coset decomposition like that for $\mathbb{U}(2,2)$ in [29, (5.7)], combined with [6, IV. (33),(35),(39)].

4.2 Global Arthur parameters

A complete description of those automorphic representations, of a quasi-split unitary group $\mathbb{G}^*$, occurring discretely in $L^2(\mathbb{G}^*(\mathbb{Q})\backslash\mathbb{G}^*(\mathbb{A}_{\mathbb{Q}}))$, was given by Mok [31, Theorem 2.5.2]. This was extended to general unitary groups (including $\mathbb{U}_{12}$) by Kaletha, Minguez, Shin and White [27, Theorem* 1.7.1], conditional on what will be written up in later papers of Kaletha, Minguez and Shin, and of Chaudouard and Laumon. (See the discussion on [27, p.6].) Part of this description is that to such an automorphic representation is attached a "global Arthur parameter", a formal unordered sum of the form $\oplus_{k=1}^m \Pi_k[d_k]$, where Π_k is a cuspidal automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A}_E)$, $d_k \geq 1$ and $\sum_{k=1}^m n_k d_k = N = 12$. Before explaining the guesses in the final column of the table, we fix some notation.

Let f be a cuspidal Hecke eigenform of weight k for $\mathrm{SL}_2(\mathbb{Z})$. There is an associated cuspidal automorphic representation Π_f of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$, with base-change $\widetilde{\Pi_f}$ of $\mathrm{GL}_2(\mathbb{A}_E)$. We have

$$c_\infty(\widetilde{\Pi_f}) : z \mapsto \mathrm{diag}\left((z/\overline{z})^{(k-1)/2}, (z/\overline{z})^{(1-k)/2}\right),$$

and

$$t_{\mathfrak{P}}(\widetilde{\Pi_f}) = \begin{cases} \mathrm{diag}(\alpha, \alpha^{-1}) & p \text{ split;} \\ \mathrm{diag}(\alpha^2, \alpha^{-2}) & p \text{ inert,} \end{cases}$$

where $a_p(f) = p^{(k-1)/2}(\alpha + \alpha^{-1})$ and $|\alpha| = 1$. In the table, $\widetilde{\Pi_f}$ is denoted Δ_{k-1}, the subscript coming from the exponents in $c_\infty(\widetilde{\Pi_f})$. For example when $k = 12$ and

$f = \Delta = \sum_{n=1}^{\infty} \tau(n)q^n$, we have Δ_{11}. Similarly for a newform $f \in S_k(\Gamma_0(3))$ we denote $\widetilde{\Pi_f}$ by $\Delta_{k-1}(3)$, e.g. $\Delta_{11}(3), \Delta_9(3), \Delta_7(3)$ and $\Delta_5(3)$. Although $S_{10}(\Gamma_0(3))$ is 2-dimensional, we reserve $\Delta_9(3)$ for the base change associated to just one of the normalised eigenforms, it being the only one that appears to actually occur in the global Arthur parameters of any of our π_i.

For a Hecke eigenform $f \in S_k(\Gamma_0(3), \chi_{-3})$ (where χ_{-3} is the quadratic character attached to E), with k odd, we have Δ_{k-1} , e.g Δ_{10} and Δ_8. Note that each of $S_{11}(\Gamma_0(3), \chi_{-3})$ and $S_9(\Gamma_0(3), \chi_{-3})$ is spanned by a conjugate pair of Hecke eigenforms, sharing the same base-change. Note also that $S_7(\Gamma_0(3), \chi_{-3})$ is spanned by a Hecke eigenform f of CM type. The base change $\widetilde{\Pi_f}$ is, in this case, not cuspidal, but we still use Δ_6 as a shorthand for $\psi_6 \oplus \overline{\psi}_6$, where ψ_6 is an everywhere-unramified, cuspidal, automorphic representation of $\mathrm{GL}_1(\mathbb{A}_E)$, given by $\psi_6(z) = z^{-6}$ for $z \in \mathbb{C}^\times$ (embedded in $\mathbb{A}_E^\times$ by putting 1 in all the other components) and $\psi_6(\pi_{\mathfrak{P}}) = \alpha_{\mathfrak{P}}^6$, where $\pi_{\mathfrak{P}} \in E_{\mathfrak{P}}^\times$ is a uniformiser at $\mathfrak{P}$ and $(\alpha_{\mathfrak{P}}) = \mathfrak{P}$. Since the group of units in $\mathcal{O}_E$ has order 6, this is well-defined, independent of the choice of $\alpha_{\mathfrak{P}}$.

For $d \geq 1$, $[d]$ is an automorphic representation of $\mathrm{GL}_d(\mathbb{A}_E)$, occurring discretely in $L^2(\mathrm{GL}_d(E)\backslash\mathrm{GL}_d(\mathbb{A}_E))$, with

$$c_\infty([d]) : z \mapsto \mathrm{Sym}^{d-1}\left(\mathrm{diag}\left((z/\overline{z})^{1/2}, (z/\overline{z})^{-1/2}\right)\right)$$

and

$$c_{\mathfrak{P}}([d]) : \mathrm{Frob}_{\mathfrak{P}} \mapsto \mathrm{Sym}^{d-1}\left(\mathrm{diag}\left((N\mathfrak{P})^{1/2}, (N\mathfrak{P})^{-1/2}\right)\right).$$

Given a cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_E)$, there is a discrete automorphic representation $\Pi[d]$ of $\mathrm{GL}_{nd}(\mathbb{A}_E)$, whose Langlands parameters are tensor products of those of Π and of $[d]$.

4.3 Recovering the Hecke eigenvalues

All the entries in the final column of the table must agree with the requirement

$$c_\infty(\widetilde{\pi_i}) : z \mapsto \mathrm{diag}\left((z/\overline{z})^{11/2}, (z/\overline{z})^{9/2}, \ldots, (z/\overline{z})^{-11/2}\right),$$

and indeed they do, as illustrated by the following examples.

For **1**, $c_\infty([12]) = \mathrm{Sym}^{11}\left(\mathrm{diag}\left((z/\overline{z})^{1/2}, (z/\overline{z})^{-1/2}\right)\right)$, which is precisely

$$\mathrm{diag}\left((z/\overline{z})^{11/2}, (z/\overline{z})^{9/2}, \ldots, (z/\overline{z})^{-11/2}\right).$$

For **2**, Δ_{11} contributes the exponents $11/2, -11/2$, while $[10]$ contributes the remaining $9/2, 7/2, \ldots, -7/2, -9/2$.

For **7**, $\Delta_{11}(3)$ gives $11/2, -11/2$, $\Delta_8[2]$ gives $9/2, 7/2, -7/2, -9/2$, and $[6]$ the remaining $5/2, 3/2, 1/2, -1/2, -3/2, -5/2$. Note that Δ_8 would have contributed $8/2, -8/2$, but with the $[2]$ these got "smeared" to either side.

From the conjectural global Arthur parameter of a π_i we may compute a putative $\lambda_i(T_{(2)})$, using the formula

$$\lambda_i(T_{\mathfrak{P}}) = (N\mathfrak{P})^{11/2}\mathrm{Tr}(t_{\mathfrak{P}}(\widetilde{\pi}_i)) + \frac{p^{12}-1}{p+1}$$

with $\mathfrak{P} = (2)$, $N\mathfrak{P} = 4$, and check that it agrees with the real one. Here are some examples.

1. We have $t_{(2)}([12]) = \mathrm{diag}(4^{11/2}, \ldots, 4^{-11/2})$, so would get

$$\lambda_1(T_{(2)}) = 1 + 4 + 4^2 + \ldots + 4^{11} + \frac{2^{12}-1}{3} = \frac{4^{12}-1}{3} + \frac{2^{12}-1}{3} = 5593770,$$

which is correct.

