$\mathrm{GL}_2 \times \mathrm{GSp}_2$ *L*-values and Hecke eigenvalue congruences

par Jonas Bergström, Neil Dummigan, David Farmer et Sally
Koutsoliotas

Résumé. Nous trouvons des examples expérimentaux de congruences des valeurs propres de Hecke entre des représentations automorphes de groupes tels que $\mathrm{GSp}_2(\mathbb{A})$, $\mathrm{SO}(4,3)(\mathbb{A})$ et $\mathrm{SO}(5,4)(\mathbb{A})$, dans lesquelles le module premier devrait, pour diverses raisons, apparaître dans la partie algébrique d'une valeur L "produit tensoriel" critique associée à des représentations automorphes cuspidales de $\mathrm{GL}_2(\mathbb{A})$ et $\mathrm{GSp}_2(\mathbb{A})$. En employant des techniques spéciales pour évaluer des fonctions L avec peu de coefficients connus, nous calculons des approximations suffisantes pour détecter le diviseur premier anticipé.

Abstract. We find experimental examples of congruences of Hecke eigenvalues between automorphic representations of groups such as $\mathrm{GSp}_2(\mathbb{A})$, $\mathrm{SO}(4,3)(\mathbb{A})$ and $\mathrm{SO}(5,4)(\mathbb{A})$, where the prime modulus should, for various reasons, appear in the algebraic part of a critical "tensor-product" *L*-value associated to cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ and $\mathrm{GSp}_2(\mathbb{A})$. Using special techniques for evaluating *L*-functions with few known coefficients, we compute sufficiently good approximations to detect the anticipated prime divisors.

1. Introduction

This paper is about divisors of critical values of *L*-functions as moduli of congruences between Hecke eigenvalues of automorphic forms. It is made possible by three separate developments in computational number theory.

(1) Mégarbané's large-scale computation of traces of Hecke operators on spaces of level-one algebraic modular forms, for $\mathrm{SO}(7)$, $\mathrm{SO}(8)$ and $\mathrm{SO}(9)$ [30], following the endoscopic classification of the associated automorphic representations by Chenevier and Renard [9].

(2) Faber and van der Geer's computation of traces of Hecke operators on spaces of vector-valued Siegel modular forms of genus 2 and level one, using point counts on hyperelliptic curves of genus 2 over finite fields [20]. When the space is 1-dimensional, this gives Hecke eigenvalues. They computed traces of Hecke operators $T(p)$ and $T(p^2)$ for $p, p^2 \leq 37$. The first-named author of this paper refined their method and extended the bound to 179. The data is available at `http://smf.compositio.nl/`.

(3) A new technique for computing good approximations to values of L-functions satisfying functional equations, given only a few coefficients in the Dirichlet series, developed by the third-named author and Ryan [21]. This combines, in such a way as to make unknown errors cancel, approximations obtained by the method of Rubinstein [37], which is related to the technique implemented in Magma [27], which is described in [15].

Conjecture 4.2 of [2] is a very wide generalisation of Ramanujan's mod 691 congruence, to "Eisenstein" congruences between Hecke eigenvalues of automorphic representations of $G(\mathbb{A})$, where $\mathbb{A}$ is the adele ring and $G/\mathbb{Q}$ is any connected, split reductive group. On one side of the congruence is a cuspidal automorphic representation $\tilde{\Pi}$. On the other is one induced from a cuspidal automorphic representation Π of the Levi subgroup M of a maximal parabolic subgroup P. The modulus of the congruence comes from a critical value of a certain L-function, associated to Π and to the adjoint representation of the L-group $\hat{M}$ on the Lie algebra $\hat{\mathfrak{n}}$ of the unipotent radical of the maximal parabolic subgroup $\hat{P}$ of $\hat{G}$. Starting from Π, we conjecture the existence of $\tilde{\Pi}$, satisfying the congruence.

In [3] we already used Mégarbané's data for $SO(7)$ and $SO(8)$, observing experimental Eisenstein congruences for the cases $G = SO(4,3)$, $M = GL_1 \times SO(3,2)$ and $G = SO(4,4)$, $M = GL_2 \times SO(2,2)$. To support the conjecture, we then needed to find the observed moduli in the corresponding L-values. In the $SO(4,4)$ case these were triple product L-values for elliptic modular forms, which were computed exactly by Ibukiyama and Katsurada, using the pull-back of a genus 3 Siegel-Eisenstein series. In the $SO(4,3)$ case they were spinor L-values for vector-valued Siegel modular forms of genus 2 and level one (note that $SO(3,2) \simeq PGSp_2$), and we resorted to sufficiently good numerical approximations. For this, Magma was good enough, and we used the Hecke eigenvalues computed by the first-named author, which went as far as the bound 149 at that point.

For this paper we used Mégarbané's data for $SO(9)$ to find an experimental Eisenstein congruence (mod $q = 17$) in the case $G = SO(5,4)$, $M = GL_2 \times SO(3,2)$ (see Example 5 in §7). In this case, the associated L-function has degree-8 Euler factors, and is the "tensor-product" of the

Hecke L-function of an elliptic modular form and the spinor L-function of a vector-valued Siegel modular form of genus 2 (both level one). We also used Mégarbané's data for $\mathrm{SO}(7)$ to find experimental "endoscopic" congruences (mod $q = 71$ and 61), between functorial lifts from $\mathrm{SO}(2,1) \times \mathrm{SO}(3,2)$ to $\mathrm{SO}(4,3)$ and non-lifts on $\mathrm{SO}(4,3)$ (see Examples 3 and 4 in §6, and note that $\mathrm{SO}(2,1) \simeq \mathrm{PGL}_2$). Here, as with the Eisenstein congruences, a generalisation of a construction of Ribet leads from the congruence, via an extension of mod q Galois representations, to an element of order q in a Bloch-Kato Selmer group, which according to the Bloch-Kato conjecture ought to show up in a certain critical L-value. For these $\mathrm{SO}(7)$ endoscopic congruences, again it is a tensor-product $\mathrm{GL}_2 \times \mathrm{GSp}_2$ L-function, for an elliptic modular form and a vector-valued Siegel modular form of genus 2. These congruences are analogous to those between Yoshida lifts and non-lifts (Siegel modular forms of genus 2) appearing in [6], where the L-function is a degree-4 tensor-product L-function for two elliptic modular forms.

To obtain sufficiently good approximations to the $\mathrm{GL}_2 \times \mathrm{GSp}_2$ L-values, Magma requires many more coefficients in the Dirichlet series than we could obtain using the computations of Hecke eigenvalues for Siegel modular forms. (See [3, §7] for a comment on the difficulty of extending these much further.) So for this the third and fourth-named authors used the kind of averaging technique described in [21]. This is described in §4.3–§4.6, and the results are in §4.2. The numerical approximations to ratios of L-values (and appropriate powers of π) are very close to rational numbers, and in these rational numbers we find the expected factors of $17, 71$ and 61. We also stumbled on some other factors of 839 and 61 (again), and realised at that point that these could also be explained using the Bloch-Kato conjecture, in terms of Eisenstein congruences for $G = \mathrm{GSp}_4$. For $q = 839$ (Example 1), P is the Klingen parabolic subgroup (Kurokawa-Mizumoto congruences) while for $q = 61$ (Example 2), P is the Siegel parabolic subgroup (Harder's conjecture). This is explained in Section 5. In fact, following basic background in §2, we begin with these examples in §3, where a rougher heuristic is given. As noted in §3.1, §3.2, the accidental discovery of these experimental divisibilities led to the *proof* of the analogous divisibilities in the scalar-valued case, where the necessary pullback formulas are known. However, the experimental congruences and divisibilities of L-values involving $\mathrm{SO}(7)$ and $\mathrm{SO}(9)$ are presumably some way beyond what can currently be proved (with the exception of the congruence in §6, Example 4). This justifies the effort made to take the experimental results as far as possible, extending to types of congruences and numerical techniques not considered or employed in [3].

In §9 we use Mégarbané's SO(9) data to observe an experimental endoscopic congruence mod $q = 37$, between a functorial lift from $SO(3,2) \times SO(3,2)$ to $SO(5,4)$ and a non-lift on $SO(5,4)$. Applying the same techniques to the associated $GSp_2 \times GSp_2$ L-values, we cannot obtain such reliable and accurate approximations as before, but are still able to see the expected factor of 37.

As should already be clear from this introduction, the data that was computed by Thomas Mégarbané was indispensable, and we are grateful to him for providing it to us before it was publicly available. We thank also Mark Watkins for pointing out errors in an earlier version of this paper, including one that had a very significant impact on §8, and for corroborating the new computation thus enabled.

2. $GL_2 \times GSp_2$ L-functions

Let f be a normalised cuspidal Hecke eigenform of weight ℓ for $SL_2(\mathbb{Z})$. Then $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^\ell f(\tau)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, and $\tau \in \mathfrak{H} = \{\tau \in \mathbb{C} : \operatorname{Im}(\tau) > 0\}$, and $f(\tau) = \sum_{n=1}^\infty a_n(f)q^n$, with $q = e^{2\pi i \tau}$ and $a_1 = 1$. The Fourier coefficients are also the eigenvalues of Hecke operators. The Hecke L-function is

$$L(s,f) = \prod_{p \text{ prime}} (1 - a_p(f)p^{-s} + p^{\ell-1-2s})^{-1}.$$

Let $1 - a_p(f)X + p^{\ell-1}X^2 =: (1 - \alpha_{p,1}X)(1 - \alpha_{p,2}X)$.

Let F be a cuspidal Hecke eigenform of weight $\operatorname{Sym}^j \otimes \det^k$ for $Sp_2(\mathbb{Z}) := \{g \in M_4(\mathbb{Z}) : {}^t g J g = J\}$, where $J = \begin{pmatrix} 0_2 & -I_2 \\ I_2 & 0_2 \end{pmatrix}$. Then $F : \mathfrak{H}_2 \to V$, where $\mathfrak{H}_2 = \{Z \in M_2(\mathbb{C}) : {}^t Z = Z, \operatorname{Im}(Z) > 0\}$ is Siegel's upper half space of genus 2, V is the space of the representation $\rho = \operatorname{Sym}^j(\mathbb{C}^2) \otimes \det^k$ of $GL_2(\mathbb{C})$, and

$$F\left((AZ+B)(CZ+D)^{-1}\right) = \rho(CZ+D)(F(Z)) \ \forall \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_2(\mathbb{Z}).$$

Let the elements $T(p), T(p^2)$ of the genus-2 Hecke algebra be as in [43, §16] (with the scaling as following Definition 8). Let $\lambda_F(p), \lambda_F(p^2)$ be the respective eigenvalues for these operators acting on F. The spinor L-function of F is $L(s, F, \operatorname{Spin}) = \prod_{p \text{ prime}} L_p(s, F, \operatorname{Spin})$, where

$$L_p(s, F, \operatorname{Spin})^{-1} = 1 - \lambda_F(p)p^{-s} + (\lambda_F(p)^2 - \lambda_F(p^2) - p^{j+2k-4})p^{-2s}$$
$$- \lambda_F(p)p^{j+2k-3-3s} + p^{2j+4k-6-4s}.$$

Let $L_p(s, F, \operatorname{Spin})^{-1} =: \prod_{j=1}^4 (1 - \beta_{p,j}p^{-s})$.