2. We have $t_{(2)}(\Delta_{11} \oplus [10]) = \mathrm{diag}(\alpha^2, 4^{9/2}, 4^{7/2}, \ldots, 4^{-7/2}, 4^{-9/2}, \alpha^{-2})$, so would get

$$\begin{aligned}\lambda_2(T_{(2)}) &= 4^{11/2}(\alpha^2 + \alpha^{-2}) + (4 + 4^2 + \ldots 4^{10}) + \frac{2^{12}-1}{3} \\ &= ((2^{11/2}(\alpha + \alpha^{-1}))^2 - 2.2^{11}) + 4\left(\frac{4^{10}-1}{3}\right) + \frac{2^{12}-1}{3} \\ &= (a_2(\Delta)^2 - 2.2^{11}) + 4\left(\frac{4^{10}-1}{3}\right) + \frac{2^{12}-1}{3} \\ &= ((-24)^2 - 2.2^{11}) + 4\left(\frac{4^{10}-1}{3}\right) + \frac{2^{12}-1}{3} = 1395495,\end{aligned}$$

also correct, using $\Delta = q - 24q^2 + 252q^3 \ldots$.

7. Similarly for $\Delta_{11}(3) \oplus \Delta_8[2] \oplus [6]$,

$$(78^2 - 2.2^{11}) + 4((6\sqrt{-14})^2 + 2.2^8)(1+4) + 4^3\left(\frac{4^6-1}{3}\right) + \frac{2^{12}-1}{3} = 90873,$$

using eigenforms $f = q + 78q^2 - 243q^3 \ldots \in S_{12}(\Gamma_0(3))$ and $g = q + 6\sqrt{-14}q^2 + (45 - 18\sqrt{-14})q^3 \ldots \in S_9(\Gamma_0(3), \chi_{-3})$.

17. Since $\psi_6((2)) = \overline{\psi}_6((2)) = 2^6$, for $\Delta_6[6]$ we get

$$(2.2^6)\left(\frac{4^6-1}{3}\right) + \frac{2^{12}-1}{3} = 176085,$$

as required.

19. The Hecke eigenform $q - 6q^2 + 9q^3 + \ldots$ spans $S_6(\Gamma_0(3))$, so for $(\Delta_5(3) \otimes \psi_6) \oplus \psi_6[4] \oplus \overline{\psi}_6[6]$ we would get

$$\lambda_{19}(T_{(2)}) = ((-6)^2 - 2.2^5)(2^6) + 4\left(\frac{4^4-1}{3}\right)(2^6) + \left(\frac{4^6-1}{3}\right)(2^6) + \frac{2^{12}-1}{3} = 108693,$$

which is correct.

Similar calculations of $T_{(2)}$ eigenvalues corroborate all the other guesses for global Arthur parameters in the table, except for **12**, **13**, **16**, **18**, which we deal with next.

4.3.1 Algebraic modular forms for $\mathbb{U}_4$

With $\mathbb{G} = \mathbb{U}_4$ and $V = V_{a,b}$ the representation of highest weight $\lambda = a(e_1 - e_4) + b(e_2 - e_3)$, where $e_i(\mathrm{diag}(t_1, t_2, t_3, t_4)) = t_i$ and $a \geq b \geq 0$, consider the space $M(V, K_L)$, where L is the standard Hermitian lattice $\mathcal{O}_E^4$, and still $E = \mathbb{Q}(\sqrt{-3})$. If v is an eigenvector for the Hecke algebra, generating an automorphic representation π of $\mathbb{U}_4(\mathbb{A}_\mathbb{Q})$, then

$$c_\infty(\widetilde{\pi}) : z \mapsto \left((z/\overline{z})^{a+3/2}, (z/\overline{z})^{b+1/2}, (z/\overline{z})^{-b-1/2}, (z/\overline{z})^{-a-3/2}\right).$$

By [36, Table 1], the class number $|\Sigma_{K_L}| = 1$, so $M(V, K_L) = V^\Gamma$, where $\Gamma := \mathbb{U}_4(\mathbb{Q}) \cap K_L$. The dimension of $M(V, K_L)$ may be computed using Weyl's character formula.

Example 1. $\mathbf{a = 3, b = 0}$ One finds that $\dim(M(V, K_L)) = 1$. Calculating the trace of $T_{(2)}$ on $M(V, K_L)$ by the method of [12], the eigenvalue of $T_{(2)}$ is 1872. The formula for this is now $(N\mathfrak{P})^{a+3/2}\mathrm{Tr}(t_\mathfrak{P}(\widetilde{\pi})) + (N\mathfrak{P})^a \left(\frac{p^4-1}{p+1}\right)$ (previously $a = 0$ and $11/2$ was in place of $3/2$), from which we deduce $4^{9/2}\mathrm{Tr}(t_\mathfrak{P}(\widetilde{\pi})) = 1872 - 2^6(2^3 - 2^2 + 2 - 1) = 1552$, and then

$$4^{11/2}\mathrm{Tr}(t_\mathfrak{P}(\widetilde{\pi})) = 4(1872 - 2^6(2^3 - 2^2 + 2 - 1)) = 6208.$$

For the cuspidal automorphic representation $\widetilde{\pi}$ of $\mathrm{GL}_4(\mathbb{A}_E)$ we write $\Delta_{9,1}$, since

$$\left(\left(\frac{z}{\overline{z}}\right)^{a+3/2}, \left(\frac{z}{\overline{z}}\right)^{b+1/2}, \left(\frac{z}{\overline{z}}\right)^{-b-1/2}, \left(\frac{z}{\overline{z}}\right)^{-a-3/2}\right) = \left(\left(\frac{z}{\overline{z}}\right)^{9/2}, \left(\frac{z}{\overline{z}}\right)^{1/2}, \left(\frac{z}{\overline{z}}\right)^{-1/2}, \left(\frac{z}{\overline{z}}\right)^{-9/2}\right).$$

Now looking back at **18**, $\Delta_{11} \oplus \Delta_{9,1} \oplus \Delta_5(3)[3]$ gives us the correct

$$\lambda_{18}(T_{(2)}) = ((-24)^2 - 2.2^{11}) + 6208 + 16((-6)^2 - 2.2^5)(1 + 4 + 4^2) + \frac{2^{12} - 1}{3} = -5355.$$

Example 2. $\mathbf{a = 3, b = 1}$. One finds that $\dim(M(V, K_L)) = 1$. The eigenvalue of $T_{(2)}$ is 0. This leads to

$$4^{11/2}\mathrm{Tr}(t_\mathfrak{P}(\widetilde{\pi})) = 4(0 - 2^6(2^3 - 2^2 + 2 - 1)) = -1280.$$

For the cuspidal automorphic representation $\widetilde{\pi}$ of $\mathrm{GL}_4(\mathbb{A}_E)$ we write $\Delta_{9,3}$. Looking back at **16**, $\Delta_{11} \oplus \Delta_{9,3} \oplus \Delta_6[2] \oplus [2]$ gives us the correct

$$\lambda_{16}(T_{(2)}) = ((-24)^2 - 2.2^{11}) - 1280 + 16(2.2^6)(1 + 4) + 2^{10}\left(\frac{4^2 - 1}{3}\right) + \frac{2^{12} - 1}{3} = 11925.$$

Remarkably, we find that, though the table poses $\Delta_{11} \oplus \Delta_{9,1} \oplus \Delta_5(3)[3]$ for **18**, $\Delta_{9,3}[3]$ would give the same $T_{(2)}$-eigenvalue, since $(-1280)(4^{-1} + 1 + 4) + \frac{2^{12}-1}{3} = -5355$. However, if $\Delta_{9,3}[3]$ were the correct global Arthur parameter, one could deduce a $T_{(\sqrt{-3})}$-contribution from $\Delta_{9,3}$ that would be incompatible with the $T_{(\sqrt{-3})}$ eigenvalue for **16**, assuming that $\Delta_{11} \oplus \Delta_{9,3} \oplus \Delta_6[2] \oplus [2]$ for **16** (which is linked to a congruence in Example 5 of §6.1) is correct.

Example 3. $\mathbf{a = 4, b = 2}$. This time $\dim(M(V, K_L)) = 2$, and $\mathrm{Tr}(T_{(2)})$ is 2628. For either of the two cuspidal automorphic representations $\widetilde{\pi}$ of $\mathrm{GL}_4(\mathbb{A}_E)$ (coming from two Hecke eigenvectors) we write $\Delta_{11,5}^{(2)}$. If $\Delta_{11,5}^{(2)} \oplus \Delta_8[2] \oplus [4]$ is correct for **12** and **13** then

$$4^{11/2}\mathrm{Tr}(t_{\mathfrak{P}}(\widetilde{\pi})) + 4((6\sqrt{-14})^2 + 2.2^8)(1+4) + 2^8\left(\frac{4^4-1}{3}\right) + \frac{2^{12}-1}{3} = 23319 \pm 162\sqrt{193},$$

which would imply that $4^{11/2}\mathrm{Tr}(t_{\mathfrak{P}}(\widetilde{\pi})) = 34 \pm 162\sqrt{193}$, then

$$\lambda_{12,13}(T_{(2)}) = 4^{11/2}\mathrm{Tr}(t_{\mathfrak{P}}(\widetilde{\pi})) + 2^8(2^3 - 2^2 + 2 - 1) = 1314 \pm 162\sqrt{193}.$$

This is consistent with $\mathrm{Tr}(T_{(2)}) = 2628$ on $M(V, K_L)$, which was confirmed independently by the method of [12].

4.3.2 $\mathfrak{P} = (\sqrt{-3})$

The formulas for $\lambda_i(T_{\mathfrak{P}})$ given at the end of §4.1 do not apply to the ramified prime $\mathfrak{P} = (\sqrt{-3})$. The following seems to work, though we have not justified it.

$$\lambda_i(T_{(\sqrt{-3})}) = 3^{11/2}\mathrm{Tr}(t_{\mathfrak{P}}(\widetilde{\pi}_i)) + 3^6 - 1.$$

We must explain what we mean by $t_{\mathfrak{P}}(\widetilde{\pi}_i)$ (with $\mathfrak{P} = (\sqrt{-3})$). We shall not attempt to apply the formula to cases involving any $\Delta_{2a+3,2b+1}$. The automorphic representation $[d]$ of $\mathrm{GL}_d(\mathbb{A}_E)$ is unramified at $\mathfrak{P}$, and we calculate $t_{\mathfrak{P}}([d])$, an actual Satake parameter, just as before. The automorphic representations ψ_6 and $\overline{\psi_6}$ of $\mathrm{GL}_1(\mathbb{A}_E)$ are unramified at $\mathfrak{P}$, with Satake parameter $(\sqrt{-3})^6 = -27$ in both cases.

For $\Delta_{11} = \widetilde{\Pi_\Delta}$, where Δ is the normalised cusp form of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$, since $\widetilde{\Pi_\Delta}$ is unramified at 3, the local representation of $W_{\mathbb{Q}_3}$ (and therefore its restriction to $W_{E_{\mathfrak{P}}}$ is unramified, and we just take $t_{\mathfrak{P}}(\Delta_{11}) = t_3(\Pi_\Delta)$, an actual Satake parameter.

For $\Delta_{k-1} = \widetilde{\Pi_f}$, where $f \in S_k(\Gamma_0(3), \chi_{-3})$ with k odd, while the local representation of $W_{\mathbb{Q}_3}$ is ramified, its restriction to $W_{E_{\mathfrak{P}}}$ is unramified, and using a theorem of Langlands and Carayol [24, Theorem 4.2.7 (3)(a)],

$$3^{(k-1)/2}\, t_{\mathfrak{P}}(\Delta_{k-1}) = \mathrm{diag}(a_3(f), 3^{k-1}/a_3(f)).$$

For $k = 9$ this is $\mathrm{diag}(45 - 18\sqrt{-14}, 45 + 18\sqrt{-14})$, and for $k = 10$ it is $\mathrm{diag}(-27 + 108\sqrt{-5}, -27 - 108\sqrt{-5})$.

For $\Delta_{k-1}(3)$, with k even and f a newform in $S_k(\Gamma_0(3))$, $\Delta_{k-1}(3)$ is ramified at $\mathfrak{P}$, but we just try using the same formula, $3^{(k-1)/2}\, t_{\mathfrak{P}}(\Delta_{k-1}) = \mathrm{diag}(a_3(f), 3^{k-1}/a_3(f))$. For $k = 6, 8, 10, 12$ this is $\mathrm{diag}(3^2, 3^3)$, $\mathrm{diag}(-3^3, -3^4)$, $\mathrm{diag}(-3^4, -3^5)$ and $\mathrm{diag}(-3^5, -3^6)$ respectively.

We now recover some of the $T_{\mathfrak{P}}$-eigenvalues in the table.

1 : [12].

$$266448 = \frac{3^{12}-1}{2} + 3^6 - 1.$$

2 : $\Delta_{11} \oplus [10]$.

$$89552 = 252 + 3\left(\frac{3^{10}-1}{2}\right) + 3^6 - 1.$$

3 : $\Delta_{11}(3) \oplus [10]$.

$$88328 = (-3^5 - 3^6) + 3\left(\frac{3^{10}-1}{2}\right) + 3^6 - 1.$$

4 : $\Delta_{10}[2] \oplus [8]$.

$$30032 = (-27-27)(1+3) + 3^2\left(\frac{3^8-1}{2}\right) + 3^6 - 1.$$

5 : $\Delta_{11} \oplus \Delta_9(3) \oplus [8]$.

$$29528 = 252 + 3(-3^4 - 3^5) + 3^2\left(\frac{3^8-1}{2}\right) + 3^6 - 1.$$

6 : $\Delta_{10}[2] \oplus \Delta_7(3) \oplus [6]$.

$$9368 = (-27-27)(1+3) + 3^2(-3^3 - 3^4) + 3^3\left(\frac{3^6-1}{2}\right) + 3^6 - 1.$$

7 : $\Delta_{11}(3) \oplus \Delta_8[2] \oplus [6]$.

$$10664 = (-3^5 - 3^6) + 3(45+45)(1+3) + 3^3\left(\frac{3^6-1}{2}\right) + 3^6 - 1.$$

8 : $\Delta_{11} \oplus \Delta_8[2] \oplus [6]$.

$$11888 = 252 + 3(45+45)(1+3) + 3^3\left(\frac{3^6-1}{2}\right) + 3^6 - 1.$$

9 : $\Delta_8[4] \oplus [4]$.

$$7568 = (45+45)(\left(\frac{3^4-1}{2}\right) + 3^4\left(\frac{3^4-1}{2}\right) + 3^6 - 1.$$

10 : $\Delta_{10}[2] \oplus \Delta_6[2] \oplus [4]$.

$$1808 = (-27-27)(1+3) + 3^2(-27-27)(1+3) + 3^4\left(\frac{3^4-1}{2}\right) + 3^6 - 1.$$

11 : $\Delta_{11} \oplus \Delta_9(3) \oplus \Delta_6[2] \oplus [4]$.

$$1304 = 252 + 3(-3^4 - 3^5) + 3^2(-27-27)(1+3) + 3^4\left(\frac{3^4-1}{2}\right) + 3^6 - 1.$$

14 : $\Delta_{11} \oplus \Delta_6[4] \oplus [2]$.

$$-4528 = 252 + 3(-27-27)(\left(\frac{3^4-1}{2}\right) + 3^5(1+3) + 3^6 - 1.$$

15 : $\Delta_{11}(3) \oplus \Delta_6[4] \oplus [2]$.

$$-5752 = (-3^5 - 3^6) + 3(-27-27)(\left(\frac{3^4-1}{2}\right) + 3^5(1+3) + 3^6 - 1.$$

17 : $\Delta_6[6]$.