Now we define $L(s, f \otimes F) := \prod_{p \, \text{prime}} L_p(s, f \otimes F)$, where

$$L_p(s, f \otimes F)^{-1} := \prod_{i=1}^{2} \prod_{j=1}^{4} (1 - \alpha_{p,i} \beta_{p,j} p^{-s}).$$

To understand the conjectured functional equation and critical values for this L-function, it is convenient to introduce the motive M_f attached to f, and the conjectured motive M_F attached to F, of ranks 2 and 4 respectively. The Betti realisations have Hodge decompositions $M_{f,B} \otimes \mathbb{C} \simeq H^{0,\ell-1} \oplus H^{\ell-1,0}$ and $M_{F,B} \otimes \mathbb{C} \simeq H^{0,j+2k-3} \oplus H^{j+2k-3,0} \oplus H^{k-2,j+k-1} \oplus H^{j+k-1,k-2}$, with each $H^{p,q}$ 1-dimensional. The L-functions associated to (q-adic realisations of) M_f and M_F are $L(s, f)$ and $L(s, F, \text{Spin})$ respectively. The L-function $L(s, f \otimes F)$ is associated to the rank-8 motive $M := M_f \otimes M_F$, which has Hodge decomposition $M_B \otimes \mathbb{C} \simeq \oplus (H^{p,q} \oplus H^{q,p})$, where $p + q = j + 2k + \ell - 4$ and $p \in \{0, k-2, \min\{\ell-1, j+2k-3\}, \min\{j+k-1, k+\ell-3\}\} =: \{p_1, p_2, p_3, p_4\}$, where we label the elements so that $p_1 \leq p_2 \leq p_3 \leq p_4$. According to [12, Table 5.3], each (p,q) contributes i^{q-p+1} to the sign in the conjectural functional equation, and using the fact that j is even, one checks easily that the sign should be $+1$. Following the recipe in [39] (or see again [12, Table 5.3]), the product of gamma factors is $\gamma(s) = \prod_{i=1}^{4} \Gamma_{\mathbb{C}}(s - p_i)$, where $\Gamma_{\mathbb{C}}(s) := (2\pi)^{-s} \Gamma(s)$. Note that, following [7, Remark 6.2], it makes no difference to replace any p_i by $q_i = j + 2k + \ell - 4 - p_i$. Anyway, the conjectured functional equation is $\Lambda(j + 2k + \ell - 3 - s) = \Lambda(s)$, where $\Lambda(s) := \gamma(s) L(s, f \otimes F)$. The meromorphic continuation and functional equation have been proved by Böcherer and Heim [7] in the case that F is scalar valued (i.e. $j = 0$), Furusawa [23] having already dealt with the even more special case $\ell = k$ (and $j = 0$).

The critical values are $L(t, f \otimes F)$ for integers t such that neither $\gamma(s)$ nor $\gamma(j + 2k + \ell - 3 - s)$ has a pole at $s = t$. This is for $p_4 < t \leq q_4$. In all our examples, the coefficient field of M_f and M_F (hence of M) is $\mathbb{Q}$, so we suppose for convenience that this is the case. (Then M_B and M_{dR} are $\mathbb{Q}$-vector spaces.) For each critical t, there is a Deligne period $c^+(M(t))$ defined as in [12], up to $\mathbb{Q}^\times$ multiples. (It is the determinant, with respect to bases of 4-dimensional $\mathbb{Q}$-vector spaces $M_B(t)^+$ and $M_{\text{dR}}(t)/\text{Fil}^0$, of an isomorphism between $M_B(t)^+ \otimes \mathbb{C}$ and $(M_{\text{dR}}(t)/\text{Fil}^0) \otimes \mathbb{C}$.) Deligne's conjecture (in this instance) is that $L(s, f \otimes F)/c^+(M(t)) \in \mathbb{Q}^\times$. Later we shall sometimes make a special choice of $c^+(M(t))$, and define $L_{\text{alg}}(t, f \otimes F) = L(t, f \otimes F)/c^+(M(t))$. If t, t' are critical points with $t \equiv t' \pmod 2$, then $c^+(M(t')) = (2\pi i)^{4(t'-t)} c^+(M(t))$, because $M_B(t') = M_B(t)(2\pi i)^{t'-t}$ while $M_{\text{dR}}(t)/\text{Fil}^0$ does not change for t within the critical range. So the ratio $\frac{L_{\text{alg}}(t', f \otimes F)}{L_{\text{alg}}(t, f \otimes F)} = \frac{L(t', f \otimes F)}{(2\pi i)^{4(t'-t)} L(t, f \otimes F)}$, which should be a rational number, is independent of any choices. Prime divisors of its numerator or denominator

will turn out to be significant. Actually, for these tensor product motives the condition $t \equiv t'$ (mod 2) is unnecessary [45, Cor.1, p.1188], but among our examples, only for the one in §8 are there so few critical points that we would need to relax it.

3. Expected consequences of congruences: the rough version

3.1. Kurokawa-Mizumoto type. Suppose that $\ell = j + k$, with ℓ as above, j, k non-negative even integers. There is a vector-valued Klingen-Eisenstein series $[f]_j$, a non-cuspidal genus-2 Siegel modular form of weight $\mathrm{Sym}^j \otimes \det^k$ for $\mathrm{Sp}_2(\mathbb{Z})$, satisfying $\Phi([f]_j) = f$, where Φ is the Siegel operator. (See [1, §1] for more details.) Let $q > 2\ell$ be a prime divisor of the numerator of $L_{\mathrm{alg}}(2\ell - 2 - j, \mathrm{Sym}^2 f)$, which we can take to be $L(2\ell - 2 - j, \mathrm{Sym}^2 f)/\pi^{2k+\ell-3}(f, f)$, where (f, f) is the Petersson norm and $L(s, \mathrm{Sym}^2 f) = \prod_{p \,\mathrm{prime}}((1 - \alpha_1^2 p^{-s})(1 - \alpha_1 \alpha_2 p^{-s})(1 - \alpha_2^2 p^{-s}))^{-1}$. Sometimes it is possible to prove a congruence (mod q) of Hecke eigenvalues between $[f]_j$ and some cuspidal Hecke eigenform F, also of weight $\mathrm{Sym}^j \otimes \det^k$ for $\mathrm{Sp}_2(\mathbb{Z})$. The first examples were proved by Kurokawa and Mizumoto [26, 32], with further examples proved by Satoh [38] and in [16].

Note that on $[f]_j$ the eigenvalue of $T(p)$ is $a_p(f)(1 + p^{k-2})$, in fact its spinor L-function (defined in terms of Hecke eigenvalues just as for the cuspidal case) is $L(s, [f]_j, \mathrm{Spin}) = L(s, f)L(s - (k - 2), f)$. Then $L(s, f \otimes [f]_j) = L(s, f \otimes f)L(s - (k - 2), f \otimes f)$. Since $L(s, f \otimes f) = \zeta(s - (\ell - 1))L(s, \mathrm{Sym}^2 f)$, we find that

$$L(s, f \otimes [f]_j) = \zeta(s-(\ell-1))\zeta(s-(\ell+k-3))L(s, \mathrm{Sym}^2 f)L(s-(k-2), \mathrm{Sym}^2 f).$$

In this situation where $\ell = j + k$, the critical range for $L(s, f \otimes F)$ is $\ell \leq t \leq \ell+k-3$. The factor $\zeta(s-(\ell+k-3))$ is non-zero at $s = \ell+k-3$, but has a trivial zero at all other odd s in the critical range, e.g. at $s = \ell+k-5$. Checking the other factors, we find that $\frac{L(s, f \otimes [f]_j)}{\pi^8 L(s-2, f \otimes [f]_j)}$ has a simple pole at $s = \ell + k - 3$.

The mod q congruence of Hecke eigenvalues between $[f]_j$ and F, hence between coefficients of the Dirichlet series for $L(s, f \otimes [f]_j)$ and $L(s, f \otimes F)$, might lead one roughly to expect that the pole of $\frac{L(s, f \otimes [f]_j)}{\pi^8 L(s-2, f \otimes [f]_j)}$ at the rightmost critical point $s = \ell + k - 3$ (for $L(s, f \otimes F)$) should cause a pole $mod\ q$ of $\frac{L(j+2k-3, f \otimes F)}{\pi^8 L(j+2k-5, f \otimes F)}$, i.e. a factor of q in its denominator. Since $j + 2k - 5$ could be replaced by any odd s in the critical range strictly to the left of $j + 2k - 3$, we can think of this q as being in the denominator of $L_{\mathrm{alg}}(j + 2k - 3, f \otimes F)$, without worrying too much about the correct scaling.

We have so far chosen the "motivic" normalisation of the L-function, but it is also convenient to consider the "unitary" normalisation $L(s + (j +$

$2k + \ell - 4)/2, f \otimes F)$, which should satisfy a functional equation relating s and $1 - s$. This normalisation is natural if we consider the L-function as arising from automorphic representations π_f and π_F of $\mathrm{GL}_2(\mathbb{A})$ and $\mathrm{GSp}_2(\mathbb{A})$ respectively, where $\mathbb{A}$ is the adele ring, so we set $L(s, \pi_f \otimes \pi_F) := L(s + (j + 2k + \ell - 4)/2, f \otimes F)$. We then expect to find q in the denominator of $\frac{L((k-2)/2, \pi_f \otimes \pi_F)}{\pi^8 L((k-6)/2, \pi_f \otimes \pi_F)}$.

Example 1. $\ell = 16, j = 4, k = 12, q = 839$. The congruence is [16, Proposition 4.1]. We expect $\frac{L(5, \pi_f \otimes \pi_F)}{\pi^8 L(3, \pi_f \otimes \pi_F)}$ to be a rational number with 839 in the denominator. In fact, if we observe it in the denominator of $\frac{L(5, \pi_f \otimes \pi_F)}{\pi^8 L(3, \pi_f \otimes \pi_F)}$ (or equivalently in the numerator of $\frac{\pi^8 L(3, \pi_f \otimes \pi_F)}{L(5, \pi_f \otimes \pi_F)}$), but not in the denominator of $\frac{\pi^8 L(1, \pi_f \otimes \pi_F)}{L(3, \pi_f \otimes \pi_F)}$, we should feel reasonably confident that it is coming from the denominator of $L_{\mathrm{alg}}(5, \pi_f \otimes \pi_F)$ rather than the numerator of $L_{\mathrm{alg}}(3, \pi_f \otimes \pi_F)$, since it seems unlikely that it would also happen to divide the numerator of $L_{\mathrm{alg}}(1, \pi_f \otimes \pi_F)$.