$$-18928 = (-27-27)\left(\frac{3^6-1}{2}\right) + 3^6 - 1.$$

19 : $(\Delta_5(3) \otimes \psi_6) \oplus \psi_6[4] \oplus \overline{\psi}_6[6]$, **20** : $(\Delta_5(3) \otimes \overline{\psi}_6) \oplus \overline{\psi}_6[4] \oplus \psi_6[6]$.

$$-13312 = (3^2+3^3)(-27) + 3(-27)(\left(\frac{3^4-1}{2}\right) + (-27)\left(\frac{3^6-1}{2}\right) + 3^6 - 1.$$

4.4 Eisenstein lattices of rank 12

Recall from Section 3.3 the genus of 5 classes of rank-12, $\sqrt{-3}$-modular lattices, for which we obtained the matrix for the Hecke operator $T_{(2)}$. One finds that the eigenvalues match those of $\mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{8}, \mathbf{9}$, so presumably the associated automorphic representations have the same global Arthur parameters. Among the conjectured global Arthur parameters on the list, these are precisely those that do not involve anything of level $\Gamma_0(3)$, ψ_6, $\overline{\psi_6}$ or some $\Delta_{2a+3,2b+1}$. The unitary group in question is isomorphic to the one we already considered (quasi-split at all finite primes), but the open compact subgroups K_L differ locally at 3.

5 Congruences of Hecke eigenvalues

Proposition 5.1. *Consider $v \in M(\mathrm{triv}, K_L)$, with values in $\mathbb{Z}$ (as a function on the 20-element set Σ_{K_L}). Suppose that $v = \sum_{i=1}^{20} c_i v_i$, with $c_i \in K = \mathbb{Q}(\sqrt{193})$ (in fact $c_i \in \mathbb{Q}$ unless $i = 12$ or 13). Suppose that $\mathfrak{q}$ is a prime of $\mathcal{O}_K$ with $\mathrm{ord}_{\mathfrak{q}}(c_i) < 0$. Then there exists some $j \neq i$ such that*

$$\lambda_i(T) \equiv \lambda_j(T) \pmod{\mathfrak{q}} \;\; \forall T \in \mathbb{T},$$

where $\mathbb{T}$ is the $\mathbb{Z}$-subalgebra of $\mathrm{End}(M(\mathrm{triv}, K_L))$ generated by all the $T(\gamma)$.

Proof. Suppose for a contradiction that this is not the case. Then for each $j \neq i$ there is some $T(j) \in \mathbb{T}$ such that $\lambda_i(T(j)) \not\equiv \lambda_j(T(j)) \pmod{\mathfrak{q}}$. Now apply $\prod_{j\neq i}(T(j) - \lambda_j(T(j)))$ to both sides of $v = \sum_{k=1}^{20} c_k v_k$. The left-hand-side remains integral. On the right-hand-side, all the terms for $k \neq i$ are killed, whereas $c_i v_i$ is multiplied by $\prod_{j\neq i}(\lambda_i(T(j)) - \lambda_j(T(j)))$, which fails to cancel the $\mathfrak{q}$ in the denominator of c_i (hence of at least one of the entries of $c_i v_i$), contradicting the integrality of the left-hand-side. □

It is clear that this proposition applies to a more general situation than that for which it is stated, but we applied it as stated, with simple v such as $(1, 0, \dots, 0)^t$, and used the computed values of $\lambda_k(T_{(2)})$ (and in one case also $\lambda_k(T_{(\sqrt{-3})})$) to find the j for a given i and $\mathfrak{q}$, thus establishing the following congruences of Hecke eigenvalues:

$$\mathbf{2} \equiv \mathbf{1} \pmod{691};$$

$$\mathbf{4} \equiv \mathbf{1} \pmod{1847};$$
$$\mathbf{9} \equiv \mathbf{1} \pmod{809};$$
$$\mathbf{8} \equiv \mathbf{2} \pmod{809};$$
$$\mathbf{7} \equiv \mathbf{3} \pmod{809};$$

$$\mathbf{3} \equiv \mathbf{1} \pmod{73};$$
$$\mathbf{5} \equiv \mathbf{2} \pmod{61};$$
$$\mathbf{6} \equiv \mathbf{4} \pmod{41};$$
$$\mathbf{17} \equiv \mathbf{19}, \mathbf{20} \pmod{13};$$

$$\mathbf{3} \equiv \mathbf{2} \pmod{17};$$
$$\mathbf{7} \equiv \mathbf{8} \pmod{17};$$
$$\mathbf{15} \equiv \mathbf{14} \pmod{17};$$

$$\mathbf{16} \equiv \mathbf{11} \pmod{11};$$
$$\mathbf{7} \equiv \mathbf{12} \pmod{\mathfrak{q}},\ \mathbf{7} \equiv \mathbf{13} \pmod{\overline{\mathfrak{q}}},\ \text{with } \mathfrak{q} \mid 59;$$
$$\mathbf{9} \equiv \mathbf{12} \pmod{\mathfrak{q}},\ \mathbf{9} \equiv \mathbf{13} \pmod{\overline{\mathfrak{q}}},\ \text{with } \mathfrak{q} \mid 23.$$

We have seen that all the conjectured global Arthur parameters are consistent with the computations of Hecke eigenvalues, and while some were found by the kind of guess-and-check process one might imagine, more often we were guided by congruences between Hecke eigenvalues.

5.1 Ramanujan-type congruences

The first batch of congruences is

$$\mathbf{2} \equiv \mathbf{1} \pmod{691};$$

$$\mathbf{4} \equiv \mathbf{1} \pmod{1847};$$

$$\mathbf{9} \equiv \mathbf{1} \pmod{809};$$

$$\mathbf{8} \equiv \mathbf{2} \pmod{809};$$

$$\mathbf{7} \equiv \mathbf{3} \pmod{809}.$$

Consider $\mathbf{2} \equiv \mathbf{1} \pmod{691}$. Assuming the global Arthur parameters really are $\mathbf{1} : [12]$ and $\mathbf{2} : \Delta_{11} \oplus [10]$, at any split prime $(p) = \mathfrak{P}\overline{\mathfrak{P}}$ we get

$$\tau(p) + (p + p^2 + \ldots + p^{10}) \equiv 1 + p + \ldots + p^{11} \pmod{691},$$

which boils down to Ramanujan's congruence $\tau(p) \equiv 1 + p^{11} \pmod{691}$. At an inert prime $(p) = \mathfrak{P}$ we get

$$(\tau(p))^2 - 2.p^{11} + (p^2 + p^4 + \ldots + p^{20}) \equiv 1 + p^2 + \ldots + p^{22} \pmod{691},$$

which becomes $(\tau(p))^2 \equiv 1 + 2.p^{11} + p^{22} = (1 + p^{11})^2 \pmod{691}$, the square of Ramanujan's congruence.

It was easy to guess that $\mathbf{1}$, which has the largest $T_{(2)}$ eigenvalue, should have global Arthur parameter [12]. The congruence mod 691, which we recognised as the modulus of Ramanujan's congruence, then suggested trying $\Delta_{11} \oplus [10]$ for $\mathbf{2}$, and when we did, it recovered the $T_{(2)}$ eigenvalue correctly. Ultimately, the 691 arises as a divisor of the Bernoulli number B_{12}, equivalently of $\zeta(1-12)$ or of $\zeta(12)/\pi^{12}$.

The space $S_{11}(\Gamma_0(3), \chi_{-3})$ is 2-dimensional, spanned by a Hecke eigenform $g =$

$$q+12\sqrt{-5}q^2+(-27+108\sqrt{-5})q^3+304q^4-1272\sqrt{-5}q^5+(-6480-324\sqrt{-5})q^6+17324q^7+\ldots$$

and its Galois conjugate. It follows, from the fact that for $p \neq 3$ the adjoint $T_p^* = \langle p \rangle T_p$, that $a_p(g)$ is real (hence rational) for split (in E) p, purely imaginary (hence with rational square) for inert p. The prime 1847 is a divisor of the generalised Bernoulli number $B_{11,\chi_{-3}}$, equivalently of $L(1-11, \chi_{-3})$ or of $L(11, \chi_{-3})/(\sqrt{3}\pi^{11})$. There is a congruence between the Hecke eigenvalues of g and an Eisenstein series $E_{11}^{1,\chi_{-3}} \in M_{11}(\Gamma_0(3), \chi_{-3})$:

$$a_p(g) \equiv 1 + \chi_{-3}(p)p^{10} \pmod{\mathfrak{q}},$$

with $(1847) = \mathfrak{q}\overline{\mathfrak{q}}$ in $\mathbb{Q}(\sqrt{-5})$. A proof of this kind of generalised Ramanujan-style congruence, presumably well-known, is recorded in [11, Proposition 2.1]. As a consequence, for p split in E we have

$$a_p(g) \equiv 1 + p^{10} \pmod{1847},$$

while for p inert in E we have

$$a_p(g)^2 \equiv (1-p^{10})^2 \pmod{1847}.$$

Now consider $\mathbf{4} \equiv \mathbf{1} \pmod{1847}$, assuming that the global Arthur parameters really are $\mathbf{1} : [12]$ and $\mathbf{4} : \Delta_{10}[2] \oplus [8]$. At a split prime p we get

$$(a_p(g))(1+p) + (p^2 + p^3 + \ldots + p^9) \equiv 1 + p + \ldots + p^{11} \pmod{1847},$$

which becomes $(1+p)$ times the above $a_p(g) \equiv 1 + p^{10} \pmod{1847}$, while at an inert p we get

$$(a_p(g)^2 + 2.p^{10})(1+p^2) + (p^4 + p^6 + \ldots + p^{18}) \equiv 1 + p^2 + \ldots + p^{22} \pmod{1847},$$

which becomes $(1+p^2)$ times the above $a_p(g)^2 \equiv (1-p^{10})^2 \pmod{1847}$.

Similarly, 809 divides $L(1-9, \chi_{-3})$, and the remaining congruences in this first batch may be accounted for by a congruence between a cusp form and an Eisenstein series in $M_9(\Gamma_0(3), \chi_{-3})$. In fact the congruence $\mathbf{9} \equiv \mathbf{1} \pmod{809}$ leads directly to the guess for the global Arthur parameter of $\mathbf{9}$, and $\mathbf{8} \equiv \mathbf{2} \pmod{809}$ to that for $\mathbf{8}$.

5.2 Ramanujan-type congruences of local origin

The next batch of congruences is

$$\mathbf{3} \equiv \mathbf{1} \pmod{73};$$

$$\mathbf{5} \equiv \mathbf{2} \pmod{61};$$

$$\mathbf{6} \equiv \mathbf{4} \pmod{41};$$

$$\mathbf{17} \equiv \mathbf{19}, \mathbf{20} \pmod{13}.$$

The space $S_{12}(\Gamma_0(3))$ is spanned by a Hecke eigenform $f = q + 78q^2 - 243q^3 + \ldots$. For all primes $p \neq 3$ there is a congruence

$$a_p(f) \equiv 1 + p^{11} \pmod{73},$$

which is of the same shape as Ramanujan's, but whereas Δ has level 1 (same as E_{12}), f has level 3. The modulus arises as a divisor of $3^{12} - 1$, and may be viewed as a divisor of $\zeta_{\{3\}}(12)/\pi^{12}$, where $\zeta_{\{3\}}(s) = (1 - 3^{-s})\zeta(s)$, the Riemann zeta function with the Euler factor at 3 omitted. Such congruences "of local origin" were anticipated by Harder in [21, §2.9], and proved in [14, Theorem 1.1] or [4, Theorem 1].

Assuming the global Arthur parameters really are $\mathbf{1} : [12]$ and $\mathbf{3} : \Delta_{11}(3) \oplus [10]$, this congruence of local origin accounts for $\mathbf{3} \equiv \mathbf{1} \pmod{73}$ in exactly the same way as Ramanujan's congruence accounts for $\mathbf{2} \equiv \mathbf{1} \pmod{691}$. In fact, recognition of the modulus in the congruence $\mathbf{3} \equiv \mathbf{1} \pmod{73}$ led to the guess $\mathbf{3} : \Delta_{11}(3) \oplus [10]$, which then produced the correct $\lambda_3(T_{(2)})$. Combined with the last congruence of the previous subsection, this then allowed us to guess the global Arthur parameter for $\mathbf{7}$ too.

The congruences $\mathbf{5} \equiv \mathbf{2} \pmod{61}$ and $\mathbf{6} \equiv \mathbf{4} \pmod{41}$ may similarly be accounted for by Ramanujan-type congruences of local origin at the prime 3, with $61 \mid (3^{10} - 1)$ (the example from [21, §2.9]) and $41 \mid (3^8 - 1)$. Note that $S_{10}(\Gamma_0(3))$ is 2-dimensional, but the Hecke eigenform $q - 36q^2 - 81q^3 \ldots$ is the one participating in the congruence.

5.3 Level-raising congruences

The next batch of congruences is

$$\mathbf{3} \equiv \mathbf{2} \pmod{17};$$

$$\mathbf{7} \equiv \mathbf{8} \pmod{17};$$

$$\mathbf{15} \equiv \mathbf{14} \pmod{17}.$$

Since $a_3(\Delta) = 252 \equiv -(3^5 + 3^6) \pmod{17}$, Δ satisfies the criterion

$$a_p(\Delta) \equiv \pm(p^{(k/2)-1} + p^{(k/2)}) \pmod{\ell}$$

(with $k = 12$, $p = 3$ and $\ell = 17$) for raising the level by p, i.e. there exists a newform $f \in S_{12}(\Gamma_0(3))$ such that

$$a_q(f) \equiv a_q(\Delta) \pmod{17} \ \forall q \neq 3.$$

This raising of the level is a theorem of Ribet [34] in the case $k = 2$, completed by Diamond in general for $k \geq 2$ [10]. In other words, Δ and f share the same residual Galois representation at $\ell = 17$. (Note that the conditions that this should be irreducible, and that $\ell > k + 1$, $\ell \neq p$ are satisfied.)

On the basis of Ramanujan-style congruences we had already guessed $\mathbf{2} : \Delta_{11} \oplus [10]$, $\mathbf{3} : \Delta_{11}(3) \oplus [10]$, $\mathbf{7} : \Delta_{11}(3) \oplus \Delta_8[2] \oplus [6]$ and $\mathbf{8} : \Delta_{11} \oplus \Delta_8[2] \oplus [6]$. The congruences $\mathbf{3} \equiv \mathbf{2} \pmod{17}$ and $\mathbf{7} \equiv \mathbf{8} \pmod{17}$ are now perfectly accounted for by the above level-raising congruence, providing further evidence for the conjectured Arthur parameters. The congruence $\mathbf{15} \equiv \mathbf{14} \pmod{17}$ now suggests the involvement of Δ_{11} and $\Delta_{11}(3)$ in $\mathbf{14}$ and $\mathbf{15}$, and a bit of guesswork aimed at filling in the gaps in $c_\infty(\widetilde{\pi}_i)$ led to proposals that produced the correct $\lambda_i(T_{(2)})$. Congruences $\mathbf{2} \equiv \mathbf{14} \pmod{17}$ and $\mathbf{3} \equiv \mathbf{15} \pmod{17}$ appear to hold if we look just at the $T_{(2)}$-eigenvalues, but the $T_{(\sqrt{-3})}$-eigenvalues rule them out.