In our numerical examples in §4, $j > 0$ and F is vector-valued. However, in the case that $j = 0$ and F is scalar-valued, a formula of Heim, for the restriction of a genus 5 Eisenstein series to $\mathfrak{H}_1 \times \mathfrak{H}_2 \times \mathfrak{H}_2$ [25], in which appears the L-value in question, allows one to actually prove the expected divisibility [18].

3.2. Harder type. Suppose that $k' = j + 2k - 2$, with k' the weight of a normalised, cuspidal Hecke eigenform g for $\mathrm{SL}_2(\mathbb{Z})$, $j > 0, k \geq 3$ integers with j even. In many examples there appears to be a cuspidal Hecke eigenform F, of weight $\mathrm{Sym}^j \otimes \det^k$ for $\mathrm{Sp}_2(\mathbb{Z})$, and a congruence $\lambda_F(p) \equiv a_p(g) + p^{k-2} + p^{j+k-1} \pmod{\mathfrak{q}}$, where $\mathfrak{q}$ divides the numerator of a suitably normalised $L_{\mathrm{alg}}(j + k, g)$. Cases where such congruences have been verified for $p \leq 37$, using Hecke eigenvalue computations by Faber and van der Geer [20], are described in [43]. The original example $(k', j, k, q) = (22, 4, 10, 41)$ used by Harder to support his conjecture [24], has subsequently been proved by Chenevier and Lannes [8, Chapter 10, Theorem* 4.4(1)].

One way of expressing the congruence is to say that $L(s, F, \mathrm{Spin})$ is congruent, coefficient by coefficient, to $L(s, g)\zeta(s - (k-2))\zeta(s - (j+k-1))$. Then, with auxiliary f of weight ℓ, $L(s, f \otimes F)$ is congruent, coefficient by coefficient, to $L(s, f \otimes g)L(s - (k-2), f)L(s - (j+k-1), f)$. Now if $\ell/2$ is odd then the sign in the functional equation of $L(s, f)$ is -1, so $L(\ell/2, f) = 0$ and the factor $L(s - (j + k - 1), f)$ vanishes at $s = (\ell/2) + j + k - 1$. We might then roughly expect $L_{\mathrm{alg}}((\ell/2) + j + k - 1, f \otimes F)$ to vanish *mod q*, so to find q in the numerator of $\frac{L((\ell/2)+j+k-1, f \otimes F)}{\pi^8 L((\ell/2)+j+k-3, f \otimes F)}$, which is the same as $\frac{L((j+2)/2, \pi_f \otimes \pi_F)}{\pi^8 L((j-2)/2, \pi_f \otimes \pi_F)}$. These will be critical values as long as we ensure that $\ell \geq j + 4$.

Example 2.   If $(j, k) = (4, 15)$, so that $k' = 32$, then the unique (up to scaling) cusp form F of weight $\mathrm{Sym}^4 \otimes \det^{15}$ appears to satisfy a congruence as above mod $\mathfrak{q}$, with $\mathfrak{q} \mid L_{\mathrm{alg}}(19, g)$ a divisor of 61. (Note that there are two conjugate choices for g and for $\mathfrak{q}$.) Now let f be the unique normalised cusp form of weight $\ell = 18$ for $\mathrm{SL}_2(\mathbb{Z})$. We expect to find 61 in the numerator of $\frac{L(3, \pi_f \otimes \pi_F)}{\pi^8 L(1, \pi_f \otimes \pi_F)}$.

In our numerical examples in §4, $j > 0$ and F is vector-valued. However, in the case that $j = 0$ and F is scalar-valued, congruent mod $\mathfrak{q}$ to the Saito-Kurokawa lift of g, a formula of Saha, for the restriction to (Siegel) $\mathfrak{H}_1 \times \mathfrak{H}_2$ of a genus 3 Hermitian Eisenstein series (non-holomorphic and non-convergent in our case), allows one to actually prove the expected divisibility. This is in the recent Sheffield Ph.D. thesis of Rendina [35].

4. Computing the L-values

4.1. Generalities. The method for computing Hecke eigenvalues of genus 2 cusp forms F, hence coefficients of $L(s, F, \mathrm{Spin})$, is described in [3, §7]. It was first carried out for $p \leq 37$ by Faber and van der Geer [20], and extended by the first-named author to obtain the first 180 coefficients in the Dirichlet series. As described in those references, computing the pth Dirichlet coefficient requires approximately p^4 operations. Thus, computing significantly more Dirichlet coefficients is not practical.

4.2. Computational results. Using the method described in Section 4.3, we have experimentally determined the following expressions involving special values.

Case 1: $\ell = \mathbf{16}, (j, k) = (\mathbf{4}, \mathbf{12})$

We have
$$\frac{\pi^8 L(3, \pi_f \otimes \pi_F)}{L(5, \pi_f \otimes \pi_F)} = \frac{7^2 \cdot 17 \cdot 839}{2^3 \cdot 3^2}$$
and
$$\frac{\pi^8 L(1, \pi_f \otimes \pi_F)}{L(3, \pi_f \otimes \pi_F)} = \frac{3^4 \cdot 7 \cdot 11^2 \cdot 71}{2^2 \cdot 5 \cdot 17}.$$
The 839 is as predicted by Example 1 in §3.1. The 71 will be explained in Example 3 in §6 below, as will the 17 in Example 5 in §7 below.

Case 2: $\ell = \mathbf{18}, (j, k) = (\mathbf{4}, \mathbf{15})$
We have
$$\frac{\pi^8 L(1, \pi_f \otimes \pi_F)}{L(3, \pi_f \otimes \pi_F)} = \frac{3 \cdot 5^4 \cdot 7 \cdot 13^2 \cdot 193}{2^4 \cdot 11 \cdot 61}.$$
The 61 is as predicted by Example 2 in §3.2. For a comment on why we are unable to account for the 193, see the end of Example 3 in §6.

Case 3: $\ell = \mathbf{16}, (j,k) = (\mathbf{6}, \mathbf{10})$

We have

$$\frac{\pi^8 L(1, \pi_f \otimes \pi_F)}{L(3, \pi_f \otimes \pi_F)} = \frac{3^4 \cdot 5^2 \cdot 61}{2^3}.$$

The 61 in the numerator will be revisited in Example 4 in §6 below.

4.3. Numerically evaluating L-functions.
We describe the numerical evaluation of the degree 8 L-function $L(s, \pi_f \otimes \pi_F)$, using the case $\ell = 16$, $(j,k) = (6, 10)$ as a representative example. We wish to evaluate that L-function at $s = 1$ and $s = 3$ to high precision so as to confidently identify a normalized ratio of those values as a rational number. (The meaning of 'high precision' depends on context. Here we will consider 30 digits to be a reasonable target.)

First we explain why this requires some effort. It is straightforward to make as many Dirichlet coefficients of $L(s, \pi_f)$ as we wish, but for $L(s, \pi_F)$ we have only the first 180 coefficients, so we have only the first 180 coefficients of $L(s) := L(s, \pi_f \otimes \pi_F)$. As remarked in Section 4.1, it is prohibitive to produce significantly more coefficients.

A common method of evaluating L-functions is to use the built-in functionality of Magma [27]. Since the functional equation satisfied by this L-function, in the unitary/analytic normalization, is

$$(4.1) \qquad \Lambda(s) := \Gamma_{\mathbb{C}}(s + 19)\Gamma_{\mathbb{C}}(s + 11)\Gamma_{\mathbb{C}}(s + 4)^2 L(s) = \Lambda(1 - s),$$

Magma can tell us how many coefficients are needed:

```
> L:=LSeries(1,[4,4,5,5,11,12,19,20],1,0: Sign:=1);
> N:=LCfRequired(L);N;
4145
```

Thus, more than 4000 coefficients are required for evaluating the L-function using standard methods, but we only have 180. Instead, we will use the methods of [21] to accurately evaluate the critical values using only the available coefficients. We summarize the ideas as applied to this example.

High-precision evaluations of general L-functions use the so-called approximate functional equation (see [37] for details and technical conditions). If $L(s) = \sum b_n n^{-s}$ has an analytic continuation to an entire function that satisfies the functional equation $\Lambda(s) = G(s)L(s) = \varepsilon \Lambda(1 - s)$, and $g(s)$ is a suitable auxiliary function, then

$$(4.2) \qquad L(s) = \sum_{n=1}^{\infty} (h_1(s, n) + \varepsilon h_2(1 - s, n)) \, b_n$$

where

$$h_1(s, n) := (g(s)G(s))^{-1} \frac{1}{2\pi i} \int_{\nu-i\infty}^{\nu+i\infty} \frac{g(s+z)G(s+z)}{n^{s+z}} \frac{dz}{z}$$

(4.3)

$$h_2(1-s, n) := (g(s)G(s))^{-1} \frac{1}{2\pi i} \int_{\nu-i\infty}^{\nu+i\infty} \frac{g(1-s+z)G(1-s+z)}{n^{1-s+z}} \frac{dz}{z}.$$

Here ν is any real number to the right of all poles of the integrand, and $g(s)$ is any entire function such that the integrals converge absolutely.

For this discussion, the important parameter is the test function $g(s)$. The idea is that we can evaluate (4.2) multiple times with different test functions. Each evaluation provides slightly different information, which we can combine to overcome the fact that we only have a few Dirichlet coefficients. We will use test functions of the form $g(s) = e^{i\beta s + \alpha s^2}$ with $\alpha > 0$, or $\alpha = 0$ and $|\beta| < d\frac{\pi}{4}$, where d is the degree of the L-function (in our example, $d = 8$).

If we insert a particular value for s, say $s = 1$, and let $g(s) = e^{i\beta s + s^2/1000}$, then (4.2) has the form

(4.4) $$L_\beta(1) = \sum_{n=1}^{\infty} c_\beta(n) b_n,$$

where $c_\beta(n)$ is a number which depends on β, n, and the parameters in the functional equation. The $L_\beta(1)$ on the left side of (4.4) is independent of β, but we use that notation to indicate which auxiliary function was used on the right side. We can evaluate $c_\beta(n)$ as accurately as we wish by numerically evaluating the integrals that appear in (4.3), which we now describe.

4.4. Numerically evaluating the integrals. This section is a summary of material from [37]. Our goal here is to provide information for someone to reproduce our calculation; our goal is not to provide a detailed exposition on numerically approximating integrals.

We wish to numerically calculate, to high precision, the numbers $c_\beta(n)$ in (4.4). This involves evaluating an integral of a product of Γ-functions, exponentials, and powers. This can be done using any computer algebra package which contains those functions and which can work to arbitrary precision; our computations were done in Mathematica [28].

The main issues involved are:

(1) Evaluating the integrand to sufficiently high precision,
(2) Truncating the improper integral, and
(3) Evaluating the resulting finite integral.