6 Eisenstein congruences for $\mathbb{U}(2,2)$

In [3], a general conjecture was made on congruences of Hecke eigenvalues, between cuspidal automorphic representations of split reductive groups G and representations parabolically induced from Levi subgroups M of maximal parabolic subgroups P, modulo divisors of critical values of L-functions associated with the latter. In the case $G = \mathrm{GL}_2$ with P a Borel subgroup, $M \simeq \mathrm{GL}_1 \times \mathrm{GL}_1$, it predicts the known Ramanujan-style congruences we have already met (including those of local origin). In the case $G = \mathrm{GSp}_2$, P the Siegel parabolic, $M \simeq \mathrm{GL}_2 \times \mathrm{GL}_1$, one recovers a conjecture of Harder [20]. With some small modifications one can relax the split condition, and for unitary groups this is explained in [13].

For $n \geq 1$ let $\mathbb{U}(n,n)$ be the linear algebraic group over $\mathbb{Q}$ whose group of A-rational points is given by

$$\mathbb{U}(n,n)(A) = \{g \in \mathrm{GL}_n(A \otimes_{\mathbb{Q}} E) \mid g^\dagger J_n g = J_n\} \tag{32}$$

for any commutative $\mathbb{Q}$-algebra A, where $J_n = \begin{bmatrix} 0_n & -I_n \\ I_n & 0_n \end{bmatrix}$. This is the unitary group associated to an Hermitian form of signature (n,n), since $\sqrt{-3}J_n$ is an Hermitian matrix. In the case $n=2$, there are two classes of maximal parabolic subgroups. There is the Siegel parabolic, with Levi subgroup $M \simeq \mathrm{GL}_{2,E}$, with $g \mapsto \begin{bmatrix} g & 0 \\ 0 & (g^\dagger)^{-1} \end{bmatrix}$, and the Klingen parabolic, with Levi subgroup $M \simeq \mathrm{GL}_{1,E} \times \mathbb{U}(1,1)$, with $\left(e, \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \mapsto$ $\begin{bmatrix} e & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & (e^\dagger)^{-1} & 0 \\ 0 & c & 0 & d \end{bmatrix}$.

6.1 Klingen parabolic

Conjecture 6.1. *Let $f \in S_{k'}(\Gamma_0(3))$ be a normalised Hecke eigenform. Suppose that*

$$\mathrm{ord}_{\mathfrak{q}}\left(\frac{L_{\{3\}}(f,(k'/2)+b+1)}{(2\pi i)^{(k'/2)+b+1}\Omega^{(-1)^{(k'/2)+b+1}}}\right) > 0$$

or

$$\mathrm{ord}_{\mathfrak{q}}\left(\frac{L_{\{3\}}(f,\chi_{-3},(k'/2)+b+1)}{i\sqrt{3}(2\pi i)^{(k'/2)+b+1}\Omega^{(-1)^{(k'/2)+b}}}\right) > 0,$$

where $\mathfrak{q}$ divides a rational prime $q > k'$, and $0 < b < (k'/2)-1$. (For the correct scaling of Deligne period $\Omega^{\pm}$, see [13, §4]. That used by the Magma command LRatio [5] is good enough for our examples, where $\mathfrak{q}$ is not a prime of congruence for f in $S_{k'}(\Gamma_0(3))$.) Then, letting $a = (k'-4)/2$, there exists a Hecke eigenform $v \in M(V_{a,b}, K_L)$ (notation as in §4.3.1) such that

$$\lambda_v(T_{\mathfrak{P}}) \equiv \begin{cases} a_p(f) + p^{a+b+2} + p^{a-b+1} & \pmod{\mathfrak{q}} \quad \text{if } (p) = \mathfrak{P}\overline{\mathfrak{P}}; \\ (a_p(f)^2 - 2p^{k'-1}) + p^{k'-4}(p^3 - p^2 + p - 1) & \pmod{\mathfrak{q}} \quad \text{if } (p) = \mathfrak{P}. \end{cases}$$

This conjecture is less general than that stated in [13, §8]. To stick to what is narrowly applicable to the situation here, we have put $E = \mathbb{Q}(\sqrt{-3})$ rather than a more general quadratic field, and restricted to f of level $\Gamma_0(3)$. This is not necessary, if we simply modify the set Σ of "bad" primes excluded from the Euler product. The general conjecture asserts the existence of a cuspidal automorphic representation $\widetilde{\Pi}$ of $\mathbb{U}(2,2)$ with Hecke eigenvalues congruent mod $\mathfrak{q}$ to those of an induced representation coming from the base-change to $\mathrm{GL}_2(\mathbb{A}_E)$ of π_f. This induced representation depends on a real parameter s, which in our case is $b+(1/2)$. The right hand side of the congruence is the Hecke eigenvalue for this induced representation. The conjecture in [13] says just that $\widetilde{\Pi}$ has set of ramified primes no bigger than Σ. We have gone a little further, in assuming that $\widetilde{\Pi}$ has the same global Arthur parameter as some automorphic representation of $\mathbb{U}_4(\mathbb{A}_{\mathbb{Q}})$, with a K_L-fixed vector.

Example 4. Let $f \in S_{12}(\Gamma_0(3))$ be the unique normalised newform $f = q + 78q^2 - 243q^3 \ldots$. We have $k' = 12, a = 4$. Let $b = 2$, so $(k'/2) + b + 1 = 9$. Using Magma, $\mathrm{LRatio}(f_{\chi_{-3}}, 9) = 59$. The conjecture predicts a congruence of Hecke eigenvalues involving one of the $\Delta^{(2)}_{11,5}$, modulo a divisor $\mathfrak{q}$ of 59 (and consequently for the other one modulo $\overline{\mathfrak{q}}$). Note that $2a+3 = 11, 2b+1 = 5$. This congruence would account for $\mathbf{7} \equiv \mathbf{12} \pmod{\mathfrak{q}}$ and $\mathbf{7} \equiv \mathbf{13} \pmod{\overline{\mathfrak{q}}}$, which therefore lend support to this instance of the above conjecture. Recall the guesses $\mathbf{7} : \Delta_{11}(3) \oplus \Delta_8[2] \oplus [6]$ and $\mathbf{12} : \Delta^{(2)}_{11,5} \oplus \Delta_8[2] \oplus [4]$. Note that $p^{a+b+2} + p^{a-b+1} = p^8 + p^3 = p^3(p^5 + 1)$, matching perfectly what is left over from the cancellation between [6] and [4].

Example 5. Let $f = q - 36q^2 - 81q^3 \ldots$, one of the normalised newforms in $S_{10}(\Gamma_0(3))$. We have $k' = 10, a = 3$. Let $b = 1$, so $(k'/2)+b+1 = 7$. Using Magma, $\mathrm{LRatio}(f_{\chi_{-3}}, 7) = 22$. The conjecture predicts a congruence mod 11 of Hecke eigenvalues, involving $\Delta_{9,3}$. Note that $2a+3 = 9, 2b+1 = 3$. This congruence would account for $\mathbf{11} \equiv \mathbf{16} \pmod{11}$, which therefore lends support to the above instance of the conjecture. Recall the guesses $\mathbf{11} : \Delta_{11} \oplus \Delta_9(3) \oplus \Delta_6[2] \oplus [4]$ and $\mathbf{16} : \Delta_{11} \oplus \Delta_{9,3} \oplus \Delta_6[2] \oplus [2]$. Note that $p^{a+b+2} + p^{a-b+1} = p^6 + p^3 = p^3(p^3 + 1)$, matching perfectly what is left over from the cancellation between [4] and [2]. Note also that the condition $q > k'$ only just holds here, with $11 > 10$.