For issue (1) the concern is possible loss of significant digits due to cata-strophic cancellation. This is a minor issue in the case at hand, although it becomes a serious problem when evaluating L-functions at $s = \sigma + it$ with t large (because the completed L-function $\Lambda(s)$ is what is actually being computed, and it decreases exponentially as a function of t). See Section 3.3 of [37] for details.

In a system such as Mathematica, issue (1) is even less of a problem, because the software keeps track of the precision of the calculation. Should there be insufficient precision at the end, one merely re-calculates, begin-ning with a larger precision.

As explained in Sections 2.4 and 3.6 of [37], these integrals can be evalu-ated by a simple Riemann sum (i.e., a sum that evaluates the integrand at equally spaced points). In particular, the step size is inversely proportional to the number of digits of accuracy in the result. That addresses issue (3), and since the integrand is decreasing exponentially, also issue (2).

4.5. A numerical example. We will evaluate (4.4) with $\beta = 0$. We set $\nu = 3$, a stepsize of $1/5$ in the Riemann sum, summing from -29 to 29, and evaluate the integrand to 40 digits of precision. We find:

$$
\begin{aligned}
L_0(1) =& 1.245 \, b_1 + 0.534 \, b_2 + 0.269 \, b_3 + \cdots + 0.000668 \, b_{17} \\
&+ \cdots + 2.10 \times 10^{-10} \, b_{101} + \cdots + 8.56 \times 10^{-14} \, b_{181} \\
&+ \cdots + 1.1 \times 10^{-21} \, b_{499} + \cdots + 5.5 \times 10^{-29} \, b_{1009} \\
&+ \cdots + 7.3 \times 10^{-34} \, b_{1499} + \cdots + 8.3 \times 10^{-38} \, b_{1999} \\
&+ \cdots + 6.7 \times 10^{-53} \, b_{4999} + \cdots .
\end{aligned}
$$

(4.5)

One sees immediately that using 180 terms gives an error of more than 10^{-13}, which is far from our goal of 30 decimal digits. Note that the numer-ical values in (4.5), and in all following equations, are truncations of the actual value. For example, the actual computed coefficient of b_1 in (4.5) is

$$1.2453392504912166301069081878583745765950,$$

for which Mathematica reports an accuracy of 37 digits.

While (4.5) makes it appear that Magma's estimate of 4000 terms is conservative, this is partially explained by the fact that we are evaluating at the point $s = 1$. If we wanted to calculate $L(\frac{1}{2} + 100i)$ then many more terms would be needed. Nevertheless, we see that 180 terms, or even 1000 terms, are not adequate for the high precision evaluation we seek.

As we will explain in Section 4.6, we can achieve high precision by eval-uating the L-function several times, and then taking a linear combination of those evaluations. Thus, we need to evaluate (4.4) for other values of β,

say $\beta = \frac{3}{2}$:

$$L_{\frac{3}{2}}(1) = 1.870\, b_1 + 0.937\, b_2 + 0.017\, b_3 + \cdots + 0.0097\, b_{17}$$
$$+ \cdots - 2.10 \times 10^{-8}\, b_{101} + \cdots - 9.44 \times 10^{-12}\, b_{181}$$
$$+ \cdots + 4.6 \times 10^{-19}\, b_{499} + \cdots + 4.7 \times 10^{-25}\, b_{1009}$$
$$+ \cdots + 1.3 \times 10^{-29}\, b_{1499} + \cdots - 4.2 \times 10^{-33}\, b_{1999}$$
$$\text{(4.6)} \qquad + \cdots + 5.2 \times 10^{-47}\, b_{4999} + \cdots .$$

The choice of $\beta = \frac{3}{2}$ seems worse, because the contributions of the coefficients are decreasing less rapidly, so more terms will be required in order to obtain a given precision. This is indeed true, for among test functions of this form, $\beta = 0$ has the contributions decreasing most rapidly.

From (4.5) or (4.6) we can determine a value for $L(1)$ by using the known coefficients and estimating the others with the Ramanujan bound $|b_p| \leq 8$. Note that the known coefficients include not only b_n for $n \leq 180$, but also some larger numbers such as $b_{875} = b_7 b_{125}$. For $\beta = 0$ we find

$$L_0(1) = 1.79890282611850360616 7865 + 8.56 \times 10^{-14}\, b_{181}$$
$$+ 3.81 \times 10^{-14}\, b_{191} + 3.25 \times 10^{-14}\, b_{193} + 2.37 \times 10^{-14}\, b_{197}$$
$$+ \cdots + 1.1 \times 10^{-21}\, b_{499} + \cdots + 5.5 \times 10^{-29}\, b_{1009}$$
$$+ \cdots + 7.3 \times 10^{-34}\, b_{1499} + \cdots + 8.3 \times 10^{-38}\, b_{1999}$$
$$+ \cdots + 6.7 \times 10^{-53}\, b_{4999} + \cdots$$
$$\text{(4.7)} \qquad = 1.798902826118503 \pm 1.8 \times 10^{-12},$$

and for $\beta = \frac{3}{2}$:

$$L_{\frac{3}{2}}(1) = 1.7989028261235553720826 51 + 9.44 \times 10^{-12}\, b_{181}$$
$$+ 9.42 \times 10^{-12}\, b_{191} + 8.85 \times 10^{-12}\, b_{193} + 7.54 \times 10^{-12}\, b_{197}$$
$$+ \cdots + 4.6 \times 10^{-19}\, b_{499} + \cdots + 4.7 \times 10^{-25}\, b_{1009}$$
$$+ \cdots + 1.3 \times 10^{-29}\, b_{1499} + \cdots + 4.2 \times 10^{-33}\, b_{1999}$$
$$+ \cdots + 5.2 \times 10^{-47}\, b_{4999} + \cdots$$
$$\text{(4.8)} \qquad = 1.7989028261235 \pm 5.5 \times 10^{-10}.$$

Those values are far from our goal of 30 decimal digits of accuracy.

In (4.7) and (4.8), and below, the expression $a = b \pm c$ means that the true value of a lies in the interval $(b - c, b + c)$. To estimate the error, we used the first $20,000$ Dirichlet coefficients.

4.6. Averaging is better. In this section we use the main idea of [21], which is that one can obtain a more precise evaluation of $L(1)$ by combining

$L_\beta(1)$ for several auxiliary functions. That is, if

$$(4.9) \qquad\qquad\qquad \sum w_j = 1,$$

then

$$(4.10) \qquad\qquad\qquad L(1) = \sum w_j L_{\beta_j}(1).$$

As we will see, suitable choices of w_j and β_j in (4.10) will give a value of $L(1)$ with a small error term. For example, consider

(4.11)

$$\{\beta_j\} = \{0,\ \tfrac{1}{2},\ 1,\ \tfrac{3}{2}\}$$
$$\{w_j\} = \{5.595844269,\ -5.074113323,\ 0.484231975,\ -0.0059629212\}.$$

The "magic numbers" in (4.11) were chosen (using a least-squares fit) to minimize the contribution of $b_{181}, b_{191}, \ldots$ in (4.12) below.

Using those values for w_j and β_j, we find:

$$\begin{aligned}
L(1) = \sum & w_j L_{\beta_j}(1) \\
= {} & 1.79890282611860339 3418629 + 1.34 \times 10^{-17} b_{181} \\
& + 4.96 \times 10^{-18} b_{191} + 3.49 \times 10^{-17} b_{193} + 5.01 \times 10^{-17} b_{197} \\
& + \cdots + 1.1 \times 10^{-20} b_{499} + \cdots + 1.9 \times 10^{-27} b_{1009} \\
& + \cdots + 9.4 \times 10^{-32} b_{1499} + \cdots + 3.1 \times 10^{-35} b_{1999} \\
& + \cdots - 3.1 \times 10^{-49} b_{4999} + \cdots
\end{aligned}$$

$$(4.12) \qquad = 1.79890282611860339 \pm 3 \times 10^{-14}.$$

The error in (4.12) compared to (4.7) and (4.8) should be somewhat surprising. The error has decreased by a factor of 60 by combining only 4 evaluations of $L(1)$. Suppose that, instead, we wanted to improve the error in (4.7) by determining more coefficients. We would need to determine the value of b_p for the 11 primes $181 \le p \le 239$ in order to have a comparable decrease in the error term.

The improved error in (4.12) indicates that the contributions from the unknown coefficients, for different auxiliary functions, are negatively correlated. The exact nature of this correlation has not been described analytically: we take it as an empirical fact.

By combining the evaluation of $L_\beta(1)$ for $\beta \in \{0, 1/10, 2/10, \ldots, 30/10\}$, with suitably chosen (by least-squares) "weights" $\{w_0, w_1, \ldots, w_{30}\}$, we find

$$(4.13) \qquad L(1) = 1.79890282611860303245572277 2619 \pm 6 \times 10^{-26}$$

and in the same way,

$$(4.14) \qquad L(3) = 1.10545688795132163036935934 1690 \pm 3 \times 10^{-27}.$$

Thus

$$\frac{\pi^8 L(1)}{L(3)} = 15440.625000000000000000000000096 \pm 6 \times 10^{-22}$$

$$\text{(4.15)} \qquad = \frac{3^4 \cdot 5^2 \cdot 61}{2^3} \pm 6 \times 10^{-22}.$$

This precision of 27 digits does not quite meet our goal of 30 digits of accuracy in the final result, but the identification of $\pi^8 L(1)/L(3)$ as a rational number seems convincing. Similar calculations produced the other values in Section 4.2.

Note that this approach to evaluating L-functions requires significantly more computation than the methods used when more coefficients are known. With known coefficients, the sum in (4.2) and the integrals in (4.3) can be interchanged, so only two integrals need to be computed numerically. For the above calculation, it was necessary to compute each of the thousands of integrals separately. Furthermore, those integrals were computed multiple times: once for each different auxiliary function. Only after evaluating all those integrals could we optimally combine them to minimize the contributions of the unknown coefficients. If the appropriate combinations could somehow be determined in advance, our ability to evaluate higher-degree L-functions would be substantially improved.

5. Expected consequences of congruences revisited: the Bloch-Kato conjecture

5.1. Statement of the conjecture. Recall the rank-8 motive $M = M_f \otimes M_F$ such that $L(M, s) = L(f \otimes F, s)$. (We shall assume at least the existence of a premotivic structure comprising realisations and comparison isomorphisms, as defined in [14, 1.1.1].) In our examples the coefficient field is $\mathbb{Q}$. Let $q > j + 2k + \ell - 3$ be a prime number. Choose a $\mathbb{Z}_{(q)}$-lattice T_B in the Betti realisation M_B in such a way that $T_q := T_B \otimes \mathbb{Z}_q$ is a $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant lattice in the q-adic realisation M_q, via the comparison isomorphism $M_B \otimes \mathbb{Q}_q \simeq M_q$. Then choose a $\mathbb{Z}_{(q)}$-lattice T_{dR} in the de Rham realisation M_{dR} in such a way that

$$\mathbb{V}(T_{\text{dR}} \otimes \mathbb{Z}_q) = T_q$$

as $\text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$-representations, where $\mathbb{V}$ is the version of the Fontaine-Lafaille functor used in [14]. Since $\mathbb{V}$ only applies to filtered ϕ-modules, where ϕ is the crystalline Frobenius, $T_{\text{dR}} \otimes \mathbb{Z}_q$ must be ϕ-stable. Anyway, this choice ensures that the q-part of the Tamagawa factor at q is trivial (by [5, Theorem 4.1(iii)]), thus simplifying the Bloch-Kato conjecture below. The condition $q > j + 2k + \ell - 3$ ensures that the condition (*) in [5, Theorem 4.1(iii)] holds.

Let t be a critical point at which we evaluate the L-function. Let $M(t)$ be the corresponding Tate twist of the motive. Let $\Omega(t)$ be a Deligne period scaled according to the above choice, i.e. the determinant of the isomorphism

$$M(t)_B^+ \otimes \mathbb{C} \simeq (M(t)_{\mathrm{dR}}/\mathrm{Fil}^0) \otimes \mathbb{C},$$

calculated with respect to bases of $(2\pi i)^t T_B^{(-1)^t}$ and $T_{\mathrm{dR}}/\mathrm{Fil}^t$, so well-defined up to $\mathbb{Z}_{(q)}^\times$.

The following formulation of the (q-part of the) Bloch-Kato conjecture, as applied to this situation, is based on [14, (59)] (where Σ was non-empty, though), using the exact sequence in their Lemma 2.1.

Conjecture 5.1 (Bloch-Kato)**.**

$$\mathrm{ord}_q \left(\frac{L(M,t)}{\Omega(t)} \right)$$

$$= \mathrm{ord}_q \left(\frac{\#H_f^1(\mathbb{Q}, T_q^*(1-t) \otimes (\mathbb{Q}_q/\mathbb{Z}_q))}{\#H^0(\mathbb{Q}, T_q^*(1-t) \otimes (\mathbb{Q}_q/\mathbb{Z}_q)) \#H^0(\mathbb{Q}, T_q(t) \otimes (\mathbb{Q}_q/\mathbb{Z}_q))} \right).$$

Here, $T_q^* = \mathrm{Hom}_{\mathbb{Z}_q}(T_q, \mathbb{Z}_q)$, with the dual action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This is an invariant $\mathbb{Z}_q$-lattice in $M_q^* \simeq M_q(j + 2k + \ell - 4)$, so $T_q^*(1-t)$ is a lattice in $M_q(j + 2k + \ell - 3 - t)$. On the right hand side, in the numerator, is a Bloch-Kato Selmer group with local conditions (unramified at $p \neq q$, crystalline at $p = q$) for all finite primes p.

5.2. Global torsion and Kurokawa-Mizumoto type congruences.
We revisit the situation of §3.1. Recall that $\lambda_F(p)$ denotes the eigenvalue of the genus-2 Hecke operator $T(p)$ acting on the cuspidal eigenform F. The q-adic realisations $M_{f,q}$ and $M_{F,q}$ should be 2-and 4-dimensional $\mathbb{Q}_q$ vector spaces with continuous linear actions ρ_f, ρ_F of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, crystalline at q, unramified at all primes $p \neq q$. For primes $p \neq q$, we should have

$$a_p(f) = \mathrm{Tr}(\rho_f(\mathrm{Frob}_p^{-1})) \quad \text{and} \quad \lambda_F(p) = \mathrm{Tr}(\rho_F(\mathrm{Frob}_p^{-1})).$$

Galois representations with these properties are known to exist, by theorems of Deligne and Weissauer [13, 44]. By Poincaré duality, $M_{f,q}^* \simeq M_{f,q}(\ell - 1)$ and $M_{F,q}^* \simeq M_{F,q}(j + 2k - 3)$. Choosing $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant $\mathbb{Z}_q$-lattices in $M_{f,q}$ and $M_{F,q}$, then reducing mod q, we obtain residual representations $\overline{\rho}_f$ and $\overline{\rho}_F$. We suppose that (as in Example 1) $\overline{\rho}_f$ is irreducible, in which case it is independent of the choice of lattice in $M_{f,q}$. The congruence

$$\lambda_F(p) \equiv a_p(f)(1 + p^{k-2}) \pmod{q},$$

interpreted as a congruence of traces of Frobenius, implies that the composition factors of $\overline{\rho}_F$ are $\overline{\rho}_f$ and $\overline{\rho}_f(2 - k)$. Which is a submodule and which is a quotient will depend on the choice of lattice in $M_{F,q}$.

Looking at the denominator of the Bloch-Kato formula, with T_q the tensor product of the $\mathbb{Z}_q$-lattices referred to above, on which $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts by $\rho_f \otimes \rho_F \simeq \rho_f^*(1 - \ell) \otimes \rho_F$, the q-torsion in $H^0(\mathbb{Q}, T_q(t) \otimes (\mathbb{Q}_q/\mathbb{Z}_q))$ is $(\overline{\rho}_f^* \otimes \overline{\rho}_F(t + 1 - \ell))^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$, which is $\mathrm{Hom}_{\mathbb{F}_q[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]}(\overline{\rho}_f, \overline{\rho}_F(t + 1 - \ell))$. Recalling that $\ell = j+k$, this is the same as $\mathrm{Hom}_{\mathbb{F}_q[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]}(\overline{\rho}_f(2-k), \overline{\rho}_F(t + 3 - j - 2k))$. This can be non-trivial only for $t \equiv \ell - 1 \pmod{q - 1}$ (if ρ_f is a submodule of ρ_F) or for $t \equiv j + 2k - 3 \pmod{q - 1}$ (if $\rho_f(2 - k)$ is a submodule of ρ_F). The only such t in the critical range $\ell \leq t \leq j + 2k - 3$ (using $q > j + 2k + \ell - 4$ from §5.1, or even just $q > 2\ell$ from §3.1) is $t = j + 2k - 3$. So, with a suitable choice of lattice, and $t = j + 2k - 3$, we can have a factor of q in the denominator of the conjectural formula for $\frac{L(M,t)}{\Omega(t)}$, which appears to provide some explanation for the q in the denominator of $\frac{L(j+2k-3, f \otimes F)}{\pi^8 L(j+2k-5, f \otimes F)}$, observed in Example 1.

Note that the factor $L(s - (k - 2), \mathrm{Sym}^2 f)$ in the expression in §3.1 for $L(f \otimes [f]_j)$ has trivial zeros at the points $s = \ell, \ell + 2$ paired with $s = j + 2k - 3, j + 2k - 5$ by the functional equation. This is because $\ell - (k - 2) = j + 2$ and $\ell + 2 - (k - 2) = j + 4$, which are even and in the range $1 \leq t \leq \ell - 1$, at least if $k > 4$. This suggests that the orders of Selmer groups may contribute cancelling factors of q to the numerators of $L_{\mathrm{alg}}(j + 2k - 3, f \otimes F)$ and $L_{\mathrm{alg}}(j + 2k - 5, f \otimes F)$, something we overlooked in the final paragraph of [18, §4.2]. Note also that one can make a similar construction of global torsion elements with respect to congruences of "Yoshida type" (which appear in [4, Conjecture 10.7]), but in that case there are no critical L-values.

5.3. Moving between Selmer groups via Harder type congruences. Now we revisit the situation of §3.2. There $t = (\ell/2) + j + k - 1$, and $T_q^*(1 - t)$ is a lattice in $M_q(j + 2k + \ell - 3 - t) = M_q((\ell/2) + k - 2)$. By an analogue of the Birch and Swinnerton-Dyer conjecture, vanishing of $L(f, \ell/2)$ should suffice for the non-triviality of $H_f^1(\mathbb{Q}, M_{f,q}(\ell/2))$ (again defined using local conditions). (See the "conjectures" $C_r(M)$ in §1 of [22], and $C_\lambda^i(M)$ in §6.5 of [22].) The sign in the functional equation of $L(f, s)$ is $(-1)^{\ell/2} = -1$, so the parity of the order of vanishing at $s = \ell/2$ is odd. Assuming that $\overline{\rho}_f$ is irreducible, the conditions of [34, Theorem B] are satisfied. Hence $H_f^1(\mathbb{Q}, M_{f,q}(\ell/2))$ is non-trivial (because the parity of its rank is also odd). If we were to impose a condition that f is ordinary at q (i.e. $q \nmid a_q(f)$), then we could alternatively get this from either [41, Théorème A] or the main theorem of [33, §12].

Anyway, from this one easily obtains a non-zero element

$$c'' \in H^1(\mathbb{Q}, \overline{\rho}_f(\ell/2)).$$

Assuming irreducibility of $\overline{\rho}_f$, it is a consequence of Harder's conjectured congruence that the composition factors of $\overline{\rho}_F$ are $\overline{\rho}_g$, $\mathbb{F}_q(2-k)$ and $\mathbb{F}_q(1-j-k)$. If we choose the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant $\mathbb{Z}_q$-lattice in $M_{F,q}$ in such a way that the composition factor $\mathbb{F}_q(2-k)$ of $\overline{\rho}_F$ is a submodule, then $\overline{\rho}_f(2-k)$ is a submodule of $\overline{\rho}_f \otimes \overline{\rho}_F$, so $\overline{\rho}_f(\ell/2)$ is a submodule of $\overline{\rho}_f \otimes \overline{\rho}_F((\ell/2)+k-2)$. Thus we may map c'' to $H^1(\mathbb{Q}, \overline{\rho}_f \otimes \overline{\rho}_F((\ell/2)+k-2))$, thence to $H^1(\mathbb{Q}, T_q^*(1-t) \otimes (\mathbb{Q}_q/\mathbb{Z}_q))$. Assuming that $\overline{\rho}_f \not\simeq \overline{\rho}_g$ (e.g. if $\ell \neq k'$ and $q > \max\{\ell, k'\}$) one easily checks that $H^0(\mathbb{Q}, \overline{\rho}_f \otimes \overline{\rho}_F((\ell/2)+k-2))$ is trivial, from which it follows that this produces a *non-zero* element of $H^1(\mathbb{Q}, T_q^*(1-t) \otimes (\mathbb{Q}_q/\mathbb{Z}_q))$. If $q > j+2k+\ell-3$ one can in fact show, as in the proof of [19, Proposition 5.1], that this element is in $H_f^1(\mathbb{Q}, T_q^*(1-t) \otimes (\mathbb{Q}_q/\mathbb{Z}_q))$. This puts a factor of q in the numerator of the conjectural formula for $\frac{L(M,t)}{\Omega(t)}$, which appears to provide some explanation for the q in the numerator of $\frac{L((\ell/2)+j+k-1, f \otimes F)}{\pi^8 L((\ell/2)+j+k-3, f \otimes F)}$, observed in Example 2.

Analogous situations were already considered in [17, §8,§11,§14] and [19, Conjecture 5.4, Corollary 8.6].