6.2 Siegel parabolic

Conjecture 6.2. *Let $f \in S_{k'}(\Gamma_0(3), \chi_{-3})$ (odd $k' > 1$) be a normalised Hecke eigenform. Suppose that*

$$\mathrm{ord}_{\mathfrak{q}} \left(\frac{L_{\{3\}}(\mathrm{Sym}^2 f, k' + s)}{(2\pi i)^{k'+2s+1} \langle f, f \rangle} \right) > 0,$$

where $\mathfrak{q}$ divides a rational prime $q > 2k'$, and s is odd with $1 < s \leq k' - 2$. Let $a = (k' - 4 + s)/2$ and $b = (k' - 2 - s)/2$, so $k' = a + b + 3, s = a - b + 1$ and $k' + s = 2a + 4$. Then there exists a Hecke eigenform $v \in M(V_{a,b}, K_L)$ such that $\pmod{\mathfrak{q}}$

$$\lambda_v(T_{\mathfrak{P}}) \equiv \begin{cases} a_p(f)(1 + p^s) & \text{if } (p) = \mathfrak{P}\overline{\mathfrak{P}}; \\ (a_p(f)^2 + 2p^{k'-1})(1 + p^{2s}) + p^{k'+s-4}(p^3 - p^2 + p - 1) & \text{if } (p) = \mathfrak{P}. \end{cases}$$

Similar remarks apply, concerning the relation of this to the conjecture in [13, §7], as in the previous subsection. The use of $(2\pi i)^{k'+2s+1} \langle f, f \rangle$ for the Deligne period is OK in our examples, where $\mathfrak{q}$ is not a prime of congruence for f in $S_{k'}(\Gamma_0(3), \chi_{-3})$.

Example 6. Let f be the Hecke eigenform $q + 6\sqrt{-14}q^2 + (45 - 18\sqrt{-14})q^3 \ldots$ which, with its Galois conjugate, spans $S_9(\Gamma_0(3), \chi_{-3})$. We have $k' = 9$. Take $s = 3$, so $k' + s = 12$, $a = 4, b = 2$, $2a + 1 = 11, 2b + 1 = 5$. The Euler factor at 3 in $L(\mathrm{Sym}^2 f, s)$ (not the same s), which is missing in $L_{\{3\}}(\mathrm{Sym}^2 f, s)$, is $P(3^{-s})^{-1}$, where $P(X) = \det(I - \mathrm{Sym}^2 \rho_f(\mathrm{Frob}_3^{-1}) | (\mathrm{Sym}^2 V)^{I_3})$, where ρ_f is a λ-adic Galois representation attached to f, on a 2-dimensional space V, and we are taking invariants for an inertia subgroup at 3. According to a theorem of Langlands and Carayol, for which a convenient reference is [24, Theorem 4.2.7 (3)(a)], the action of I_3 on V is diagonalisable, with the trivial character and the character of order 2 appearing. On the unramified part, Frob_3^{-1} acts by the U_3 eigenvalue, which is the coefficient of q^3, and $\det \rho_f$ is the product of the $(k'-1)$ power of the cyclotomic character and the Galois character associated to χ_{-3}. It follows that

$$P(X) = 1 + 5022X + 43046721X^2,$$

noting that $43046721 = 3^{16}$, $-5022 = (45-18\sqrt{-14})^2 + (45+18\sqrt{-14})^2$ and $(45-18\sqrt{-14})(45+18\sqrt{-14}) = 3^8$. Now $P(3^{-12}) = \frac{2^5 23}{3^6}$, and $23 > 2k' = 18$, so the conjecture predicts a congruence mod $\mathfrak{q}$ of Hecke eigenvalues, involving $\Delta^{(2)}_{11,5}$, where $\mathfrak{q} \mid 23$. This congruence would account for $\mathbf{9} \equiv \mathbf{12} \pmod{\mathfrak{q}}$ and $\mathbf{9} \equiv \mathbf{13} \pmod{\overline{\mathfrak{q}}}$, which therefore lend support to this instance of the above conjecture. Recall the guesses $\mathbf{9} : \Delta_8[4] \oplus [4]$ and $\mathbf{12} : \Delta^{(2)}_{11,5} \oplus \Delta_8[2] \oplus [4]$. The $a_p(f)(1+p^3)$ is exactly what is left after cancellation between $\Delta_8[4]$ and $\Delta_8[2]$.

Remark 6.3. In [12] (repeated in [13]), we looked at the case $k' = 9, s = 5$, with $q = 19$ or 37, gathering a scrap of evidence for the conjectured congruences by computing the Hecke eigenvalues for $T_{(2)}$ on the 2-dimensional space $M(V_{5,1}, K_L)$. Contrary to what was stated there, due to a misunderstanding about the Euler factor at 3 in the L-value computed by the formula, the case $q = 19$ actually comes from the missing Euler factor at 3 rather than the complete L-value:

$$P(3^{-14}) = \frac{2^5.5^3.7.19}{3^{12}}.$$

7 Hermitian theta series

Let

$$\mathcal{H}_m = \{Z \in M_n(\mathbb{C}) \mid i(Z^\dagger - Z) > 0\}$$

be the Hermitian upper half space of degree m. Given an integral Hermitian lattice $L \subseteq V_{12}$, we define its Hermitian theta series of degree m, $\vartheta^{(m)}(L) : \mathcal{H}_m \to \mathbb{C}$ by

$$\vartheta^{(m)}(L, Z) := \sum_{\mathbf{x} \in L^m} \exp(\pi i \mathrm{Tr}(\langle \mathbf{x}, \mathbf{x} \rangle Z)).$$

Applying [35, Lemma 2.1], we find that if L is unimodular (e.g. $L = \mathcal{O}_E^{12}$) then $\vartheta^{(m)}(L)$ is a modular form of weight 12 for the group $\widetilde{\Gamma}^{(m)} :=$

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{U}(m,m)(\mathbb{Q}) \mid A, D \in M_m(\mathcal{O}_E), B \in (\sqrt{-3})^{-1} M_m(\mathcal{O}_E), C \in 3\sqrt{-3} M_m(\mathcal{O}_E) \right\},$$

i.e.

$$\vartheta^{(m)}(L, (AZ+B)(CZ+D)^{-1}) = |CZ+D|^{12}\, \vartheta^{(m)}(L, Z) \quad \forall \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \widetilde{\Gamma}^{(m)}.$$

Extending by linearity, we obtain $\Theta^{(m)} : M(\mathrm{triv}, K_L) \to M_{12}^{(m)}(\widetilde{\Gamma}^{(m)})$, i.e.

$$\Theta^{(m)}(\sum x_i e_i) = \sum x_i \, \vartheta^{(m)}(L_i).$$

Given a Hecke eigenform $f \in S_{2k+1}(\Gamma_0(3), \chi_{-3})$ with $2k < 12$, we may apply a theorem of Ikeda [26] to obtain a Hecke eigenform $I^{(12-2k)}(f) \in S_{12}(\Gamma^{(12-2k)})$, where $\Gamma^{(m)} := \mathbb{U}(m,m)(\mathbb{Q}) \cap M_{2m}(\mathcal{O}_E)$ is the standard Hermitian modular group. It is easy to show that if $f(Z) \in M_{12}(\Gamma^{(m)})$ then for any $N \in \mathbb{Z}_{>0}$, $f(NZ) \in M_{12}(\Gamma_N^{(m)})$, where

$$\Gamma_N^{(m)} := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{U}(m,m)(\mathbb{Q}) \mid A, D \in M_m(\mathcal{O}_E), B \in N^{-1}M_m(\mathcal{O}_E), C \in NM_m(\mathcal{O}_E) \right\}.$$

Then $\widetilde{\Gamma}^{(m)} \subseteq \Gamma_3^{(m)}$. It follows that $I^{(12-2k)}(f)(3\,Z) \in S_{12}(\widetilde{\Gamma}^{(12-2k)})$. Following [7, VII,Corollaire 3.4], and looking back at the conjectured global Arthur parameters in our table, it is natural to guess something like the following.

Conjecture 7.1. .