6. Endoscopic congruences for SO(7).

Example 3. Recall that when $\ell = 16$ and $(j,k) = (4,12)$, we found an apparent factor of 71 in the numerator of

$$\frac{\pi^8 L(1, \pi_f \otimes \pi_F)}{L(3, \pi_f \otimes \pi_F)} = \frac{\pi^8 L((\ell+j+2k-2)/2, f \otimes F)}{L((\ell+j+2k+2)/2, f \otimes F)}.$$

With $q = 71$ and $t = (\ell+j+2k-2)/2$ (which is the integer immediately to the right of the centre of the functional equation), we would like to construct a non-zero element in $H_f^1(\mathbb{Q}, T_q^*(1-t) \otimes (\mathbb{Q}_q/\mathbb{Z}_q))$ to explain this. The q-torsion in $T_q^*(1-t) \otimes (\mathbb{Q}_q/\mathbb{Z}_q)$ is (the space of) $\overline{\rho}_f \otimes \overline{\rho}_F((\ell+j+2k-4)/2) \simeq \mathrm{Hom}_{\mathbb{F}_q}(\overline{\rho}_f((\ell+2-j-2k)/2), \overline{\rho}_F)$, using $\overline{\rho}_f^* \simeq \overline{\rho}_f(\ell-1)$. Note that the Hodge type of $M_f((\ell+2-j-2k)/2)$ is $\{((j+2k-\ell-2)/2, \ell-1+(j+2k-\ell-2)/2), (\ell-1+(j+2k-\ell-2)/2, (j+2k-\ell-2)/2)\}$, which is $\{(5,20),(20,5)\}$ in this case, and the effect of the twist is to make the "weight" $w = "p+q" = \ell-1+(j+2k-\ell-2) = j+2k-3$, equal to that of the Hodge type $\{(0, j+2k-3), (k-2, j+k-1), (j+k-1, k-2), (j+2k-3, 0)\}$ of M_F, which is $\{(0,25),(10,15),(15,10),(25,0)\}$ in this case. This raises the possibility that $\overline{\rho}_f((\ell+2-j-2k)/2) = \overline{\rho}_f(-5)$ and $\overline{\rho}_F$ could both occur as composition factors in the reduction mod q of an invariant $\mathbb{Z}_q$-lattice in a 6-dimensional Galois representation coming from the q-adic realisation of a rank 6 motive, pure of weight $j+2k-3$, with Hodge type the union of those of $M_f((\ell+2-j-2k)/2)$ and M_F.

If such a 6-dimensional q-adic Galois representation $\tilde{\rho}$ exists, and if it is irreducible, then adapting a well-known construction of Ribet [36], there exists an invariant $\mathbb{Z}_q$-lattice whose reduction mod q is a non-trivial extension of $\overline{\rho}_f((\ell+2-j-2k)/2)$ by $\overline{\rho}_F$ (both of which we suppose to be irreducible). This gives a non-zero element in $H^1(\mathbb{Q}, \operatorname{Hom}_{\mathbb{F}_q}(\overline{\rho}_f((\ell+2-j-2k)/2), \overline{\rho}_F))$. Since $H^0(\mathbb{Q}, \operatorname{Hom}_{\mathbb{F}_q}(\overline{\rho}_f((\ell+2-j-2k)/2), \overline{\rho}_F))$ is trivial, the image of this element in $H^1(\mathbb{Q}, T_q^*(1-t) \otimes (\mathbb{Q}_q/\mathbb{Z}_q))$ is non-zero. If we also suppose that $\tilde{\rho}$ is unramified at all $p \neq q$, crystalline at q, then one can show (using $q > (3j + 6k + \ell - 8)/2$) that it lies in fact in $H_f^1(\mathbb{Q}, T_q^*(1-t) \otimes (\mathbb{Q}_q/\mathbb{Z}_q))$, as desired. It remains to explain where $\tilde{\rho}$ comes from.

Given a self-dual, cuspidal, automorphic representation π of $\operatorname{GL}_6(\mathbb{A})$, there is an associated $\tilde{\rho} : \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{GL}_6(\mathbb{Q}_q)$ (see [40, Remark 7.6]). If π is unramified at all finite places then $\tilde{\rho}$ is unramified at all $p \neq q$ and crystalline at q. It is not currently known to be irreducible, but we shall assume that, as expected, it is, so that the above construction applies. The infinitesimal character of π_∞ determines the Hodge-Type of the conjectural motive of which $\tilde{\rho}$ should be the q-adic realisation [11] (and the Hodge-Tate weights of $\tilde{\rho}|_{\operatorname{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)}$). A self-dual, cuspidal automorphic representation of $\operatorname{PGL}_6(\mathbb{A})$ discovered by Chenevier and Renard, denoted $\Delta_{25,15,5}$ in [9, Table 13], has the correct infinitesimal character. By Arthur's symplectic-orthogonal alternative [9, Theorem* 3.9], it is the functorial lift of a discrete automorphic representation of $\operatorname{SO}(4,3)(\mathbb{A})$.

Let $\operatorname{SO}(7)$ be the special orthogonal group of the E_7 root lattice, the even, positive-definite lattice of discriminant 2, unique up to isomorphism. This is a semi-simple group over $\mathbb{Z}$, and $\operatorname{SO}(7)(\mathbb{Z}) \simeq W(E_7)^+$, the even subgroup of the Weyl group, of order 1451520. For $\mu = a_1 e_1 + a_2 e_2 + a_3 e_3$ (in the notation of [9, 5.2]), with $a_1, a_2, a_3 \in \mathbb{Z}$ and $a_1 \geq a_2 \geq a_3 \geq 0$, let V_μ be the space of the complex representation θ_μ of $\operatorname{SO}(7)$ with highest weight μ, and let $\rho := \frac{5}{2}e_1 + \frac{3}{2}e_2 + \frac{1}{2}e_3$. The infinitesimal character of the representation θ_μ of $\operatorname{SO}(7)(\mathbb{R})$ is $\mu + \rho$. Let K be the open compact subgroup $\prod_p \operatorname{SO}(7)(\mathbb{Z}_p)$ of $\operatorname{SO}(7)(\mathbb{A}_f)$, and let

$$M(V_\mu, K) := \{f : \operatorname{SO}(7)(\mathbb{A}_f) \to V_\mu : f(gk) = f(g) \ \forall k \in K,$$
$$f(\gamma g) = \theta_\mu(\gamma)(f(g)) \ \forall \gamma \in \operatorname{SO}(7)(\mathbb{Q})\}$$

be the space of V_μ-valued algebraic modular forms with level K (i.e. "level 1"), where $\mathbb{A}_f$ is the "finite" part of the adele ring. Since

$$\#(\operatorname{SO}(7)(\mathbb{Q})\backslash \operatorname{SO}(7)(\mathbb{A}_f)/K) = 1,$$

$M(V_\mu, K)$ can be identified with the fixed subspace $V_\mu^{\operatorname{SO}(7)(\mathbb{Z})}$.

If we let $\mu = 10e_1 + 6e_2 + 2e_3$, so that $\mu + \rho = (1/2)(25e_1 + 15e_2 + 5e_3)$, then $M(V_\mu, K)$ is 2-dimensional, spanned by K-fixed vectors of automorphic

representations of SO(7)($\mathbb{A}$) whose Arthur parameters are $\Delta_{25,15,5}$ and an endoscopic lift denoted $\Delta_{25,5} \oplus \Delta_{15}$ in [9, Table 13]. Note that $(25,5) = (j + 2k - 3, j + 1)$ and $15 = \ell - 1$, with $(j,k) = (4,12)$ and $\ell = 16$. Mégarbané has calculated the traces of certain Hecke operators $T(p)$ (for $p \leq 53$) on spaces including this one [31]. The contribution $p^5 a_p(f) + \lambda_F(p)$ from $\Delta_{25,5} \oplus \Delta_{15}$ is easily subtracted off (as below) to find the eigenvalue denoted $T(p)(\Delta_{25,15,5})$, in fact Mégarbané has recorded the results in [30, Tables 2,4,5,6]. A congruence of Hecke eigenvalues

$$T(p)(\Delta_{25,15,5}) \equiv p^5 a_p(f) + \lambda_F(p) \pmod{71}$$

for all primes p would, viewing them as traces of Frobenius, imply that $\overline{\tilde{\rho}}$ has composition factors $\overline{\rho}_f((\ell + 2 - j - 2k)/2)$ and $\overline{\rho}_F$, which is what we need. We confirmed this congruence for all $p \leq 53$. In the tables below, we show the results for $2 \leq p \leq 11$ and $p = 53$.

p	$a_p(f)$	$\lambda_F(p)$	$\mathrm{Tr}(T(p)|V_\mu^{\mathrm{SO}(7)(\mathbb{Z})})$
2	216	-96	6816
3	-3348	-527688	-474120
5	52110	596139180	145932324
7	2822456	-3608884496	49205357040
11	20586852	3047542095144	3229012641000
53	6797151655902	-3921035060705523617268	-8934610079
			5036491708

p	$T(p)(\Delta_{25,15,5})$	$-T(p)(\Delta_{25,15,5}) + p^5 a_p(f) + \lambda_F(p)$
2	0	$2^5.3.\mathbf{71}$
3	867132	$-2^7.3^5.\mathbf{71}$
5	-613050606	$2^9.3.23.\mathbf{71}.547$
7	5377223544	$2^8.3^4.7^2.13.41.\mathbf{71}$
11	-3134062555596	$2^7.3.23.\mathbf{71}.15145211$
53	989150772174783875874	$-2^9.3^3.\mathbf{71}.73.1031.27990002153$

Recall from §4.2 that when $\ell = 18$ and $(j,k) = (4,15)$, we see a factor of 193 in the numerator of $\frac{\pi^8 L(1,\pi_f \otimes \pi_F)}{L(3,\pi_f \otimes \pi_F)}$. Were we to try to account for this in a similar manner, we would have to look at $\mu + \rho = (1/2)(31e_1 + 17e_2 + 5e_3)$, in which case $M(V_\mu, K)$ is 8-dimensional and $\Delta_{31,17,5}$ is not unique (using [10]). Hence we cannot extract Hecke eigenvalues from traces of Hecke operators in the same way, so are unable to test the expected mod 193 congruence.

Example 4. Recall that when $\ell = 16$ and $(j, k) = (6, 10)$, we found an apparent factor of 61 in the numerator of

$$\frac{\pi^8 L(1, \pi_f \otimes \pi_F)}{L(3, \pi_f \otimes \pi_F)} = \frac{\pi^8 L((\ell + j + 2k - 2)/2, f \otimes F)}{L((\ell + j + 2k + 2)/2, f \otimes F)}.$$

In the same way as for the previous example, this could be explained by a certain mod 61 congruence of Hecke eigenvalues of algebraic modular forms for SO(7), this time with $\mu = 9e_1 + 6e_2 + 3e_3$. We have again used Mégarbané's data to verify the congruence for $p \leq 53$. Similarly, it is necessary to subtract from the trace a contribution from $\Delta_{23,7} \oplus \Delta_{15}$, to get the Hecke eigenvalue for $\Delta_{23,15,7}$.

p	$\lambda_F(p)$	$\text{Tr}(T(p)|V_\mu^{\text{SO}(7)(\mathbb{Z})})$
2	1680	4416
3	-6120	148104
5	2718300	-89271276
7	6916898800	10652657232
11	-1417797110136	-764339838888
53	-15111411349636553220	86535126376033794804

p	$T(p)(\Delta_{23,15,7})$	$-T(p)(\Delta_{23,15,7}) + p^4 a_p(f) + \lambda_F(p)$
2	-720	$2^5.3.\mathbf{61}$
3	425412	$-2^8.3^2.5.\mathbf{61}$
5	-124558326	$2^{10}.3.\mathbf{61}.853$
7	-3040958424	$2^9.3^7.5.7^2.\mathbf{61}$
11	352045171116	$-2^8.3.5.7^2.31.\mathbf{61}.4127$
53	48013741730657079162	$-2^{10}.3^3.5.17.\mathbf{61}.66215793179$

In fact, Mégarbané has very recently proved this congruence unconditionally for all p [29, Théorème 1.0.3(x)]. He uses scalar-valued algebraic modular forms for SO(25), in the manner of Chenevier and Lannes's proof of Harder's mod 41 congruence using O(24) [8, Chapter 10, Theorem* 4.4(1)]. He found that the modulus of the congruence is in fact $5856 = 2^5.3.61$.

7. An Eisenstein congruence for SO(9).

Example 5. Recall that when $\ell = 16$ and $(j, k) = (4, 12)$, we found an apparent factor of 17 in the denominator of

$$\frac{\pi^8 L(1, \pi_f \otimes \pi_F)}{L(3, \pi_f \otimes \pi_F)} = \frac{\pi^8 L((\ell + j + 2k - 2)/2, f \otimes F)}{L((\ell + j + 2k + 2)/2, f \otimes F)},$$

and in the numerator of $\frac{\pi^8 L(3, \pi_f \otimes \pi_F)}{L(5, \pi_f \otimes \pi_F)}$, so apparently in the numerator of $L_{\text{alg}}((\ell + j + 2k + 2)/2, f \otimes F)$. With $q = 17$ and $t = (\ell + j + 2k + 2)/2$

(which is no longer immediately to the right of the centre of the functional equation), we would like to construct a non-zero element in $H^1_f(\mathbb{Q}, T^*_q(1 - t) \otimes (\mathbb{Q}_q/\mathbb{Z}_q))$ to try to explain this, though the condition $q > j + 2k + \ell - 3$ does not hold here. The q-torsion in $T^*_q(1 - t) \otimes (\mathbb{Q}_q/\mathbb{Z}_q)$ is (the space of) $\overline{\rho}_f \otimes \overline{\rho}_F((\ell + j + 2k - 8)/2) \simeq \mathrm{Hom}_{\mathbb{F}_q}(\overline{\rho}_f((\ell + 6 - j - 2k)/2), \overline{\rho}_F)$, using $\overline{\rho}^*_f \simeq \overline{\rho}_f(\ell - 1)$.

We would like to see $\overline{\rho}_f((\ell + 6 - j - 2k)/2) = \overline{\rho}_f(-3)$ and $\overline{\rho}_F$ both occurring as composition factors in the reduction mod q of an invariant $\mathbb{Z}_q$-lattice in a 6-dimensional Galois representation coming from the q-adic realisation of a rank 6 motive. Then we could apply the construction of Ribet again (though q is not large enough now for us to prove the local condition at q). The problem is, M_F still has Hodge type $\{(0, 25), (10, 15), (15, 10), (25, 0)\}$, of weight 25, while the Hodge type of $M_f(-3)$ is $\{(3, 18), (18, 3)\}$, of weight only 21. What we need to do is to balance $\overline{\rho}_f(-3)$ with another composition factor $\overline{\rho}_f(-7)$, noting that the Hodge type of $M_f(-7)$ is $\{(7, 22), (22, 7)\}$, and $3 + 22 = 7 + 18 = 25$. Now $\overline{\rho}_f(-3)$, $\overline{\rho}_f(-7)$ and $\overline{\rho}_F$ could all occur as composition factors in the reduction mod q of an invariant $\mathbb{Z}_q$-lattice in an 8-dimensional Galois representation coming from the q-adic realisation of a rank 8 motive, pure of weight 25, with Hodge type $\{(0, 25), (3, 22), (7, 18), (10, 15), (15, 10), (18, 7), (22, 3), (25, 0)\}$. Although this is not the union of the Hodge types of $M_f(-3)$, $M_f(-7)$ and M_F, the union of the sets of Hodge-Tate weights of their q-adic realisations restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$ is $\{0, 3, 7, 10, 15, 18, 22, 25\}$.

This time a self-dual, cuspidal, automorphic representation of $\mathrm{PGL}_8(\mathbb{A})$ discovered by Chenevier and Renard [9, Corollary**6.5, Table 8], denoted $\Delta_{25,19,11,5}$ in their notation, has the correct infinitesimal character. By Arthur's symplectic-orthogonal alternative [9, Theorem* 3.9], it is the functorial lift of a discrete automorphic representation of $\mathrm{SO}(5, 4)(\mathbb{A})$. Again, there is an associated $\tilde{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_8(\mathbb{Q}_q)$ (see [40, Remark 7.6]). The relevant space of algebraic modular forms for $\mathrm{SO}(9)$ is 3-dimensional, spanned by Hecke eigenforms that are vectors in automorphic representations of $\mathrm{SO}(9)(\mathbb{A})$ with Arthur parameters $\Delta_{25,19,11,5}$ and $\Delta^2_{25,19,5} \oplus \Delta_{11}$, [30, Table 1]. Here $\Delta^2_{25,19,5}$ stands for a pair of self-dual, cuspidal, automorphic representations of $\mathrm{PGL}_6(\mathbb{A})$, and $\Delta^2_{25,19,5} \oplus \Delta_{11}$ for a pair of endoscopic lifts. To get the Hecke eigenvalues we want, for $\Delta_{25,19,11,5}$, one must subtract the endoscopic contributions from the traces computed by Mégarbané for $p \leq 7$ [31]. Also, computing the trace of the $\mathrm{SO}(7)$ $T(p)$ on $\Delta^2_{25,19,5}$ similarly requires the subtraction of an endoscopic contribution by $\Delta_{25,5} \oplus \Delta_{19}$ from a trace on a whole space of algebraic modular forms. One can obtain the $T(p)(\Delta_{25,19,11,5})$ directly from [30, Table 7].

The congruence verified in the second table for $p \leq 7$, if it held for all p, would imply that (with $q = 17$) $\tilde{\overline{\rho}}$ has composition factors $\overline{\rho}_f(-3)$,

$\bar{\rho}_f(-7)$ and $\bar{\rho}_F$. Note that $\bar{\rho}_f$ is certainly irreducible, by [42, Corollary to Theorem 4], and the irreducibility of $\bar{\rho}_F$ can presumably be checked as in [8, Proposition 4.10]. This congruence (disregarding the smallness of q) is an instance of the kind considered in [2], in the case $G = \mathrm{SO}(5,4)$. The expression $(p^3+p^7)a_p(f)+\lambda_F(p)$ is the eigenvalue of $T(p)$ on an automorphic representation of $G(\mathbb{A})$ induced from a maximal parabolic subgroup with Levi subgroup $M \simeq \mathrm{GL}(2) \times \mathrm{SO}(3,2)$. (Harder's congruences above are a different instance, as explained in [2, §7] and [3, §3.2].)

p	$\mathrm{Tr}(T(p)|V_\mu^{\mathrm{SO}(7)(\mathbb{Z})})$	$\lambda_F(p) = $ $T(p)(\Delta_{25,5})$	$T(p)(\Delta_{19})$	$\mathrm{Tr}(T(p))(\Delta_{25,19,5}^2)$
2	10176	-96	456	6624
3	929988	-527688	50652	90072
5	-36016170	596139180	-2377410	-334979100
7	-40517568504	-3608884496	-16917544	-31105966416

p	$\mathrm{Tr}(T(p)|V_{\mu'}^{\mathrm{SO}(9)(\mathbb{Z})})$	$T(p)(\Delta_{25,19,11,5})$	$-T(p)(\Delta_{25,19,11,5})$ $+(p^3+p^7)a_p(f) + \lambda_F(p)$
2	5280	4800	$2^5.3^2.5.\mathbf{17}$
3	889920	-302400	$-2^8.3^3.5.13.\mathbf{17}$
5	-345413400	-765121800	$2^{10}.3^2.5.\mathbf{17}.53.131$
7	-29042227200	29642547200	$2^9.3^3.5.7.\mathbf{17}.191.1459$

8. An endoscopic congruence for $\mathrm{SO}(9)$.

Example 6. Now consider a self-dual, cuspidal, automorphic representation of $\mathrm{PGL}_8(\mathbb{A})$ discovered by Chenevier and Renard [9, Corollary**6.5, Table 8], denoted $\Delta_{25,21,15,9}$ in their notation. By Arthur's symplectic-orthogonal alternative [9, Theorem* 3.9], it is the functorial lift of a discrete automorphic representation of $\mathrm{SO}(5,4)(\mathbb{A})$. There is an associated $\tilde{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_8(\mathbb{Q}_q)$ (see [40, Remark 7.6]). The relevant space of algebraic modular forms for $\mathrm{SO}(9)$ is 3-dimensional, spanned by Hecke eigenforms that are vectors in automorphic representations of $\mathrm{SO}(9)(\mathbb{A})$ with Arthur parameters $\Delta_{25,21,15,9}$, $\Delta_{25} \oplus \Delta_{15} \oplus \Delta_{21,9}$ and $\Delta_{21,9} \oplus \Delta_{25,15}$ [30, Table 1]. To get the Hecke eigenvalues we want, for $\Delta_{25,19,11,5}$, one must subtract the endoscopic contributions from the traces calculated by Mégarbané for $p \le 7$, in fact he has done that and listed the results in [30, Table 7]. Here F and G are genus 2 cuspidal, Hecke eigenforms for $\mathrm{Sp}_2(\mathbb{Z})$, with $(j,k) = (8,8)$ and $(14,7)$ respectively.

p	$T(p)(\Delta_{25,21,15,9})$	$\lambda_F(p) := T(p)(\Delta_{21,9})$	$\lambda_G(p) := T(p)(\Delta_{25,15})$
2	-7200	1344	-3696
3	631200	-6408	511272
5	6175800	-30774900	118996620
7	25981995200	451366384	-82574511536

p	$p^2\lambda_F(p) + \lambda_G(p) - T(p)(\Delta_{25,21,15,9})$
2	$2^4.3.5.\mathbf{37}$
3	$-2^6.3.5^2.\mathbf{37}$
5	$-2^8.3.5.\mathbf{37}.4621$
7	$-2^7.3^3.5.\mathbf{37}.135197$

The congruence verified in the second table for $p \leq 7$, if it held for all p, would imply that (with $q = 37$) $\tilde{\bar{\rho}}$ has composition factors $\bar{\rho}_F(-2)$ and $\bar{\rho}_G$. The irreducibility of $\bar{\rho}_F$ and $\bar{\rho}_G$ can presumably be checked as in [8, Proposition 4.10]. Now reasoning as in Example 3, but in the opposite direction, we should expect to find the prime factor 37 in the numerator of $\pi^8 \frac{L(1,\pi_F \otimes \pi_G)}{L(2,\pi_F \otimes \pi_G)}$, a ratio of critical values for a degree-16 GSp$_2 \times$ GSp$_2$ *L*-function. (As was pointed out to us by M. Watkins, $L(3, \pi_F \otimes \pi_G)$ is not critical.) Using the same techniques as in §4, we obtain the approximation $\pi^8 \frac{L(1,\pi_F \otimes \pi_G)}{L(2,\pi_F \otimes \pi_G)} \approx 6243.7501$, likely to be correct to 7 or 8 significant figures. This approximation is not as accurate as what we obtained in our degree 8 examples, but we note that $6243.75 = \frac{3^3.5^2.\mathbf{37}}{2^2}$. This is suggestive of the expected prime factor of 37. However, this evidence is not as convincing as, for example, equation (4.15).