1. *$\Theta^{(2)}(v_4) \in \mathrm{Span}\{I^{(2)}(f)(3\,Z)\}$, with $f \in S_{11}(\Gamma_0(3), \chi_{-3})$;*
2. *$\Theta^{(4)}(v_9) \in \mathrm{Span}\{I^{(4)}(f)(3\,Z)\}$, with $f \in S_9(\Gamma_0(3), \chi_{-3})$;*
3. *$\Theta^{(6)}(v_{17}) \in \mathrm{Span}\{I^{(6)}(f)(3\,Z)\}$, with $f \in S_7(\Gamma_0(3), \chi_{-3})$.*

Further, comparing with [25, §7], we should expect $\Theta^{(3)}(v_6)$, $\Theta^{(3)}(v_7)$,$\Theta^{(3)}(v_8)$, $\Theta^{(4)}(v_{10})$, $\Theta^{(5)}(v_{14})$, $\Theta^{(5)}(v_{15})$ all to come from some kind of Hermitian Miyawaki lifts, and $\Theta^{(4)}(v_{11})$ from an iterated Hermitian Miyawaki lift.

Recall from §4.4 that the space of scalar-valued algebraic modular forms for the genus of 5 classes of rank-12, $\sqrt{-3}$-modular lattices has a basis of eigenvectors $\{w_1, w_2, w_4, w_8, w_9\}$, with $T_{(2)}$-eigenvalues matching those of $\{v_1, v_2, v_4, v_8, v_9\}$. Their Hermitian theta series $\Theta^{(m)}(w_i)$ lie in $S_{12}(\Gamma^{(m)})$ [23, Theorem 2.1],[9] and it was conjectured in [22, Remark 3(b)] that $\Theta^{(4)}(w_9)$ is an Hermitian Ikeda lift.

Acknowledgements

We thank G. Chenevier and G. Nebe for helpful communications.

References

[1] J. Arthur. *The endoscopic classification of representations*, volume 61 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2013. Orthogonal and symplectic groups.

[2] C. Bachoc and G. Nebe. Classification of two genera of 32-dimensional lattices of rank 8 over the Hurwitz order. *Experiment. Math.*, 6(2):151–162, 1997.

[3] J. Bergström and N. Dummigan. Eisenstein congruences for split reductive groups. *Selecta Math. (N.S.)*, 22(3):1073–1115, 2016.

[4] N. Billerey and R. Menares. On the modularity of reducible m$od\,l$ Galois representations. *Math. Res. Lett.*, 23(1):15–41, 2016.

[5] W. Bosma, J. Cannon, and C. Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993).

[6] P. Cartier. Representations of p-adic groups: a survey. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 111–155. Amer. Math. Soc., Providence, R.I., 1979.

[7] G. Chenevier and J. Lannes. Formes automorphes et voisins de Kneser des r\'eseaux de Niemeier. *ArXiv e-prints*, Sept. 2014.

[8] A. M. Cohen, G. Nebe, and W. Plesken. Maximal integral forms of the algebraic group G_2 defined by finite subgroups. *Journal of Number Theory*, 72(2):282–308, 1998.

[9] D. M. Cohen and H. L. Resnikoff. Hermitian quadratic forms and Hermitian modular forms. *Pacific J. Math.*, 76(2):329–337, 1978.

[10] F. Diamond. Congruence primes for cusp forms of weight $k \geq 2$. *Astérisque*, (196-197):6, 205–213 (1992), 1991. Courbes modulaires et courbes de Shimura (Orsay, 1987/1988).

[11] N. Dummigan. Eisenstein primes, critical values and global torsion. *Pacific J. Math.*, 233(2):291–308, 2007.

[12] N. Dummigan. A simple trace formula for algebraic modular forms. *Exp. Math.*, 22(2):123–131, 2013.

[13] N. Dummigan. Eisenstein congruences for unitary groups. `http://neil-dummigan.staff.shef.ac.uk/unitarycong5.pdf`, 2015, preprint.

[14] N. Dummigan and D. Fretwell. Ramanujan-style congruences of local origin. *J. Number Theory*, 143:248–261, 2014.

[15] W. Feit. Some lattices over $\mathbb{Q}(\sqrt{-3})$. *J. Algebra*, 52(1):248–263, 1978.

[16] W. T. Gan and J.-K. Yu. Group schemes and local densities. *Duke mathematical journal*, 105(3):497–524, 2000.

[17] M. Greenberg and J. Voight. Lattice methods for algebraic modular forms on classical groups. In *Computations with Modular Forms*, pages 147–179. Springer, 2014.

[18] B. H. Gross. On the Satake isomorphism. In *Galois representations in arithmetic algebraic geometry (Durham, 1996)*, volume 254 of *London Math. Soc. Lecture Note Ser.*, pages 223–237. Cambridge Univ. Press, Cambridge, 1998.

[19] B. H. Gross. Algebraic modular forms. *Israel J. Math.*, 113:61–93, 1999.

[20] G. Harder. A congruence between a Siegel and an elliptic modular form. In *The 1-2-3 of modular forms*, Universitext, pages 247–262. Springer, Berlin, 2008.

[21] G. Harder. Secondary operations in the cohomology of Harish-Chandra modules. `http://www.math.uni-bonn.de/people/harder/Manuscripts/Eisenstein/SecOPs.pdf`, 2013, preprint.

[22] M. Hentschel, A. Krieg, and G. Nebe. On the classification of lattices over $\mathbb{Q}(\sqrt{-3})$, which are even unimodular $\mathbb{Z}$-lattices. *Abh. Math. Semin. Univ. Hambg.*, 80(2):183–192, 2010.

[23] M. Hentschel and G. Nebe. Hermitian modular forms congruent to 1 modulo p. *Arch. Math. (Basel)*, 92(3):251–256, 2009.

[24] H. Hida. *Geometric modular forms and elliptic curves.* World Scientific Publishing Co., Inc., River Edge, NJ, 2000.

[25] T. Ikeda. Pullback of the lifting of elliptic cusp forms and Miyawaki's conjecture. *Duke Math. J.*, 131(3):469–497, 2006.

[26] T. Ikeda. On the lifting of Hermitian modular forms. *Compos. Math.*, 144(5):1107–1154, 2008.

[27] T. Kaletha, A. Minguez, S. W. Shin, and P.-J. White. Endoscopic Classification of Representations: Inner Forms of Unitary Groups. *ArXiv e-prints*, Sept. 2014.

[28] M. Kirschmer. Definite quadratic and hermitian forms with small class number. Habilitation thesis, 2016. RWTH Aachen University.

[29] K. Klosin. Congruences among modular forms on $\mathrm{U}(2,2)$ and the Bloch-Kato conjecture. *Ann. Inst. Fourier (Grenoble)*, 59(1):81–166, 2009.

[30] T. Mégarbané. Calcul des op\'erateurs de Hecke sur les classes d'isomorphisme de r\'eseaux pairs de d\'eterminant 2 en dimension 23 et 25. *ArXiv e-prints*, July 2016.

[31] C. P. Mok. Endoscopic classification of representations of quasi-split unitary groups. *Mem. Amer. Math. Soc.*, 235(1108):vi+248, 2015.

[32] G. Nebe and B. Venkov. On Siegel modular forms of weight 12. *J. Reine Angew. Math.*, 531:49–60, 2001.

[33] W. Plesken and B. Souvignier. Computing isometries of lattices. *Journal of Symbolic Computation*, 24(3):327–334, 1997.

[34] K. A. Ribet. Congruence relations between modular forms. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983)*, pages 503–514. PWN, Warsaw, 1984.

[35] R. Scharlau, A. Schiemann, and R. Schulze-Pillot. Theta series of modular, extremal, and Hermitian lattices. In *Integral quadratic forms and lattices (Seoul, 1998)*, volume 249 of *Contemp. Math.*, pages 221–233. Amer. Math. Soc., Providence, RI, 1999.

[36] A. Schiemann. Classification of Hermitian forms with the neighbour method. *J. Symbolic Comput.*, 26(4):487–508, 1998.

[37] S. Schönnenbeck. Simultaneous computation of Hecke operators. *ArXiv e-prints*, Dec. 2016.

[38] T. Tamagawa. On the ζ-functions of a division algebra. *Ann. of Math. (2)*, 77:387–405, 1963.