Though it was the narrowness of the critical range that forced evaluation of the *L*-function at adjacent points, doing so had a fortuitous side-benefit. The quantity $\pi^8 \frac{L(1,\pi_F \otimes \pi_G)}{L(2,\pi_F \otimes \pi_G)}$ is small enough that 7 or 8 significant figures are enough to go comfortably beyond approximating just the integer part, and to reveal what is presumably the correct rational number. This would not have been the case for the much larger $\pi^{16} \frac{L(1,\pi_F \otimes \pi_G)}{L(3,\pi_F \otimes \pi_G)}$, even had $L(3, \pi_F \otimes \pi_G)$ been critical.

References

[1] T. ARAKAWA, "Vector valued Siegel's modular forms of degree two and the associated Andrianov *L*-functions", *Manuscripta math.* **44** (1983), p. 155-185.

[2] J. BERGSTRÖM, N. DUMMIGAN, "Eisenstein congruences for split reductive groups", *Selecta Math.* **22** (2016), p. 1073-1115.

[3] J. BERGSTRÖM, N. DUMMIGAN, T. MÉGARBANÉ, T. IBUKIYAMA, H. KATSURADA, "Eisenstein congruences for SO(4, 3), SO(4, 4), spinor and triple product *L*-values", *Experimental Mathematics* **27** (2018), p. 230-250.

[4] J. BERGSTRÖM, C. FABER, G. VAN DER GEER, "Siegel modular forms of degree three and the cohomology of local systems", *Selecta Math.* **20** (2014), p. 83-124.

[5] S. Bloch, K. Kato, "L-functions and Tamagawa numbers of motives", *The Grothendieck Festschrift Volume I*, Progress in Mathematics, 86, Birkhäuser, Boston, 1990, p. 333-400.

[6] S. Böcherer, N. Dummigan, R. Schulze-Pillot, "Yoshida lifts and Selmer groups", *J. Math. Soc. Japan* **64** (2012), p. 1353-1405.

[7] S. Böcherer, B. Heim, "L-functions on $GSp_2 \times GL_2$ of mixed weights", *Math. Z.* **235** (2000), p. 11-51.

[8] G. Chenevier, J. Lannes, Automorphic forms and even unimodular lattices, Ergebnisse der Mathematik und ihre Grenzgebiete 69, Springer, Cham, 2019.

[9] G. Chenevier, D. Renard, "Level one algebraic cusp forms of classical groups of small rank", *Mem. Amer. Math. Soc.* **1121**, vol. 237, 2015.

[10] G. Chenevier, D. Renard, Level one algebraic cusp forms of classical groups, website, http://gaetan.chenevier.perso.math.cnrs.fr/levelone.html

[11] L. Clozel, "Motifs et formes automorphes: applications du principe de functorialité", *Automorphic forms, Shimura varieties and L-functions, Vol. I*, Academic Press, London, 1990, p. 77-159.

[12] P. Deligne, "Valeurs de Fonctions L et Périodes d'Intégrales", *AMS Proc. Symp. Pure Math.*, Vol. 33 part 2, 1979, p. 313-346.

[13] P. Deligne, "Formes modulaires et représentations l-adiques", *Sém. Bourbaki*, éxp. 355, Lect. Notes Math., Vol. 179, Springer, Berlin, 1969, p. 139-172.

[14] F. Diamond, M. Flach, L. Guo, "The Tamagawa number conjecture of adjoint motives of modular forms", *Ann. Sci. École Norm. Sup. (4)* **37** (2004), p. 663-727.

[15] T. Dokchitser, "Computing special values of motivic L-functions", *Experiment. Math.* **13** (2004), p. 137-149.

[16] N. Dummigan, "Symmetric square L-functions and Shafarevich-Tate groups, II", *Int. J. Number Theory* **5** (2009), p. 1321-1345.

[17] N. Dummigan, "Symmetric square L-functions and Shafarevich-Tate groups", *Experiment. Math.* **10** (2001), p. 383-400.

[18] N. Dummigan, B. Heim, A. Rendina, "Kurokawa-Mizumoto congruences and degree-8 L-functions", *Manuscripta math.* **160** (2019), p. 217-237.

[19] N. Dummigan, T. Ibukiyama, H. Katsurada, "Some Siegel modular standard L-values, and Shafarevich-Tate groups", *J. Number Theory* **131**(2011), p. 1296-1330.

[20] C. Faber, G. van der Geer, "Sur la cohomologie des systèmes locaux sur les espaces de modules des courbes de genre 2 et des surfaces abéliennes, I, II", *C. R. Math. Acad. Sci. Paris* **338** (2004), p. 381-384 and p. 467-470.

[21] D. W. Farmer, N. C. Ryan, "Evaluation of L-functions with few known coefficients", *LMS J. Comput. Math.* **17** (2014), p. 245-258.

[22] J.-M. Fontaine, "Valeurs spéciales des fonctions L des motifs", Séminaire Bourbaki, Vol. 1991/92, *Astérisque* **206** (1992), Exp. No. 751, 4, p. 205-249.

[23] M. Furusawa, "On L-functions for $GSp(4) \times GL(2)$ and their special values", *J. f. d. reine u. angew. Math.* **438** (1993), p. 187-218.

[24] G. Harder, "A congruence between a Siegel and an elliptic modular form, manuscript", 2003, reproduced in *The 1-2-3 of Modular Forms*, Springer-Verlag, Berlin Heidelberg, 2008, p. 247-262.

[25] B. Heim, "Pullbacks of Eisenstein series, Hecke-Jacobi theory and automorphic L-functions", *AMS Proc. Symp. Pure Math.*, Vol. 66 (1999), p. 201-238 and p. 313-346.

[26] N. Kurokawa, "Congruences between Siegel modular forms of degree 2", *Proc. Japan Acad.* **55A** (1979), p. 417-422.

[27] The Magma computer algebra system http://magma.maths.usyd.edu.au/magma/

[28] The Mathematica computer algebra system http://www.wolfram.com/mathematica/

[29] T. Mégarbané, "Calcul des opérateurs de Hecke sur les classes d'isomorphisme de réseaux pairs de déterminant 2 en dimension 23 et 25", *J. Number Theory* **186** (2018), p. 370-416.

[30] T. Mégarbané, "Traces des opérators de Hecke sur les espaces de formes automorphes de SO_7, SO_8 ou SO_9 en niveau 1 et poids arbitraire", *J. Théor. Nombres Bordeaux* **30** (2018), p. 239-306.

[31] T. Mégarbané, "Calcul des traces d'opérators de Hecke sur les espaces de formes automorphes", website, http://megarban.perso.math.cnrs.fr/

[32] S. Mizumoto, "Congruences for eigenvalues of Hecke operators on Siegel modular forms of degree two", *Math. Ann.* **275** (1986), p. 149-161.

[33] J. Nekovář, "Selmer Complexes", *Astérisque* **310** (2006).

[34] J. Nekovář, "Some consequences of a formula of Mazur and Rubin for arithmetic local constants", *Algebra Number Theory* **7** (2013), p. 1101-1120.

[35] A. Rendina, "Congruences of Saito-Kurokawa lifts and divisibility of degree-8 *L*-values", Ph.D. thesis, University of Sheffield, August 2019.

[36] K. Ribet, "A modular construction of unramified *p*-extensions of $\mathbb{Q}(\mu_p)$", *Invent. math.* **34** (1976), p. 151-162.

[37] M. Rubinstein, "Computational methods and experiments in analytic number theory", *Recent perspectives in random matrix theory and number theory*, London Math. Soc. Lecture Note Ser., 322, Cambridge Univ. Press, Cambridge, 2005, p. 425-506.

[38] T. Satoh, "On certain vector valued Siegel modular forms of degree two", *Math. Ann.* **274** (1986), p. 335-352.

[39] J.-P. Serre, "Facteurs locaux des fontions zêta des variétés algébriques (définitions et conjectures)", *Séminaire Delange-Pisot-Poitou*, 1969/70, no. **19**.

[40] S. W. Shin, "Galois representations arising from some compact Shimura varieties", *Ann. of Math.* (2) **173** (2011), p. 1645-1741.

[41] C. Skinner, E. Urban, "Sur les déformations *p*-adiques de certaines représentations automorphes", *J. Inst. Math. Jussieu* **5** (2006), p. 629-698.

[42] H. P. F. Swinnerton-Dyer, "On *l*-adic representations and congruences for coefficients of modular forms", *Modular functions of one variable, III*, Lecture Notes in Math., Vol. 350, Springer, Berlin, 1973, p. 1-55.

[43] G. van der Geer, "Siegel Modular Forms and Their Applications", *The 1-2-3 of Modular Forms*, Springer-Verlag, Berlin Heidelberg, 2008, p. 181-245.

[44] R. Weissauer, "Four dimensional Galois representations", *Astérisque* **302** (2005), p. 67-150.

[45] H. Yoshida, "Motives and Siegel modular forms", *Amer. J. Math.* **123** (2001), 1171–1197.

Matematiska institutionen
Stockholms universitet
106 91 Stockholm
Sweden.
E-mail : jonasb@math.su.se

University of Sheffield
School of Mathematics and Statistics
Hicks Building
Hounsfield Road
Sheffield, S3 7RH
U.K.
E-mail : n.p.dummigan@shef.ac.uk

American Institute of Mathematics
600 East Brokaw Road
San Jose, CA 95112
U.S.A.
E-mail : farmer@aimath.org

Department of Physics and Astronomy
Bucknell University
Lewisburg, PA 17837
U.S.A.
E-mail : koutslts@bucknell.edu